An Efficient Variational Multiphase Motion for the Mumford-Shah Segmentation Model

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Abstract

We propose here a new multiphase level set model for image segmentation, using Mumford-Shah techniques in the piecewise-constant case. The proposed model is also a generalization of our active contour model without edges based on a two-phase segmentation. In order to handle multiple individual segments and complex topologies (such as multiple junctions), we propose a new and efficient multiphase representation by level sets: the necessary number of level set functions is considerably reduced (we need only n level set functions to represent 2^n phases or segments), and in addition we have overcomed the problems of vacuum and overlap, naturally arising in multiphase problems. Finally, we show how the model can be used in image segmentation, for synthetic and real (possible noisy) pictures.

1. Introduction: motivations and terminology

An important problem in image processing is the segmentation of a picture representing a real scene, into classes or categories, corresponding to different objects and the background in the image. In the end, each pixel should belong to one class and only one. In other words, we look for a partition of the image into distinct segments.

In this paper, we propose an efficient multiphase-level set algorithm for image segmentation, based on the piecewise-constant Mumford-Shah model. This is also a generalization of our active contour model without edges, from [8] and [7], based on a two-phase segmentation.

To illustrate our algorithm, we only present the two-dimensional case for grey-level images, but the model can be easily formulated and applied in any dimension (such as volumetric images), as well as for vector-valued images (such as color RGB images, multichannels, etc). For more details, we refer the reader to [8], [6], [9] and [10].

Let us first fix the notations. Let $\Omega \subset \mathbf{R}^2$ be an open and bounded domain (usually a rectangle in $\mathbf{R}^2$), and $u_0 : \Omega \to \mathbf{R}$ be a given image-function (usually taking values between 0 and 255). We will denote by C a set of curves in Ω, and $|C|$ will be the total length of curves making up C. For such C, we can denote by Ω_i the connected components of $\Omega \setminus C$, such that

$$\Omega = \cup_i \Omega_i \cup C.$$

The piecewise-constant Mumford-Shah segmentation model [16] is formulated as follows: given u_0, find an optimal set of curves C and a piecewise-constant image-function $u : \Omega \to \mathbf{R}$, such that $u = c_i$ in each Ω_i, minimizing the energy:

$$F(u, C) = \sum_i \int_{\Omega_i} |u_0 - c_i|^2 dxdy + \nu |C|, \qquad (1)$$

where $\nu > 0$ is a fixed parameter. By minimizing the above energy, we look for the best global approximation of u_0 by the simplest function u (piecewise-constant). The constraint on the set of edges C requires the regions Ω_i to have smooth boundaries. The segmented image-function u can be expressed as: $u(x, y) = \sum_i c_i \chi_{\Omega_i}(x, y)$, where χ_{Ω_i} is the characteristic function of the set Ω_i. For the piecewise-constant function u, we also have:

$$\int_\Omega |u - u_0|^2 dxdy = \sum_i \int_{\Omega_i} |u_0 - c_i|^2 dxdy,$$

by the definition of u and because Ω_i is a partition of Ω.

Minimizing $F(u, C)$ with respect to c_i, it is easy to obtain that

$$c_i = mean_{\Omega_i}(u_0) = \frac{\int_{\Omega_i} u_0(x, y) dxdy}{\text{area}(\Omega_i)}.$$

In [8] and [7], we have proposed an active contour model without edges, based on a two-phase segmentation. Let us

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assume that C (the snake or the active contour) is the boundary of an open subset of Ω, and consider the interior and the exterior sets of C, called $inside(C)$ and $outside(C)$. The active contour model is defined via an energy minimization, as follows:

$$\begin{aligned} \inf_{c_1,c_2,C} F_2(c_1,c_2,C) &= \int_{inside(C)} |u_0 - c_1|^2 dxdy \\ &+ \int_{outside(C)} |u_0 - c_2|^2 dxdy \\ &+ \nu |C|. \end{aligned}$$

This model performs as active contours, looking for a 2-phase segmentation of the image. The segmented image will be $u = c_1$ inside C, and $u = c_2$ outside C.

