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Image recovery via total variation minimization and related problems

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Summary. We study here a classical image denoising technique introduced by L. Rudin and S. Osher a few years ago, namely the constrained minimization of the total variation (TV) of the image. First, we give results of existence and uniqueness and prove the link between the constrained minimization problem and the minimization of an associated Lagrangian functional. Then we describe a relaxation method for computing the solution, and give a proof of convergence. After this, we explain why the TV-based model is well suited to the recovery of some images and not of others. We eventually propose an alternative approach whose purpose is to handle the minimization of the minimum of several convex functionals. We propose for instance a variant of the original TV minimization problem that handles correctly some situations where TV fails.

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1. Introduction

L. Rudin and S. Osher have proposed, quite a few years ago, the following method for image reconstruction (see [14], [16], [17], [18] and the references in these papers). Suppose your image (or your data) u_0 is a function defined on a bounded and smooth (or piecewise smooth) open subset Ω of $\mathbb{R}^N$ – very often Ω will simply be a rectangle in $\mathbb{R}^2$ – , and suppose that this data is a "nice," say, piecewise smooth image u that has been transformed via a linear operator A (for instance, a blur) and to which a random noise n has then been added:

$$u_0 = Au + n \tag{1}$$

You wish to recover u, knowing u_0. Of course we must assume some knowledge of A and n in order to be able to solve the problem. Rudin and Osher's approach

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consists in solving the following constrained minimization problem:

$$\text{(2)}\qquad \begin{aligned}&\text{Minimize } \int_\Omega |\nabla u|\\ &\text{with } \int_\Omega Au = \int_\Omega u_0 \text{ and } \int_\Omega |Au - u_0|^2 = \sigma^2.\end{aligned}$$

The first constraint corresponds to the assumption that the noise has zero-mean, and the second that its standard deviation is σ.

This problem is naturally linked to the following unconstrained problem:

$$\text{(3)}\qquad \text{Minimize/Find a critical point of } \int_\Omega |\nabla u| + \frac{\lambda}{2}|Au - u_0|^2$$

for a given Lagrange multiplier λ. As long as λ is non-negative, this is just a minimization problem, but if $\lambda < 0$, not much can be said about it. Notice that in both problems (2) and (3), $\int_\Omega |\nabla u|$ is just an alternative notation for $|Du|(\Omega)$, common in image processing papers. We will not use it any longer, but prefer to denote the total variation of a function $u \in \mathrm{BV}(\Omega)$ by $J(u) = |Du|(\Omega)$. In the remaining of the paper, J will be considered as a convex and lower semicontinuous function on $L^p(\Omega)$ (taking value $+\infty$ everytime $u \notin \mathrm{BV}(\Omega)$).

In the next section, we present an existence and uniqueness result for (2), as well as a proof of the link between (2) and (3), under the assumptions stated below. Then, in Sect. 3, we will explain how a classical relaxation method can be used to solve (3), and show a proof of convergence for this method.

In the following sections, we will comment some results, and consider as well variants of (3). In particular, we will propose to replace J with a functional that takes into account several properties of the images, that we need to keep or recover. This new functional will correspond to the convexification of the minimum of two or more functionals, each one of those corresponding to one desired property, so that $J(u) \simeq 0$ each time u decomposes into functions having the right properties. This general framework may prove useful in a wide class of image reconstruction problems. For instance, the TV functional does not act upon constants but does on affine functions. We explain how to construct, by this general procedure, a simple modification of the TV that does not act on affine functions.

First we need to state a few assumptions, that are necessary for our study, but also quite natural.

H1. A is a continuous and linear operator of $L^p(\Omega)$,
H2. $A \cdot 1 = 1$ ($\Leftrightarrow \int_\Omega A^* u = \int_\Omega u$ for all $u \in L^p(\Omega)'$),
H3. $n(x)$ is an oscillatory function, representing a white noise added to the "clean" image,
H4. $\int_\Omega n = 0$, and $\sigma^2 = \int_\Omega |n|^2$ is known.

Here $p = 2$ if the dimension is 1 or 2, and $p = N/(N-1)$ if $N \geq 3$. In special cases p may have some other value in $[N/(N-1), 2]$ (for instance, when A is the identity operator, we may stay in $L^2(\Omega)$ even when $N \geq 3$).

Remark 1. The second assumption may be seen as technical, as it ensures that $|Du|(\Omega)+\|Au\|_2$ is coercive on $\mathrm{BV}(\Omega)$ [13], the space of functions with bounded total variation (here $|Du|(\Omega)$ denotes the variation of function u over Ω), but it is also a natural assumption in the case where A represents a blur (or any mean-preserving linear filtering) of u – for instance when u_0 is a picture taken with a defocused camera. This may not be suited to other interesting image recontruction cases, like tomography or IRM reconstruction (see for instance [6]): in these situations another hypothesis has to be made to ensure some control on the L^1-norm of u (for instance, $A1 \neq 0$).

Remark 2. Notice that

$$\int_\Omega |u_0|^2 = \int_\Omega |Au+n|^2 = \int_\Omega |Au|^2 + \int_\Omega |n|^2 + 2\int_\Omega Au.n,$$

and as it is reasonable to assume that $\int_\Omega Au.n = 0$ (which means that the "noise" n and Au are totally uncorrelated signals), this implies that $\int_\Omega |u_0|^2 = \int_\Omega |Au|^2 + \sigma^2$ (to simplify we assume that the measure of Ω is 1), and $\|u_0\|_2^2 = \int_\Omega |u_0|^2 \geq \sigma^2$. Moreover, n also has to be orthogonal to constant functions ($\int_\Omega n = 0$), and the same argument shows now that for all $c \in \mathbb{R}$, $\|u_0 - c\|_2^2$ has to be greater than σ^2.

