# Nonlocal methods for image processing

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# Outline

## General Model

$$v(x) = u(x) + n(x), x \in \Omega$$

- $v(x)$ observed image
- $u(x)$ true image
- $n(x)$ i.i.d gaussian noise (white noise)

Gaussian kernel

$$x \to G_h(x) = \frac{1}{4\pi h^2} e^{-\frac{|x|^2}{4h^2}}$$

# Outline

# Linear low-pass filter

Idea: average in a local spatial neighborhood

$$GF_h(v)(x) = G_h * v(x) = \frac{1}{C(x)} \int_{y \in \Omega} v(y) \exp^{\frac{\|y-x\|^2}{4h^2}} dy$$

where $C(x) = 4\pi h^2$
Pro: work well for harmonic function (homogenous region)
Con: perform poorly on singular part, namely edge and texture

# Anisotropic filter

Idea: average only in the direction orthogonal to $Dv(x)(\frac{\partial v(x)}{\partial x}, \frac{\partial v(y)}{\partial y})$.

$$AF_h(v)(x) = \frac{1}{C(x)} \int_t v(x + f\frac{Dv(x)^\perp}{|Dv(x)|}) \exp^{\frac{-t^2}{h^2}} dt$$

where $C(x) = 4\pi h^2$.
Pro: Avoid blurring effect of Gaussian filter, maintaining edges.
Con: perform poorly on flat region, worse there than a Gaussian blur.

# Neighboring filter

Spatial neighborhood

$$B_\rho(x) = \{y \in \Omega | \|y - x\| \leq \rho\}$$

Gray-level neighborhood

$$B_(x, h) = \{y \in \Omega | \|v(y) - v(x)\| \leq \rho\}$$

for a given image $v$. Yaroslavsky filter

$$YNF_{h,\rho} = \frac{1}{C(x)} \int_{B_\rho(x)} u(y) e^{-\frac{|u(y)-u(x)|^2}{4h^2}} dy$$

Bilateral(SUSAN) filter

$$SUSAN_{h,\rho} = \frac{1}{C(x)} \int u(y) e^{-\frac{|u(y)-u(x)|^2}{4h^2}} e^{-\frac{|y-x|^2}{4\rho^2}} dy$$

Behave like weighted heat equation, enhancing the edges

# Denoising example

FIG. 3. *Denoising experience on a natural image. From left to right and from top to bottom: noisy image (standard deviation 20), Gaussian convolution, anisotropic filter, total variation minimization, Tadmor–Nezzar–Vese iterated total variation, Osher et al. iterated total variation, and the Yaroslavsky neighborhood filter.*

# Outline

# Nonlocal mean filter[1]

Idea: Take advantage of high degree of redundancy of natural images

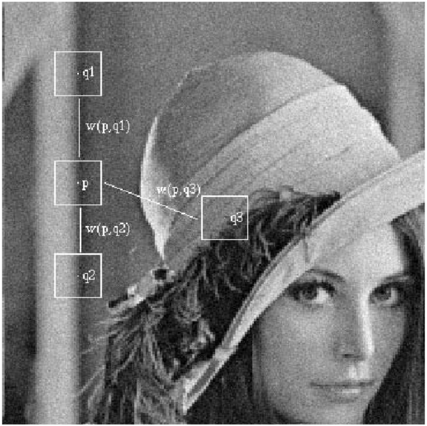



FIG. 6. *q1 and q2 have a large weight because their similarity windows are similar to that of p. On the other side the weight $w(p, q3)$ is much smaller because the intensity grey values in the similarity windows are very different.*

# Denoising formula

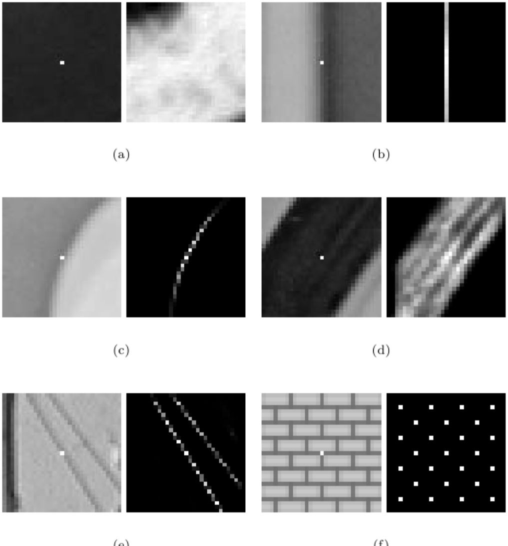
$$NLM(v)(x) := \frac{1}{C(x)} \int_\Omega w(x,y)v(y)dy,$$