In order to minimize in practice $F_2(c_1, c_2, C)$, the level set method of S. Osher and J. Sethian [17] was used, combined with techniques from [24]. The curve C is represented implicitly, via a level set function $\phi : \Omega \to \boldsymbol{R}$ (assumed to be Lipschitz), such that

$$\begin{cases} \phi(x,y) > 0 \text{ inside } C, \\ \phi(x,y) < 0 \text{ outside } C, \\ \phi(x,y) = 0 \text{ on the curve } C. \end{cases}$$

Then, using the Heaviside function $H(\phi) = 1$ if $\phi \geq 0$ and $H(\phi) = 0$ if $\phi < 0$, the minimization of $F_2(c_1, c_2, C)$ can be written as:

$$\begin{aligned} \inf_{c_1,c_2,\phi} F_2(c_1,c_2,\phi) &= \int_\Omega |u_0 - c_1|^2 H(\phi) dxdy \\ &+ \int_\Omega |u_0 - c_2|^2 (1 - H(\phi)) dxdy \\ &+ \nu \int_\Omega |\nabla H(\phi)| dxdy. \end{aligned}$$

In this formulation, the segmented image u can be expressed as: $u(x,y) = c_1 H(\phi) + c_2(1 - H(\phi))$.

Considering H_ε and δ_ε any C^1 approximations and regularizations of the Heaviside function H and Dirac function δ_0 (concentrated at 0), as $\varepsilon \to 0$, such that $H'_\varepsilon = \delta_\varepsilon$, and minimizing the energy $F_2(c_1, c_2, \phi)$ with respect to c_1, c_2 and ϕ, we obtain the equations:

$$c_1 = \frac{\int_\Omega u_0 H(\phi) dxdy}{\int_\Omega H(\phi) dxdy}, \quad c_2 = \frac{\int_\Omega u_0 (1 - H(\phi)) dxdy}{\int_\Omega (1 - H(\phi)) dxdy},$$

$$\frac{\partial \phi}{\partial t} = \delta_\varepsilon(\phi) \Big[\mathrm{div}\Big(\frac{\nabla \phi}{|\nabla \phi|} \Big) - \nu (|u_0 - c_1|^2 - |u_0 - c_2|^2) \Big],$$

where div denotes the divergence operator for the spatial variables. Evolving the above equation in ϕ, moves the curve $C = \{\phi = 0\}$ implicitly.

This active contour model is a particular case of the Mumford-Shah piecewise-constant model, as a two-phase segmentation. It has the advantage of detecting contours both with or without gradient, automatically detects interior contours, and is robust in the presence of noise. These advantages are due to the fact that the stopping criteria is different than the one usually used in more classical active contour models or snakes (see [3], [4], [11], [12], [14], [22]): here, the stopping criteria is based on a global approximation of the image, and not only on local information (the gradient, for instance).

Many other authors have done work on image segmentation. We cite here only a few of them, more related with the present paper: [25], [1], [2], [5], [13], [20], [21]. For a detailed exposition of the segmentation problem by variational methods, we refer the reader to [15]. Finally, we would like to refer the reader to two related works: [23] and [18].

2. Description of the multiphase image segmentation model

The main goal of this paper is to extend the previous active contour model without edges to images with more than two individual segments, triple junctions, etc. The basic idea is to use several level set functions. In this context, we refer the reader to [24] and [19], where to each phase or segment, a level set function is associated, and provide a solution to these problems (multiple phases, triple junctions, etc.). The work [24] was proposed for motion of multiple junctions (triple junctions for instance) in fluid dynamics, and the work [19] is an application of the previous one to supervised image classification. But this can be computationally expensive, especially in image segmentation, where we often deal with many distinct intensity levels, which requires a large number of level set functions.

In addition, when working with multiple phases, the problems of vacuum (the union of phases is not the entire domain), and overlap (the phases are not disjoint) arise, and should be avoided. In [24] (and then in [19]), these problems are solved by using additional constraints, making the numerical problem more complicated. Our main idea here is to propose a different multiphase formulation, having the following advantages: the number of level set functions is considerably reduced (only n level set functions to represent up to 2^n phases); the phases partition the domain Ω into disjoint sets (no overlap), whose union is Ω (no vacuum); while still representing triple junctions, multiple segments, etc.