Therefore, we always will assume that:

H5. $$\|u_0 - \int_\Omega u_0\|_2 \geq \sigma.$$

2. Existence and uniqueness for (2)

The following theorem shows that problem (2) is a well-posed problem:

Theorem 2.1. *Assume H1-H2 hold, as well as H5:* $\sigma \in (0, \|u_0 - \int_\Omega u_0\|_2]$. *Assume also that* $u_0 \in X$, *where* X *is the closure in* $L^2(\Omega)$ *of* $L^2(\Omega) \cap A(L^p(\Omega) \cap \mathrm{BV}(\Omega))$.[1] *Then (2) has a solution* $u \in L^p(\Omega) \cap \mathrm{BV}(\Omega)$, *and* $Au \in L^2(\Omega)$ *is unique. Moreover, problem (2) is equivalent to (3) for a unique (if* $\sigma < \|u_0 - \int_\Omega u_0\|_2$*) and* non-negative *Lagrange multiplier* λ *(that depends on* σ*, and of course on* u_0*,* Ω*). If* A *is injective, then the solution* u *of both problems is unique.*

Here as previously, $p = 2$ if $N = 1$ or 2, and $p = N/(N-1)$ for $N \geq 3$, so that $\mathrm{BV}(\Omega)$ is continuously embedded in $L^p(\Omega)$. However, in some particular cases we also may suppose $p > N/(N-1)$ and then $L^p(\Omega) \cap \mathrm{BV}(\Omega) \subsetneq \mathrm{BV}(\Omega)$.

[1] If $u_0 \notin X$ then one also has to assume that $\sigma \geq \delta$, where δ is the distance between X and u_0, but everything else remains true. In fact, it suffices in this case to replace u_0 by its $L^2(\Omega)$-projection on X, u_0', and σ by $\sigma' = \sqrt{\sigma^2 - \delta^2}$, and all statements and proofs in the sequel turn correct

Remark. In the sequel we will sometimes assume that $\int_\Omega u_0 = 0$. This may be done without loss of generality: actually, if $\overline{u}_0 = u_0 - \int_\Omega u_0$, we have the following obvious fact that is due to Assumption *H2* on A:

$$u \text{ is a solution of (2) (resp., (3))}$$
$$\Leftrightarrow$$
$$\overline{u} = u - \int_\Omega u \text{ is a solution of (2) (resp., (3)) with } \overline{u}_0 \text{ instead of } u_0.$$

Therefore, in almost all the proofs that follow we could assume that all functions stay in $\{u \in L^p(\Omega) \ : \ \int_\Omega u = 0\}$, which is a closed subspace – with respect to both strong and weak topologies – of $L^p(\Omega)$.

The following sections are devoted to the proof of Theorem (2.1).

2.1. Existence of a solution

The existence of a solution to problem (2) is proven in [14], in the case where A is a compact operator and for any σ. Here we adapt the proof to the case where A is not compact (for instance, if A is the identity operator), provided *H5* holds.

Suppose $\int_\Omega u_0 = 0$ and consider a minimizing sequence for (2), that we denote by u_n, $n \geq 1$. We assume the constraints are satisfied by all u_n. Therefore u_n is bounded in BV(Ω) (as $|Du|(\Omega) + \|Au\|_2$ is greater than the BV(Ω)-norm – see the Poincaré inequalities in [13], [21]) and in $L^p(\Omega)$ by Sobolev embedding, with $p \in [1, +\infty]$ if $N = 1$, and $p = N/(N-1)$ if $N \geq 3$ (we choose $p = 2$ for both cases $N = 1$ or 2). Thus we can assume that u_n converges weakly in $L^p(\Omega)$ to u, while Du_n converges weakly as a measure to Du. As Au_n is bounded in $L^2(\Omega)$, we also assume that Au_n weakly converges in $L^2(\Omega)$ to some function which has to be Au because of Assumption *H1*.

We have,

$$J(u) \leq \liminf_{n\to\infty} J(u_n),$$

$$\int_\Omega Au = \lim_{n\to\infty} \int_\Omega Au_n = 0,$$

$$\|Au - u_0\|_2 \leq \lim_{n\to\infty} \|Au_n - u_0\|_2 = \sigma.$$

Consider now the continuous function $f(t) = \|t.Au - u_0\|_2$ for $t \in [0,1]$. As $f(1) \leq \sigma$ and $f(0) \geq \sigma$, there exists some $t \in [0,1]$ such that $f(t) = \sigma$. Function $u' = t.u$ satisfies $\int_\Omega u' = 0$, $\|Au' - u_0\|_2 = \sigma$, and

$$J(u') = tJ(u) \leq \liminf_{n\to\infty} J(u_n);$$

and provides a solution for the problem. (Moreover we see that in fact $t = 1$ and $u' = u$, as we cannot have $J(u') < J(u)$). $\square$

Remark 1. In this proof, instead of Assumption *H2*, we could just assume that $A1 \neq 0$ (which implies that $|\int_\Omega u_n|$ is bounded and therefore still ensures the weak compactness of (u_n) in $\mathrm{BV}(\Omega)$). However, when $\int_\Omega u_0 \neq 0$ it becomes more difficult to show that the limit of the minimizing sequence satisfies $\|Au - u_0\|_2 = \sigma$. One has to assume, for instance, that $\int_\Omega A1 \neq 0$ and $\sigma \le \|u_0 - (A1/\int_\Omega A1)\int_\Omega u_0\|_2$. If $A1 = 1$, and $\sigma \le \|u_0 - \int_\Omega u_0\|_2$, it is straightforward to adapt the proof to the case $\int_\Omega u_0 \neq 0$.

Remark 2. If we drop the constraint $\int_\Omega Au = \int_\Omega u_0$, then for any $\sigma \le \|u_0\|_2$, we can find a minimizer u of J with $\|Au - u_0\|_2 = \sigma$ (even if $1 \neq A1 \neq 0$). Moreover, we necessarily have that $\min_{c\in\mathbb{R}} \|A(u+c) - u_0\|_2 = \sigma = \|Au - u_0\|_2$, which implies that $\langle A1, Au - u_0\rangle = 0$. Therefore, Assumption *H2 automatically* ensures that the minimizer u satisfies $\int_\Omega Au = \int_\Omega u_0$. In the sequel we will always assume that $A1 = 1$ and forget the constraint on $\int_\Omega Au$.

