Math 285J, L. Vese. **Assignment 1:**

[**1**] Consider in two dimensions the functional minimization problem

$$\inf_u F(u) = F_1(u) + \lambda F_2(u_0 - Ku),$$

where $u_0 : \Omega \to \mathbb{R}$ is a given degraded version of a true (unknown) image $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^2$, and K is a linear and continuous operator on $L^2(\Omega)$. Here, F_1 represents the regularization term while F_2 represents the data fidelity term. Recall that $|\nabla u| = \sqrt{(u_x)^2 + (u_y)^2}$.

Assume $u_0, u \in L^2(\Omega)$, $F_1(u) = \int_\Omega \phi_1(|\nabla u|)dxdy$, $F_2(u_0 - Ku) = \int_\Omega \phi_2(u_0 - Ku)dxdy$, where $\nabla u = (u_x, u_y)$ is the spatial gradient operator, $\phi_i : \mathbb{R} \to \mathbb{R}$ are functions of class C^1 $(i = 1, 2)$, and that $\phi_2'(u_0 - Ku) \in L^2(\Omega)$, as long as $u_0 - Ku \in L^2(\Omega)$.

(i) For $u \in W^{1,1}(\Omega)$, obtain the Euler-Lagrange equation associated with the minimization problem in u, in the stationary and time-dependent cases, together with the appropriate boundary conditions in u on $\partial\Omega$. For the time-dependent case, show that the energy $E(t) = F(u(x, y, t))$ is decreasing in time. [1]

(ii) Show that, if ϕ_i, $i = 1, 2$ are both convex, and ϕ_1 is in addition non-decreasing from $[0, \infty)$ to $[0, \infty)$, then the functional $F(u)$ is convex.

[**2**] Consider in two dimensions $f \in L^2(\Omega)$, and $u(\cdot, \lambda)$ the minimizer of

$$F(u) = \lambda \int_\Omega |\nabla u|dxdy + \frac{1}{2}\int_\Omega (u - f)^2 dxdy,$$

with $\lambda > 0$. Recall that $|\nabla u| = \sqrt{(u_x)^2 + (u_y)^2}$ can be made differentiable substituting it by $\sqrt{\epsilon^2 + (u_x)^2 + (u_y)^2}$.

(i) Using the result from the previous problem, give the associated Euler-Lagrange equation with the corresponding boundary conditions for a minimizer $u = u(\cdot, \lambda) \in W^{1,1}(\Omega)$.

(ii) Show that the L^2-norm of $u(\cdot, \lambda)$, given by $\sqrt{\int_\Omega (u(x, y, \lambda))^2 dxdy}$ is bounded by a constant independent of λ.

(iii) Show (e.g. using the obtained stationary E.-L. equation and associated boundary condition), that

$$\int_\Omega u(x, y, \lambda)dxdy = \int_\Omega f(x, y)dxdy.$$

(iv) Show that $u(\cdot, \lambda)$ converges in the $L^1(\Omega) - \textit{strong}$ topology to the average of the initial data. In other words, show that

$$\lim_{\lambda \to \infty} \int_\Omega \left| u(x, y, \lambda) - \frac{\int_\Omega f(x, y)dxdy}{|\Omega|} \right| dxdy = 0.$$

[1] We may need to formally assume, in addition, that $(Ku)_t = K(u_t)$; this is natural for a linear and continuous operator K that does not depend on t, for instance if $Ku = k * u$ and $k = k(x, y)$ does not depend on t.

[3] Discretize and implement the stationary or the non-stationary E.-L. equation from **[2]** by the method of your choice using finite differences, for the denoising case. More details will be discussed in class. Choose an original true image $\hat{u}$, and define a noisy version $f = \hat{u} + noise$ (see matlab sample codes on the class web-page, or in matlab you can add noise of zero mean to an image using "imnoise"). Give the optimal λ (may be different in each case) and the RMSE between the original clean image $\hat{u}$ and the reconstructed image u:

$$RMSE = \sqrt{\frac{\sum_{i=1,j=1}^{i=M,j=N} (\hat{u}(i,j) - u(i,j))^2}{MN}}.$$

Plot the energy versus iterations.

Optional: You can make additional tests by substituting the data fidelity term $\|f - u\|_{L^2(\Omega)}^2$ above by $\|f - u\|_{L^2(\Omega)}$ or by $\|f - u\|_{L^1(\Omega)}$, and compare the results. Each method may require different λ for the same image. λ can also be automatically selected if we know the noise variance in the form $\|f - u\|^2 = \sigma^2$. Using a norm, instead of the norm square for the data fidelity term avoids the intensity loss drawback of the ROF model.