Let us explain our multiphase formulation next:

- using one level set function ϕ defined as above, we can represent up to two phases, given by the disjoint regions $\{\phi > 0\}$ and $\{\phi < 0\}$, simply denoted by "+" and "−";

- using two level set functions ϕ_1 and ϕ_2, we can represent up to four phases, given by the disjoint regions $\{\phi_1 > 0, \phi_2 > 0\}$, $\{\phi_1 > 0, \phi_2 < 0\}$, $\{\phi_1 < 0, \phi_2 > 0\}$ and

$\{\phi_1 < 0, \phi_2 < 0\}$, again simply denoted by "++", "+−", "−+", "−−";

- using three level set functions ϕ_1, ϕ_2, ϕ_3, we can define up to eight phases, given by the disjoint regions "+ + +", "+ + −", "+ − +", "+ − −", "− + +", "− + −", "− − +", "− − −"; and so on,

- giving n level set functions $\phi_1, ..., \phi_n$, we can define in the same way up to 2^n disjoint regions.

In all cases, the set of curves C is defined by:

$$C = \cup\{(x, y) \in \Omega | \phi_i(x, y) = 0\},$$

and the union of the disjoint regions and of the curves $\{\phi_i = 0\}$ is a covering of Ω (clearly, there is no vacuum and no overlap, by definition).

Now let us show how the two-phase active contour model, minimizing the energy $F(c_1, c_2, \phi)$, can be extended to more than two phases, using the previous multiphase formulation. For instance, the corresponding energy for 4 phases or classes (using two level set functions ϕ_1, ϕ_2), is:

$$\begin{aligned} F_4(c, \Phi) &= \int_\Omega |u_0 - c_{11}|^2 H(\phi_1) H(\phi_2) dxdy \\ &+ \int_\Omega |u_0 - c_{10}|^2 H(\phi_1)(1 - H(\phi_2)) dxdy \\ &+ \int_\Omega |u_0 - c_{01}|^2 (1 - H(\phi_1)) H(\phi_2) dxdy \\ &+ \int_\Omega |u_0 - c_{00}|^2 (1 - H(\phi_1))(1 - H(\phi_2)) dxdy \\ &+ \nu \int_\Omega |\nabla H(\phi_1)| + \nu \int_\Omega |\nabla H(\phi_2)|, \end{aligned}$$

where $c = (c_{11}, c_{10}, c_{01}, c_{00})$, and $\Phi = (\phi_1, \phi_2)$.

With these notations, we can express, similarly with the general case, the segmented image-function u in a linear combination of the characteristic functions of the four phases:

$$\begin{aligned} u = c_{11} H(\phi_1) H(\phi_2) + c_{10} H(\phi_1)(1 - H(\phi_2)) \\ + c_{01}(1 - H(\phi_1)) H(\phi_2) + c_{00}(1 - H(\phi_1))(1 - H(\phi_2)). \end{aligned}$$

The Euler-Lagrange equations obtained by minimizing $F_4(c, \Phi)$ with respect to c and Φ are:

$$\begin{cases} c_{11} = mean(u_0) \text{ in } \{\phi_1 > 0, \phi_2 > 0\} ='' ++'' \\ c_{10} = mean(u_0) \text{ in } \{\phi_1 > 0, \phi_2 < 0\} ='' +-'' \\ c_{01} = mean(u_0) \text{ in } \{\phi_1 < 0, \phi_2 > 0\} ='' -+'' \\ c_{00} = mean(u_0) \text{ in } \{\phi_1 < 0, \phi_2 < 0\} ='' --'', \end{cases}$$

$$\begin{aligned} \frac{\partial \phi_1}{\partial t} &= \delta_\varepsilon(\phi_1)\Big[\nu \text{div}\Big(\frac{\nabla \phi_1}{|\nabla \phi_1|}\Big) \\ &- \Big((u_0 - c_{11})^2 - (u_0 - c_{01})^2\Big) H(\phi_2) \\ &- \Big((u_0 - c_{10})^2 - (u_0 - c_{00})^2\Big)(1 - H(\phi_2))\Big], \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \phi_2}{\partial t} &= \delta_\varepsilon(\phi_2)\Big[\nu \text{div}\Big(\frac{\nabla \phi_2}{|\nabla \phi_2|}\Big) \\ &- \Big((u_0 - c_{11})^2 - (u_0 - c_{01})^2\Big) H(\phi_1) \\ &- \Big((u_0 - c_{10})^2 - (u_0 - c_{00})^2\Big)(1 - H(\phi_1))\Big]. \end{aligned}$$

