Consider in two dimensions $f \in L^2(\Omega)$, and $u(\cdot, \lambda)$ the minimizer of

$$F(u) = \lambda \int_{\Omega} |\nabla u| dxdy + \frac{1}{2} \int_{\Omega} (u - f)^2 dxdy,$$

with $\lambda > 0$. Recall that $|\nabla u| = \sqrt{(u_x)^2 + (u_y)^2}$ can be made differentiable substituting it by $\sqrt{\varepsilon^2 + (u_x)^2 + (u_y)^2}$.

(i) Give the Euler-Lagrange equation associated with the minimization, along with the corresponding boundary conditions for a minimizer $u = u(\cdot, \lambda) \in W^{1,1}(\Omega)$.

(ii) Show that the $L^2$-norm of $u(\cdot, \lambda)$, given by $\sqrt{\int_{\Omega} (u(x, y, \lambda))^2 dxdy}$ is bounded by a constant independent of $\lambda$.

(iii) Show (e.g. using the obtained stationary E.-L. equation and associated boundary condition), that

$$\int_{\Omega} u(x, y, \lambda) dxdy = \int_{\Omega} f(x, y) dxdy.$$

(iv) Show that $u(\cdot, \lambda)$ converges in the $L^1(\Omega) - strong$ topology to the average of the initial data. In other words, show that

$$\lim_{\lambda \to \infty} \int_{\Omega} |u(x, y, \lambda) - \frac{\int_{\Omega} f(x, y) dxdy}{|\Omega|}| dxdy = 0.$$

Discretize and implement the stationary or the non-stationary E.-L. equation from [1] by the method of your choice using finite differences, for the denoising case (discretization posted online). Choose an original true image $\hat{u}$, and define a noisy version $f = \hat{u} + \text{noise}$ (see matlab sample codes on the class web-page, or in matlab you can add noise of zero mean to an image using “imnoise”). Give the optimal $\lambda$ (may be different in each case) and the RMSE between the original clean image $\hat{u}$ and the reconstructed image $u$:

$$RMSE = \sqrt{\frac{\sum_{i=1}^{M} \sum_{j=1}^{N} (\hat{u}(i, j) - u(i, j))^2}{MN}}.$$

Plot the energy versus iterations.

Optional: You can make additional tests by substituting the data fidelity term $\|f - u\|_{L^2(\Omega)}^2$ above by $\|f - u\|_{L^2(\Omega)}$ or by $\|f - u\|_{L^1(\Omega)}$, and compare the results. Each method may require different $\lambda$ for the same image. $\lambda$ can also be automatically selected if we know the noise variance in the form $\|f - u\|^2 = \sigma^2$. Using a norm, instead of the norm square for the data fidelity term avoids the intensity loss drawback of the ROF model.