Remark 3. Notice that if for some p within $(N/(N-1), 2]$ we have $\|u\|_p \le C\|Au\|_2$, for instance if A is the identity and $p = 2$, then the same existence result holds with $u \in L^p(\Omega)$.

The previous proof shows in fact that, as long as $\sigma \le \|u_0 - \int_\Omega u_0\|_2$, the minimum of J in the set $\{\|Au - u_0\|_2 \le \sigma\}$ is reached for some u with $\|Au - u_0\|_2 = \sigma$, that satisfies $\int_\Omega Au = \int_\Omega u_0$. Therefore, problem (2) is equivalent to the constrained minimization problem

$$\text{(4)} \qquad \begin{aligned} &\text{Minimize } J(u) \\ &\text{with } \int_\Omega |Au - u_0|^2 \le \sigma^2. \end{aligned}$$

in which the constraint is convex.

Moreover, if both u and v are solutions to (2), we deduce that $Au = Av$. Actually, we have $J(\frac{u+v}{2}) \le \frac{1}{2}(J(u)+J(v)) = \min J$ and $\|A\frac{u+v}{2} - u_0\|_2 \le \sigma$, with equality iff $Au = Av$. As we cannot have $\|A\frac{u+v}{2} - u_0\|_2 < \sigma$, then $Au = Av$.

2.2. Characterization of the solutions

The equivalence between (2) and (4) has interesting consequences. We first study the simpler case where A is a continuous operator from $L^p(\Omega)$ into $L^2(\Omega)$, and will treat the general case later.

Proposition 2.1. *If u is a solution of (2), and $A : L^p(\Omega) \to L^2(\Omega)$ is continuous, then there exists $\lambda \ge 0$ such that*

$$-\lambda A^*(Au - u_0) \in \partial J(u).$$

Here $\partial J(u) \subset L^p(\Omega)'$ is the subdifferential of J at u [11], [3].

Proof. Set

$$G(u) = \chi_{u_0+\sigma\overline{B}}(u) = \begin{cases} +\infty & \text{if } u \notin u_0 + \sigma\overline{B} \Leftrightarrow \|u-u_0\|_2 > \sigma \\ 0 & \text{if } u \in u_0 + \sigma\overline{B} \Leftrightarrow \|u-u_0\|_2 \le \sigma \end{cases}$$

($\overline{B}$ denotes the closed unit ball in $L^2(\Omega)$). J and G are convex lower semi-continuous functions and problem (4) is equivalent to minimizing $J(u)+G(Au)$. We have Dom$J = \{u : J(u) < +\infty\} = \mathrm{BV}(\Omega) \cap L^p(\Omega)$ and Dom$G = \{u : G(u) < +\infty\} = u_0 + \sigma\overline{B}$, and as we assumed that $u_0 \in \overline{A\mathrm{Dom}J}$, there exists $\tilde{u} \in \mathrm{Dom}J$ with $\|A\tilde{u} - u_0\|_2 < \sigma/2$. Then, as A is continuous from $L^p(\Omega)$ into $L^2(\Omega)$, $G \circ A$ is continuous at $\tilde{u}$ ($\tilde{u} \in \mathrm{Int}(\mathrm{Dom}G \circ A)$) and therefore for all u,

$$\partial(J + G \circ A)(u) = \partial J(u) + \partial(G \circ A)(u).$$

Moreover, as G is continuous at $A\tilde{u}$, we have for all u,

$$\partial(G \circ A)(u) = A^*\partial G(Au)$$

with $\partial G(u) = \{0\}$ if $\|u-u_0\|_2 < \sigma$ and $\partial G(u) = \{\lambda(u-u_0),\ \lambda \ge 0\}$ if $\|u-u_0\|_2 = \sigma$. Thus,

$$\partial(J + G \circ A)(u) = \partial J(u) + A^*\partial G(Au).$$

(See [11, Prop 5.6 and 5.7] or [3, Thm 4.4].)

If u is a solution of (2) and thus of (4), then $0 \in \partial(J + G \circ A)(u)$. As any solution of (2) satisfies $\|Au - u_0\|_2 = \sigma$, this shows that

$$\begin{aligned} &\exists\lambda \ge 0, \quad 0 \in \partial J(u) + \lambda A^*(Au - u_0) \\ \Leftrightarrow\ &\exists\lambda \ge 0, \ -\lambda A^*(Au-u_0) \in \partial J(u). \end{aligned} \qquad \square$$

Note that it implies that for this $\lambda \ge 0$, u is a minimizer of the convex functional $J(u) + (\lambda/2)\|Au - u_0\|_2^2$ (which is the functional of problem (3)). Conversely, a minimizer u of this functional is obviously a solution of (2) for $\sigma = \|Au - u_0\|_2$. This establishes the equivalence between problems (2) (with $0 < \sigma \le \|u_0 - \int_\Omega u_0\|_2$) and (3) (with $\lambda \ge 0$). We later on will show that the correspondence between σ and λ is (almost) one-to-one, but before we have to prove the equivalence between (2) and (3) in the general case (when A is not necessarily continuous from $L^p(\Omega)$ into $L^2(\Omega)$).

Remark. The previous proof is still valid if $N \ge 3$, $N/(N-1) < p \le 2$ and $C_1\|u\|_p \le \|Au\|_2 \le C_2\|u\|_p$. In the sequel we will assume $p = N/(N-1)$.