The above coupled equations for ϕ_1 and ϕ_2 govern the evolution of the curves $\{\phi_1 = 0\} \cup \{\phi_2 = 0\}$.

In an analogue fashion, we can write the energy for three level set functions, and so on.

We will see that we can successfully segment images with multiple segments, triple junctions, and complex topologies. the general model also inherits the advantages of the active contour model without edges: detects edges with or without gradient, automatically detects interior contours, and is robust with respect to noise.

3. Numerical results

We present here numerical result on synthetic and real images. We show on the left column the evolving contours over the original image, and on the right column the detected means, providing the segmentation of the image.

In Figure 1 we show a result with the 2-phase segmentation (i.e. our active contour model without edges), using one level set function. We see how interior contours are detected automatically.

In Figure 2 we show a result with a 4-phase segmentation, using two level set functions; again, interior contours are automatically detected. Here we detect four distinct segments.

In Figure 3, we use an 8-phase segmentation model, with three level set functions. The algorithm depicts six final segments. We see that triple junctions can be represented using our multiphase formulation.

Finally, in Figure 4, we show a result on a real picture, using the four-phase segmentation model. We plot the final four segments individually. Here, we have used the same type of initial curves as in the previous example.

4. Conclusions

In this work, we have proposed an efficient multiphase level set formulation for segmentation of images, based on the piecewise-constant Mumford-Shah model. By our formulation, we can handle triple junctions, multiple segments, and in addition, the phases cannot overlap or produce vacuum, by construction. We use a reduced number of level set functions to represent the phases: we only need n level set functions, to represent up to 2^n phases or segments. Finally, we validated our segmentation model on various images.

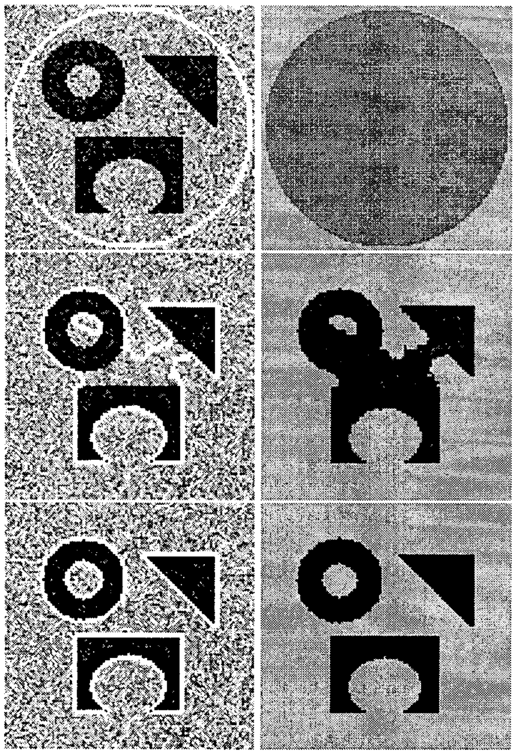

Figure 1. Two-phase segmentation with one level set function.