where

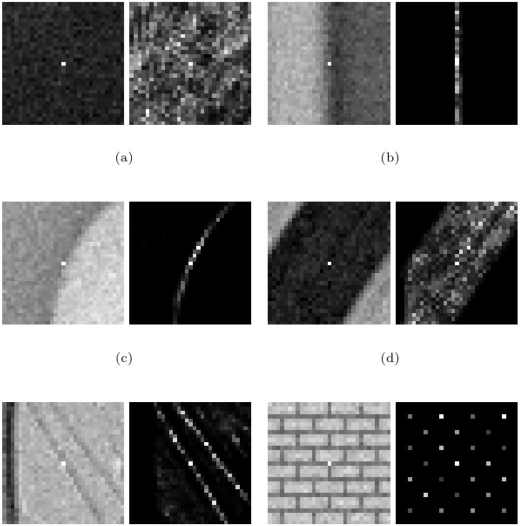
$$w(x,y) = \exp\{-\frac{G_a * (||v(x+\cdot) - v(y+\cdot)||^2)(0)}{2h_0^2}\},$$

$$C(x) = \int_\Omega w_v(x,y)dy$$

# Weight from clean image



# Weight from noisy image



# Example

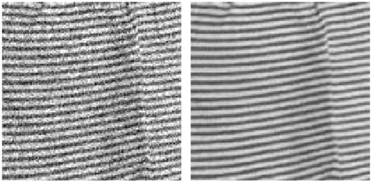


FIG. 7. *NL-means denoising experiment with a nearly periodic image. Left: Noisy image with standard deviation 30. Right: NL-means restored image.*

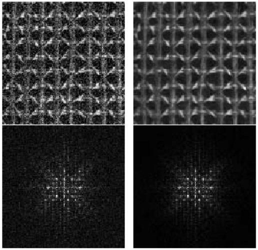


FIG. 8. *NL-means denoising experiment with a Brodatz texture image. Left: Noisy image with standard deviation 30. Right: NL-means restored image. The Fourier transform of the noisy and*

# Comparison with other methods

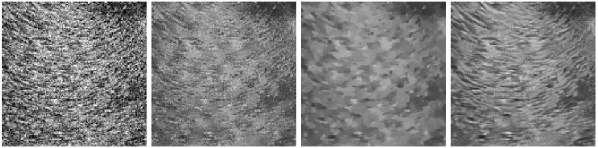
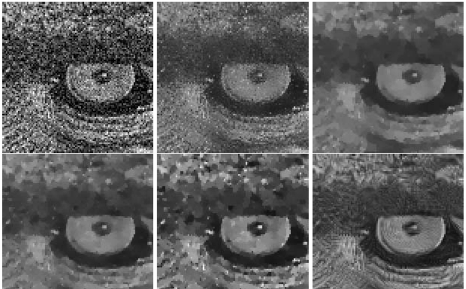


FIG. 20. *Denoising experience on a natural image. From left to right and from top to bottom: noisy image (standard deviation 35), neighborhood filter, total variation, and the NL-means algorithm.*



# Outline

# Nonlocal operators[2]/Graph based Regularization

Given a nonnegative and symmetric weight function $w(x,y)$ for each pair of points $(x,y) \in \Omega \times \Omega$:

- Nonlocal gradient of an image $u(x)$:

$$\nabla_w u(x,y) = (u(y) - u(x))\sqrt{w(x,y)} : \quad \Omega \times \Omega \to \Omega$$

- Nonlocal divergence of a gradient filed $p(x,y) : \Omega \times \Omega \to \mathcal{R}$ is defined by

$$< \nabla_w u, p >= - < u, \mathrm{div}_w p >, \forall u(x), p(x,y)$$

$$\Longrightarrow \mathrm{div}_w p(x) = \int_\Omega (p(x,y) - p(y,x))\sqrt{w(x,y)} dy.$$