Now we suppose that $N \ge 3$, A is an arbitrary continuous operator from $L^p(\Omega)$ into $L^p(\Omega)$ ($p = N/(N-1)$), satisfying Assumption *H2*, and such that u_0 belongs to the closure of $L^2(\Omega) \cap A(\mathrm{BV}(\Omega))$ in $L^2(\Omega)$. Let ρ_ε be a symmetric smoothing kernel and set for any $u \in L^p(\Omega)$, $A_\varepsilon u = \rho_\varepsilon * Au$, Au and all other functions being extended to $\mathbb{R}^N$ by the value zero outside Ω. If $\phi \in L^2(\Omega)$ and $\|\phi\|_2 \le 1$, we can write, letting $p' = p/(p-1)$

$$\begin{aligned}\langle \phi, A_\varepsilon u\rangle_{L^2(\Omega),L^2(\Omega)} &= \langle \phi, \rho_\varepsilon * Au\rangle_{L^2(\mathbb{R}^N),L^2(\mathbb{R}^N)}\\ &= \langle \rho_\varepsilon * \phi, Au\rangle\\ &\leq \|\rho_\varepsilon * \phi\|_{p'}\|Au\|_p\\ &\leq \|\rho_\varepsilon\|_{p'}\|\phi\|_1\|Au\|_p\end{aligned}$$

and as $\|\phi\|_1 \leq C\|\phi\|_2 \leq C$ and $\|Au\|_p \leq \|A\|_p\|u\|_p$, we have for any ϕ with $\|\phi\|_2 \leq 1$

$$\langle \phi, A_\varepsilon u\rangle_{L^2(\Omega),L^2(\Omega)} \leq C\|u\|_p$$

and thus

$$\|A_\varepsilon u\|_2 \leq C\|u\|_p,$$

with C a finite constant, showing that A_ε is a continuous operator from $L^p(\Omega)$ into $L^2(\Omega)$.

We will now consider a solution u_ε of the problem

$$\text{(5)}\qquad \begin{array}{l}\text{Minimize } J(u)\\ \text{with } \|A_\varepsilon u - u_{0,\varepsilon}\|_2 \leq \sigma\end{array}$$

for a given $u_{0,\varepsilon} \in L^2(\Omega)$ that converges to u_0 as ε goes to zero. We first define this $u_{0,\varepsilon}$ as follows. Notice that if $\underline{u}$ is a solution of (2) and c_ε a constant that goes to zero with ε, we have

$$\|A_\varepsilon(\underline{u} + c_\varepsilon) - \rho_\varepsilon * (u_0 + c_\varepsilon)\|_2 = \|\rho_\varepsilon * (A\underline{u} - u_0)\|_2 = \sigma_\varepsilon \to \sigma$$

as ε goes to zero (and σ_ε does not depend on c_ε). We set $u_{0,\varepsilon} = t_\varepsilon\rho_\varepsilon * (u_0 + c_\varepsilon)$ with $t_\varepsilon = \sigma/\sigma_\varepsilon$. c_ε is chosen to be 0 when $\|t_\varepsilon\rho_\varepsilon * u_0\|_2 \geq \sigma$, otherwise we choose $c_\varepsilon \geq 0$ such that $\|t_\varepsilon\rho_\varepsilon * (u_0 + c_\varepsilon)\|_2 = \sigma$ (notice that $\|t_\varepsilon\rho_\varepsilon * (u_0 + c)\|_2 \to +\infty$ as $c \to +\infty$). As

$$\|t_\varepsilon\rho_\varepsilon * (u_0 + c_\varepsilon)\|_2 = \sigma \geq t_\varepsilon\left|(\|\rho_\varepsilon * u_0\|_2 - c_\varepsilon\|\rho_\varepsilon * 1\|_2)\right|$$

c_ε is bounded and any limit point c_0 as ε goes to 0 satisfies $\|u_0 + c_0\|_2 = \sigma$ and therefore $c_0 = 0$. Thus, $\lim_{\varepsilon\to 0} c_\varepsilon = 0$, and $u_{0,\varepsilon}$ goes to u_0 in $L^2(\Omega)$ as $\varepsilon \to 0$.[2]

Now consider problem (5): the same proof as in Sect. 2.1 may be adapted to show that there exists a minimizer u_ε, moreover as $\|u_{0,\varepsilon}\|_2 \geq \sigma$ we have $\|A_\varepsilon u_\varepsilon - u_{0,\varepsilon}\|_2 = \sigma$. As $u_{0,\varepsilon}$ was built in order to have $\|A_\varepsilon t_\varepsilon(\underline{u} + c_\varepsilon) - u_{0,\varepsilon}\|_2 = \sigma$, we have

$$J(u_\varepsilon) \leq J(t_\varepsilon(\underline{u} + c_\varepsilon)) = t_\varepsilon J(\underline{u}).$$

Therefore we may extract a subsequence (still denoted u_ε) such that u_ε goes to some $\overline{u}$ weakly in $\mathrm{BV}(\Omega)$ as well as in $L^p(\Omega)$. Moreover we may assume that $A_\varepsilon u_\varepsilon$ weakly converges to some $v \in L^2(\Omega)$ and if $\phi \in C_c^\infty(\Omega)$,

$$\langle \phi, A_\varepsilon u_\varepsilon\rangle = \langle \rho_\varepsilon * \phi, Au_\varepsilon\rangle \xrightarrow{\varepsilon\to 0} \langle \phi, A\overline{u}\rangle$$

[2] In fact, c_ε is just defined to ensure that $\|u_{0,\varepsilon}\|_2 \geq \sigma$ and therefore is useful only in the case where $\sigma = \|u_0\|_2$. If $\sigma < \|u_0\|_2$, then as soon as ε is small enough we have $\|u_{0,\varepsilon}\|_2 \geq \sigma$, even if $c_\varepsilon = 0$ for all ε

(as $\rho_\varepsilon * \phi$ goes strongly to ϕ in $L^p(\Omega)'$ and Au_ε weakly to Au in $L^p(\Omega)$) which shows that $v = A\overline{u}$. As $A_\varepsilon u_\varepsilon - u_{0,\varepsilon}$ weakly converges to $Au - u_0$ in $L^2(\Omega)$, we have

$$\|A\overline{u} - u_0\|_2 \leq \sigma = \lim_{\varepsilon\to 0} \|A_\varepsilon u_\varepsilon - u_{0,\varepsilon}\|_2$$

and we also have

$$J(\overline{u}) \leq \liminf_{\varepsilon\to 0} J(u_\varepsilon) \leq \limsup_{\varepsilon\to 0} J(u_\varepsilon) \leq J(\underline{u}), \tag{6}$$

showing that u is also a minimizer for problem (2). In particular, $A\overline{u} = A\underline{u}$, $J(\overline{u}) = J(\underline{u})$, which shows that the liminf and the limsup in (6) are in fact limits, and $\|A\overline{u} - u_0\|_2 = \sigma$, which shows that $A\overline{u}$ is in fact the strong limit of $A_\varepsilon u_\varepsilon$ in $L^2(\Omega)$.