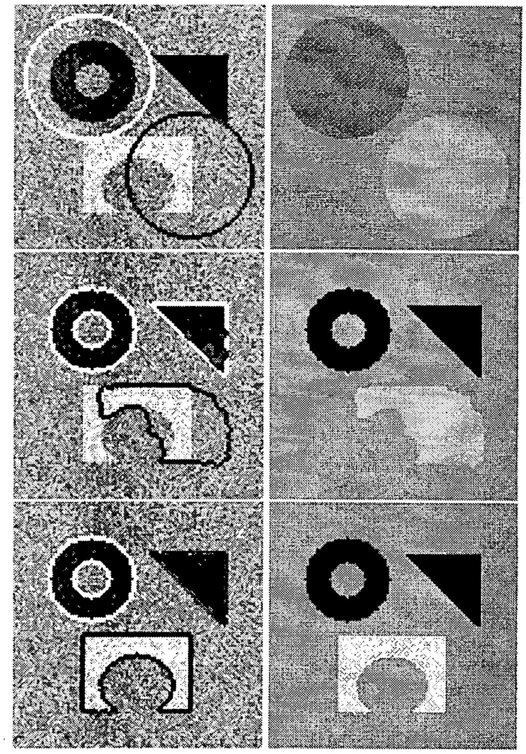

Figure 2. Four-phase segmentation with two level set functions.

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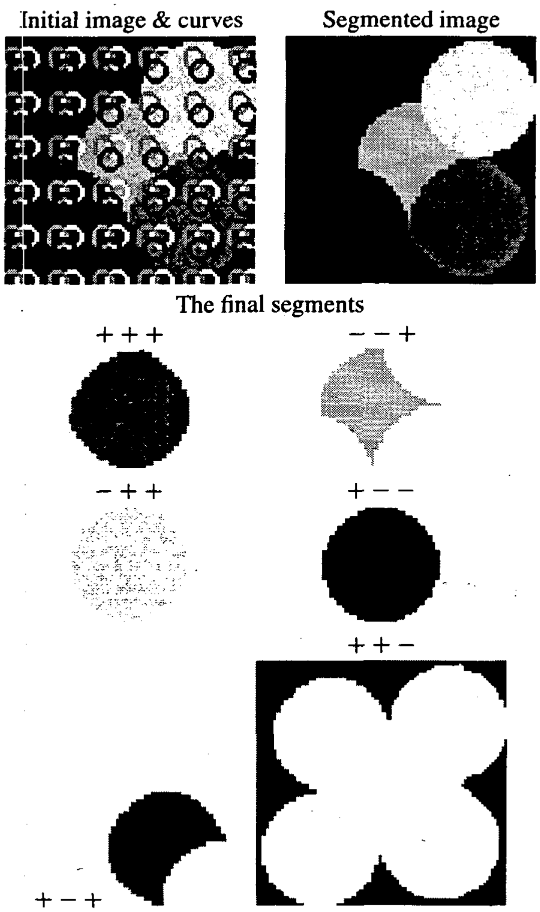

Figure 3. Eight-phase segmentation with three level set functions. Six phases and several triple junctions are detected.

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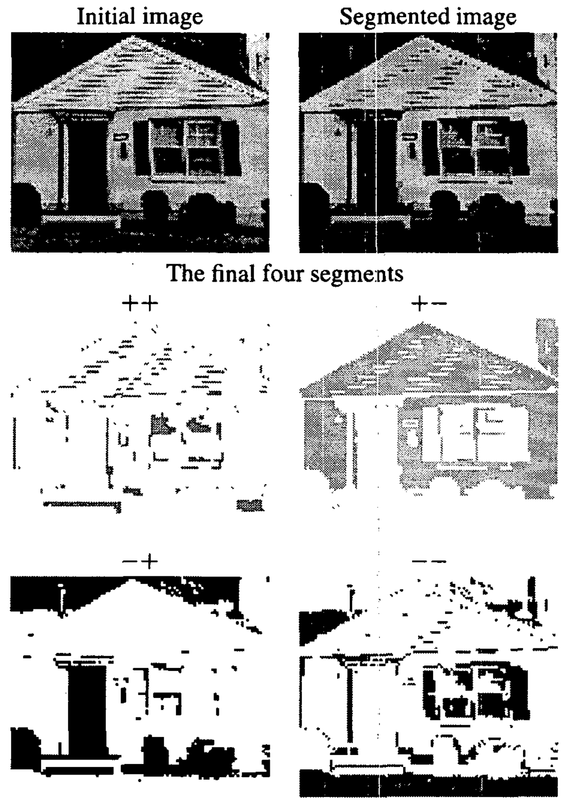

Figure 4. Four-phase segmentation with two level set functions.

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