- Nonlocal functionals of $u$:

$$J_{NL/H^1}(f) = \frac{1}{4}\int_\Omega |\nabla_w u(x)|^2 : \quad \frac{1}{4}\int_x \int_y |\nabla_w u(x,y)|^2$$

$$J_{NL/TV}(f) = \int_\Omega |\nabla_w u(x)|_1 : \quad \int_x \sqrt{\int_y |\nabla_w u(x,y)|^2}.$$

# Nonlocal $H^1$ regularization by non-local means

- Model:$\min J_{NL/H1}(u) + \frac{\mu}{2}||u-f||^2$
- Euler-Lagrange equation: $L_w(u)u + \mu(u-f) = 0$, where $L_w$ is unnormalized graph laplacian :

$$L_w(u) = \int_\Omega w(x,y)(u(x)-u(y)).$$

- We can replace $L_w(u)$ by normalized graph laplacian[3]

$$L_w^0 = \frac{1}{C(x)} L_w = Id - \text{NLM}_w(u).$$

- Semi-explicit iteration: for a time step $\tau > 0, s = 1 + \tau + \tau\mu, \alpha_1 = \frac{\tau}{s}, \alpha_2 = \frac{\tau\mu}{s}$:

$$u^{k+1} = (1-\alpha_1)u^k + \alpha_1 \text{NLM}_w(u^k) + \alpha_2 f.$$

[3] When $N \to \infty$ and $h_0 \to 0$, then $L_w^0$ converges to the continuous manifold Laplace - Beltrami operator.

# Nonlocal TV regularization by Chambolle's algorithm

- Model: $\min_u J_{NL/TV,w}(u) + \frac{\mu}{2}||u - f||^2$
- Extension of Chambolle's projection method for Nonlocal TV:

$$\inf_u \sup_{||p||\leq 1} \int_{\Omega\times\Omega} < \nabla_w u, p > + \frac{\mu}{2}||u - f||^2,$$

where the solution can be solved by a projected solution $u^* = f - \frac{1}{\mu} \div_w p^*$. and the dual variable $p^*$ is obtained by

$$\sup_{||p||\leq 1} \int_{\Omega\times\Omega} < \nabla_w u, p > + \frac{1}{2\mu}||\text{div}_w p||^2.$$

Algorithm:

$$p^{n+1} = \frac{p^n + \tau\nabla_w(\text{div}_w p^n - \mu f)}{1 + \tau|\nabla_w(\text{div}_w p^n - \mu f)|}, \quad \tau > 0$$

# Deblurring by Nonlocal Means[4]

Problem: $f = Au + n$, $A$ linear operator, $n$ Gaussian noise. **Idea:** Use initial blurry and noisy image $f$ to compute the weight.

$$J_{\mathsf{NLM},w(f)} := \min ||u - \mathsf{NLM}_f u||^2 + \frac{\lambda}{2}||Au - f||^2 \tag{1}$$

which is equivalent to

$$J_{\mathsf{NLM},w(f)} := \min ||L^0_{w_f}(u)||^2 + \frac{\lambda}{2}||Au - f||^2 \tag{2}$$

where $L^0_{w_f}$ is the normalized graph laplacian with the weight computed from $f$.

Gradient descents flow:

$$((L^0_{w_f})^* L^0_{w_f})u + \lambda A^*(Au - f) = 0$$

[4] A. Buades, B. Coll, and J-M. Morel. 2006

# Image recovery via nonlocal operators

**Idea:** Use a deblurred image to compute the weight.

1. Preprocessing:
   - Compute a deblurred image via a fast method:
     $$u_0 = \min \frac{1}{2}||Au - f||^2 + \delta||u||^2 \Longleftrightarrow u_0 = (A^*A + \delta)^{-1}A^*f.$$
     where $\delta$ is chosen optimally by respecting the condition
     $$\sigma^2 = ||Au_0 - f||^2$$
     where $\sigma^2$ is the noise level in blurry image.
   - Compute the nonlocal weight $w_0$ by using $u_0$ as a reference image (set $h_0 = \sigma^2||(A^*A + \delta)^{-1}A^*||^2$.)
2. Nonlocal regularization with the fixed weight $w_0$:
   $$\min J_{w_0}(u) + \frac{\lambda}{2}||Au - f||^2$$
   by gradient descent.