Now, for reasons similar to those in the previous section (because A_ε is continuous from $L^p(\Omega)$ into $L^2(\Omega)$), there exists for all ε a positive λ_ε such that

$$-\lambda_\varepsilon A_\varepsilon^*(A_\varepsilon u_\varepsilon - u_{0,\varepsilon}) \in \partial J(u_\varepsilon) \subset L^p(\Omega)'.$$

(Here the assumption that $\|u_{0,\varepsilon}\|_2 \geq 0$ and thus $\|A_\varepsilon u_\varepsilon - u_{0,\varepsilon}\|_2 = \sigma$ is essential.) Among other things this implies that u_ε is a minimizer for

$$J(u) + \frac{\lambda_\varepsilon}{2}\|A_\varepsilon u - u_{0,\varepsilon}\|_2^2, \tag{7}$$

and allows us to show that λ_ε is bounded. Actually, we have for all $u \in L^p(\Omega)$,

$$\frac{\lambda_\varepsilon}{2}\sigma^2 \leq J(u_\varepsilon) + \frac{\lambda_\varepsilon}{2}\|A_\varepsilon u_\varepsilon - u_{0,\varepsilon}\|_2^2 \leq J(u) + \frac{\lambda_\varepsilon}{2}\|A_\varepsilon u - u_{0,\varepsilon}\|_2^2. \tag{8}$$

As $\sigma > 0$, there is a $u \in \mathrm{BV}(\Omega)$ such that, if ε is small enough,

$$\|A_\varepsilon u - u_{0,\varepsilon}\|_2^2 = \|\rho_\varepsilon * (Au - t_\varepsilon(u_0 + c_\varepsilon))\|_2^2 \leq \sigma^2/2$$

(as $u_0 \in \overline{L^2(\Omega) \cap A(\mathrm{BV}(\Omega))}$). It follows that

$$\lambda_\varepsilon \leq \frac{4J(u)}{\sigma^2} < +\infty.$$

Consider now a limit point $\overline{\lambda} \geq 0$ of λ_ε. The variational inequality (8) converges as ε goes to zero to

$$J(\overline{u}) + \frac{\overline{\lambda}}{2}\|A\overline{u} - u_0\|_2^2 \leq J(u) + \frac{\overline{\lambda}}{2}\|Au - u_0\|_2^2 \tag{9}$$

as either $Au \notin L^2(\Omega)$ and the right-hand term of (9) is $+\infty$, or $Au \in L^2(\Omega)$ and it is the L^2-limit of $A_\varepsilon u$. Notice that even if we have $\underline{u} \neq \overline{u}$, still $A\underline{u} = A\overline{u}$ and $J(\underline{u}) = J(\overline{u})$ and (9) is also satisfied by $\underline{u}$. Conversely we will see in the next section that any minimizer u' of the functional in (9) satisfies $Au' = A\overline{u}$ and $J(u') = J(\overline{u})$ and therefore is also a solution of (2) with the same σ. This shows the equivalence between problems (2) with $0 < \sigma \leq \|u_0 - \int_\Omega u_0\|_2$ and (3) with $\lambda \geq 0$ in the general case.

We will now study the Lagrange problem and show that for a given σ, with $\sigma < \|u_0 - \int_\Omega u_0\|_2$, there is a unique corresponding λ.

2.3. Study of problem (3) for $\lambda \geq 0$

First of all, it is simple to show that when $\lambda \geq 0$, problem (3) has a solution $u^\lambda \in L^2(\Omega) \cap \mathrm{BV}(\Omega)$, which is unique as soon as A is injective. Acar and Vogel [1] have analyzed this problem in detail and have also studied perturbations of the system, regularizations, etc. For $\lambda = 0$ we need to add explicitely the condition $\int_\Omega Au^\lambda = \int_\Omega u_0$, otherwise any constant function is a solution, on the other hand when $\lambda > 0$, Assumption *H2* automatically ensures that the minimizers of the energy in (3) satisfy $\int_\Omega Au^\lambda = \int_\Omega u_0$.

Notice that, because of the strict convexity of the term $\|Au - u_0\|_2^2$ with respect to Au, it is straightforward to check that Au^λ is unique, even if u^λ is not. This implies that we can define a function $\sigma(\lambda) = \|Au^\lambda - u_0\|_2$. We then have the following lemma:

Lemma 2.3. *The function $\sigma(\lambda)$ is a nonincreasing and continuous function. It maps $\mathbb{R}_+$ onto $(0, \|u_0 - \int_\Omega u_0\|_2]$. Moreover, there exists $\underline{\lambda} \geq 0$ such that $\sigma(\lambda)$ is strictly decreasing on $[\underline{\lambda}, +\infty)$, and $\sigma(\lambda) = \|u_0 - \int_\Omega u_0\|_2$ if $0 \leq \lambda \leq \underline{\lambda}$.*

Proof. Consider first $\lambda > \mu \geq 0$. We have

$$J(u^\lambda) + \frac{\lambda}{2}\|Au^\lambda - u_0\|_2^2 \leq J(u^\mu) + \frac{\lambda}{2}\|Au^\mu - u_0\|_2^2 \tag{10}$$

and

$$J(u^\mu) + \frac{\mu}{2}\|Au^\mu - u_0\|_2^2 \leq J(u^\lambda) + \frac{\mu}{2}\|Au^\lambda - u_0\|_2^2. \tag{11}$$

Combining both inequalities, we get $(\lambda - \mu)\sigma(\lambda)^2 \leq (\lambda - \mu)\sigma(\mu)^2$ and this shows that $\sigma(.)$ is nonincreasing.