# Nonlocal regularization for inverse problems

- **Idea**: nonlocal weight updating during nonlocal regularization by operator splitting.
- Model :

$$\min_u J_{w(u)}(u) + \frac{\lambda}{2}||Au - v||^2$$

**Approximated Algorithm:**

$$\left\{ \begin{array}{lcl} v^{k+1} & = & u^k + \frac{1}{\mu}A^*(f - Au^k) \\ w^{k+1} & = & w(v^{k+1})(optional) \\ u^{k+1} & = & \arg\min J_{NL/TV, w^{k+1}} + \frac{\lambda\mu}{2}||u - v^{k+1}||^2 \end{array} \right. \quad (3)$$

where $u^{k+1}$ is solved by Chamobelle's method for NLTV.

# Nonlocal regularization with Bregmanized methods

With/without weight updating:

**Algorithm:**

$$\left\{\begin{array}{lcl} f^{k+1} & = & f^k + f - Au^k \\ v^{k+1} & = & u^k + \frac{1}{\mu}A^*(f^{k+1} - Au^k) \\ w^{k+1} & = & w(v^{k+1})(\text{optional}) \\ u^{k+1} & = & \arg\min J_{NL/TV,w^{k+1}} + \frac{\lambda\mu}{2}||u - v^{k+1}||^2 \end{array}\right. \quad (4)$$

# Outline

# Compressive sampling : $Au = RFu$

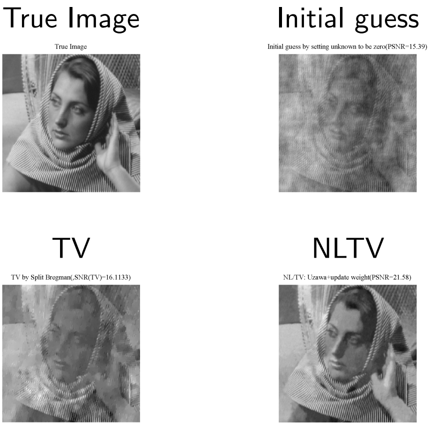



Figure: Data: $30\%$ random Fourier measurements

# Deconvolution: $Au = k * u$

True Image

Blurry and noisy Image

Fix weight

NL/TV: Uzawa($w_0$)(steps=20,lambda=0.200,mu=1.0,time=151.11 s,SNR=13.28)

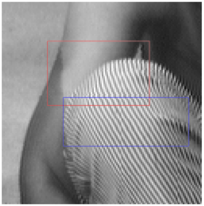
Update weight

NL/TV: Uzawa(Updating weight)(SNR=13.21)

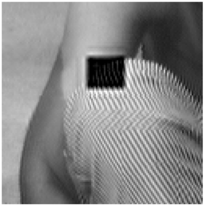
Figure: $9 \times 9$ box average blur kernel, $\sigma = 3$

# Wavelet Inpainting: $Au = RWu$

Original



Received, PSNR= 17.51



TV, PSNR=28.64

NLTV, PSNR= 36.06

Figure: Block loss(including low-low frequencies loss). For both TV and NLTV, the initial guess is the received image

# Outline

# References


- A Review of Image Denoising Algorithms, with a New One, A. Buades, B. Coll, J. M. Morel Multiscale Modeling and Simulation, Vol. 4, No. 2. (2005), pp. 490-530.
- Guy Gilboa and Stanley Osher, Nonlocal Operators with Applications to Image Processing, UCLA CAM report, July 2007
- Image Recovery via Nonlocal Operators, Journal of Scientific Computing, Yifei Lou, Xiaoqun Zhang, Stanley Osher, Andrea Bertozzi
- Xiaoqun Zhang, Martin Burger, Xavier Bresson and Stanley Osher, Bregmanized Nonlocal Regularization for Deconvolution and Sparse Reconstruction, UCLA CAM Report, January 2009
- Xiaoqun Zhang and Tony F. Chan, Wavelet Inpainting by Nonlocal Total Variation, UCLA CAM Report, July 2009
- Miyoun Jung and Luminita A. Vese, Image Restoration via