As for any $\sigma_0 \in (0, \|u_0 - \int_\Omega u_0\|_2]$ problem (2) admits a solution, proposition (2.1) shows that there exists a $\lambda_0 \geq 0$ such that $\sigma(\lambda_0) = \sigma_0$, and this implies the continuity of the mapping $\sigma(\lambda)$, as well as the fact that σ goes to zero as λ goes to ∞.

We want to prove now that this mapping is strictly decreasing. Suppose there exists $\lambda < \mu$ such that $\sigma(\lambda) = \sigma(\mu)$. Equations (10) and (11) show this time that $J(u^\lambda) = J(u^\mu)$ and, in fact, that u^λ is a solution of (3) for any $\lambda' \in [\lambda, \mu]$. If A is continuous from $L^p(\Omega)$ into $L^2(\Omega)$, this means that

$$\forall \lambda' \in [\lambda, \mu],\ -\lambda' A^*(Au^\lambda - u_0) \in \partial J(u^\lambda) \neq \emptyset. \tag{12}$$

Remember that $p \in \partial J(u)$ is equivalent to

$$\langle p, u \rangle = J(u) + J^*(p)$$

where J^* is the Legendre-Fenchel transform of the convex function J (see for instance [3, Prop 4.2]). Here $J^*(p) = \chi_V(p)$ (= 0 if $p \in V$ and $+\infty$ if $p \notin V$) where V is a convex closed set (the $L^p(\Omega)'$-closure of $\{\psi = \operatorname{div}\phi \ : \ \phi \in C_c^\infty(\Omega)$ and $\|\phi\|_\infty \leq 1\}$).

Therefore,

(13) $$\forall \lambda' \in [\lambda, \mu],\ -\lambda' \left\langle Au^{\lambda} - u_0, Au^{\lambda} \right\rangle = J(u^{\lambda})$$

which implies that $J(u^{\lambda}) = 0$ and $u^{\lambda} = \int_{\Omega} u_0$ (and $\sigma(\lambda) = \|u_0 - \int_{\Omega} u_0\|_2$).

Now, in the general case (A continuous operator of $L^p(\Omega)$), we cannot say that (12) holds. If we consider the proof in Sect. 2.2, we can see that for the approximated problems, we have for each ε

$$-\lambda_{\varepsilon}(Au_{\varepsilon} - u_{0,\varepsilon}) \in \partial J(u_{\varepsilon}),$$

thus

$$-\lambda_{\varepsilon} \left\langle Au_{\varepsilon} - u_{0,\varepsilon}, Au_{\varepsilon} \right\rangle = J(u_{\varepsilon})$$

and this converges to

$$-\overline{\lambda} \left\langle A\overline{u} - u_0, A\overline{u} \right\rangle = J(\overline{u}).$$

However, we need this result for *any* λ such that $\overline{u}$ minimizes $J(u) + (\lambda/2)||Au - u_0||_2$, and not just for the limit point $\overline{\lambda}$ of λ_{ε}. In order to show this, we have to consider now the approximated problem

(14) $$\text{Minimize } J(u) + \frac{\lambda}{2}\|A_{\varepsilon}u - u_0\|_2^2.$$

If u_{ε} is a solution of (14), we have

$$-\lambda \left\langle A_{\varepsilon}u_{\varepsilon} - u_0, A_{\varepsilon}u_{\varepsilon} \right\rangle = J(u_{\varepsilon}),$$

and we must check that $A_{\varepsilon}u_{\varepsilon}$ strongly converges to Au in $L^2(\Omega)$, and $J(u_{\varepsilon})$ converges to $J(u)$, as ε goes to zero – where u is a minimizer of (3). (We recall that, given λ, neither $J(u)$ nor Au depend on this particular minimizer u.)

As $J(u_{\varepsilon})$ and $\|A_{\varepsilon}u_{\varepsilon}\|_2$ are bounded, u_{ε} converges weakly to some u in BV(Ω) and in $L^p(\Omega)$, and $A_{\varepsilon}u_{\varepsilon}$ converges weakly in $L^2(\Omega)$ to some limit which must be Au. We have

(15) $$J(u) + \frac{\lambda}{2}\|Au - u_0\|_2^2 \leq \liminf_{\varepsilon \to 0} J(u_{\varepsilon}) + \frac{\lambda}{2}\|A_{\varepsilon}u_{\varepsilon} - u_0\|_2^2$$

and therefore u minimizes (3).

Now call $m = J(u) + \frac{\lambda}{2}\|Au - u_0\|_2^2$. For any $\eta > 0$, if ε is small enough, we have $\|A_{\varepsilon}u - Au\|_2 \leq \eta$. As

$$\begin{aligned} \|A_{\varepsilon}u - u_0\|_2^2 &= \|Au - u_0\|_2^2 + \|A_{\varepsilon}u - Au\|_2^2 + 2\left\langle A_{\varepsilon}u - Au, Au - u_0 \right\rangle \\ &\leq \|Au - u_0\|_2^2 + \eta^2 + C\eta, \end{aligned}$$

we get

$$m_{\varepsilon} = J(u_{\varepsilon}) + \frac{\lambda}{2}\|A_{\varepsilon}u_{\varepsilon} - u_0\|_2^2 \leq J(u) + \frac{\lambda}{2}\|A_{\varepsilon}u - u_0\|_2^2 \leq m + \frac{\lambda}{2}(C + \eta)\eta.$$

This implies with (15) that $m_{\varepsilon} \to m$ as ε goes to zero, moreover, as the liminf in (15) is in fact valid independently for the term in J and for the term in $\|\cdot\|_2^2$, it also implies that $J(u_{\varepsilon}) \to J(u)$ and $\|A_{\varepsilon}u_{\varepsilon} - u_0\|_2 \to \|Au - u_0\|_2$, showing that the limit of $A_{\varepsilon}u_{\varepsilon}$ is strong in $L^2(\Omega)$. Thus (13) holds in the general case.

Therefore in any case we have established that if u is a solution of (3) for both λ and μ, $\lambda < \mu$, then $J(u) = 0$. The consequence of this fact is that $\sigma(\lambda)$ has to be strictly decreasing, except possibly on $[0, \underline{\lambda}]$ for some $\underline{\lambda} \geq 0$, where it takes the value $\|u_0 - \int_\Omega u_0\|_2$. The proof of Lemma 2.3 is complete. □

It is possible to have $\underline{\lambda} > 0$: actually, (we assume $\int_\Omega u_0 = 0$), 0 is a solution of (3) if (assuming A is continuous from $L^p(\Omega)$ into $L^2(\Omega)$ – or $u_0 \in L^p(\Omega)'$),

$$\lambda A^* u_0 \in \partial J(0) = V$$

Where $V = \mathrm{Dom} J^*$ is defined above. Therefore, $\underline{\lambda}$ may be defined as

$$\underline{\lambda} = \max\{\lambda \ : \ \lambda A^* u_0 \in V\}.$$

Notice that if we solve the problem (for any smooth function f on $\partial\Omega$)

$$\begin{cases} \Delta v = A^* u_0 & \text{in } \Omega \\ v = f & \text{on } \partial\Omega \end{cases}$$

we get a lower bound: $\underline{\lambda} \geq 1/\|\nabla v\|_\infty$ which is non-zero as soon as $A^* u_0 \in L^q(\Omega)$ with $q > N$ (as $v \in W^{2,q}(\Omega) \subset C^1(\overline{\Omega})$ in this case).

3. A relaxation algorithm for solving (3)

For numerical reasons, we do not solve exactly (3) but an approximation. We are going to prove the convergence of a general relaxation algorithm, described in [6], [19], and inspired mainly by works by D. Geman (for instance see [12]). [2] gives a proof of convergence in the discrete case, while [20] proposes a similar method for minimizing (via a gradient method descent) the total variation of an image[3]. See also [7] for a wide review of this kind of "Auxiliary Variables" approaches in computer vision, with applications to many energy-based reconstruction or edge detection problems.

Let Φ_ε be the following C^1 function:

$$\Phi_\varepsilon : x \mapsto \begin{cases} \frac{1}{2\varepsilon} x^2 & \text{if } |x| \leq \varepsilon \\ |x| - \frac{\varepsilon}{2} & \text{if } \varepsilon \leq |x| \leq \frac{1}{\varepsilon} \\ \frac{\varepsilon}{2} x^2 + \frac{1}{2}(\frac{1}{\varepsilon} - \varepsilon) & \text{if } |x| \geq \frac{1}{\varepsilon} \end{cases} \tag{16}$$

and consider the problem

$$\text{Minimize } \int_\Omega \Phi_\varepsilon(|\nabla u|) + \frac{\lambda}{2}|Au - u_0|^2 \tag{17}$$

where $u \in H^1(\Omega) = W^{1,2}(\Omega)$. If we introduce $\tilde{J}(u) = J(u)$ when $u \in H^1(\Omega)$ and $+\infty$ when $u \notin H^1(\Omega)$, then we can show that $\int_\Omega (\Phi_\varepsilon(|\nabla u|) + \frac{\varepsilon}{2})$ decreases and

[3] We have just learned that Dobson and Vogel had also independently found a proof of convergence for a similar iterative approach [10]

converges pointwise to $\tilde{J}(u)$ as ε goes to zero. As J is the lower semi-continuous envelope of $\tilde{J}$, this implies (see [8, Prop 5.7]) that $\int_\Omega (\Phi_\varepsilon(|\nabla u|) + \frac{\varepsilon}{2})$ Γ-converges to J and that the functional in (17) Γ-converges to the one in (3) as ε goes to zero. Therefore, if A is injective, the unique solution of (17) will converge as ε goes to zero to the solution of (3) (or, if A is not injective, the solutions of (17) will have limit points that are solutions of (3)).

In the sequel we show how to minimize (17). We will set $\lambda = 1$ and, as ε will be fixed, we will denote Φ_ε simply by Φ. Also, we will only treat the case where A is the identity operator, the general case being similar.

Consider the following functional:

$$E(u,v) = \int_\Omega v|\nabla u|^2 + \frac{1}{v} + |u - u_0|^2 \tag{18}$$

where $u \in H^1(\Omega)$ and $v \in L^2(\Omega)$, $\varepsilon \le v \le 1/\varepsilon$.

Start from any u^1 and v^1 (for instance $v^1 \equiv 1$) and let:

$$\begin{aligned} u^{n+1} &= \arg\min_{u \in H^1(\Omega)} E(u, v^n) \\ v^{n+1} &= \arg\min_{\varepsilon \le v \le 1/\varepsilon} E(u^{n+1}, v) = \varepsilon \vee \frac{1}{|\nabla u^{n+1}|} \wedge \frac{1}{\varepsilon} \end{aligned} \tag{19}$$

where we used the notations $a \vee b = \max(a,b)$ and $a \wedge b = \min(a,b)$. u^{n+1} is therefore characterized by

$$\forall \phi \in H^1(\Omega), \int_\Omega v^n \nabla u^{n+1} \cdot \nabla\phi + (u^{n+1} - u_0)\phi = 0,$$

i.e. $-\mathrm{div}\,(v^n \nabla u^{n+1}) + u^{n+1} = u_0$ in $H^1(\Omega)'$.[4] We have the following result.

Proposition 3.1. *The sequence (u_n) converges (strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$) to the minimizer of (17).*

Proof. It is easy to establish that

$$\begin{aligned} E(u^n, v^n) - E(u^{n+1}, v^n) &\ge \varepsilon\|\nabla(u^n - u^{n+1})\|_2^2 + \|u^n - u^{n+1}\|_2^2 \\ &\ge \min(1,\varepsilon)\|u^n - u^{n+1}\|_{H^1(\Omega)}^2 \end{aligned} \tag{20}$$

and that

$$E(u^{n+1}, v^n) - E(u^{n+1}, v^{n+1}) \ge \varepsilon^3\|v^n - v^{n+1}\|_2^2. \tag{21}$$

This implies that for all $p \in [1, +\infty)$,

[4] Note here that the divergence operator is in this case has to be understood in a weak sense, *i.e.* as a notation for the injection $L^2(\Omega, \mathbb{R}^N) \to H^1(\Omega)'$ that maps a function $\mathbf{v}$ to the linear form $\phi \mapsto -\int \mathbf{v}\nabla\phi$; in fact, it will be always applied to functions $\mathbf{v}$ with $\mathrm{div}\,\mathbf{v} \in L^2(\Omega)$ that satisfy an homogeneous Neumann condition ($\mathbf{v} \cdot \mathbf{n} = 0$) on $\partial\Omega$. The functions $u \in L^2(\Omega)$ are also identified with the linear forms $\phi \mapsto \int u\phi$. We do not need in this work to know anything more about the structure of the dual space $H^1(\Omega)'$ which is simply endowed with its standard weak-$*$ topology

(22) $$\lim_{n\to\infty} \|v^n - v^{n+1}\|_{L^p(\Omega)} = 0$$

(for $p < 2$, as Ω is bounded, and for $p > 2$, as v^n is bounded).

Now take any test function $\phi \in H^1(\Omega)$. For all n we have

$$\int_\Omega v^n \nabla u^{n+1} \cdot \nabla\phi + (u^{n+1} - u_0)\phi = 0,$$

which may be written

(23) $$\int_\Omega v^{n+1}\nabla u^{n+1} \cdot \nabla\phi + (u^{n+1} - u_0)\phi = \int_\Omega (v^{n+1} - v^n)\nabla u^{n+1} \cdot \nabla\phi,$$

and we have for all p, p' with $\frac{1}{p} + \frac{1}{p'} + \frac{1}{2} = 1$,

$$\left|\int_\Omega (v^{n+1} - v^n)\nabla u^{n+1} \cdot \nabla\phi\right| \le \|v^n - v^{n+1}\|_{L^p} \|\nabla u^{n+1}\|_{L^{p'}} \|\nabla\phi\|_2.$$

The last expression goes to zero as long as we can choose p' such that $\|\nabla u^{n+1}\|_{L^{p'}(\Omega)}$ is bounded.

This follows from a result by Meyers [15]: as u^{n+1} is the solution of the equation in $H^1(\Omega)'$

$$-\mathrm{div}\,(v^n\nabla u) + u = u_0 \in L^2(\Omega),$$

with $0 < \varepsilon \le v^n \le 1/\varepsilon < +\infty$, there exists a $p' > 2$ (depending on ε and the dimension N), and $C', C < \infty$ such that

$$\|\nabla u^{n+1}\|_{p'} \le C'\|u_0 - u^{n+1}\|_2 \le C.$$

Using this result if we choose $p = \frac{2p'}{p'-2} < \infty$, the right hand term of (23) is bounded by $C\|v^n - v^{n+1}\|_{L^p}\|\nabla\phi\|_2$ and goes to zero as n goes to infinity. This proves that

$$-\mathrm{div}\, v^n\nabla u^n + u^n - u_0$$

goes to zero in $H^1(\Omega)'$.

Now, as the sequence u^n is bounded in $H^1(\Omega)$, and therefore compact in $L^2(\Omega)$ (remember Ω is bounded and smooth), we can extract a subsequence u^{n_k} that converges strongly to some $u \in L^2(\Omega)$. We also may assume that ∇u is the weak limit of ∇u^{n_k} as k goes to infinity.

As $v^n = \varepsilon \vee \frac{1}{|\nabla u^n|} \wedge \frac{1}{\varepsilon} = \Phi'(|\nabla u^n|)/|\nabla u^n|$, all this leads to

$$-\mathrm{div}\left\{\frac{\Phi'(|\nabla u^{n_k}|)}{|\nabla u^{n_k}|}\nabla u^{n_k}\right\} \rightharpoonup u_0 - u$$

in $H^1(\Omega)'$.

For all $\phi \in H^1(\Omega)$, we denote

$$\mathscr{A}(\phi) = -\mathrm{div}\left\{\frac{\Phi'(|\nabla\phi|)}{|\nabla\phi|}\nabla\phi\right\} \in H^1(\Omega)'.$$

We know that $\mathscr{A}(u^{n_k})$ goes to $T = u_0 - u$ and we need to check that this limit is exactly $\mathscr{A}(u)$: this would ensure that u satisfies the Euler equation for problem (17) (i.e., $\mathscr{A}(u)+u-u_0 = 0$) and is therefore the unique solution of (17). In order to do so we shall use the celebrated trick due to Minty for monotone equations (see for instance [5]).

Consider any $\phi \in H^1(\Omega)$. As $\mathscr{A}$ is the derivative of a convex functional, it is a monotone operator and we can write:

$$\langle \mathscr{A}(u^{n_k}) - \mathscr{A}(\phi), u^{n_k} - \phi \rangle \geq 0 \tag{24}$$

As $n \to \infty$,

$$\langle \mathscr{A}(\phi), u^{n_k} \rangle = \int_\Omega \frac{\Phi'(|\nabla\phi|)}{|\nabla\phi|} \nabla\phi \cdot \nabla u^{n_k} \to \int_\Omega \frac{\Phi'(|\nabla\phi|)}{|\nabla\phi|} \nabla\phi \cdot \nabla u = \langle \mathscr{A}(\phi), u \rangle$$

as ∇u^{n_k} converges to ∇u weakly in $L^2(\Omega)$, and

$$\langle \mathscr{A}(u^{n_k}), u^{n_k} \rangle = \int_\Omega (u_0 - u^{n_k})u^{n_k} + (v^{n_k} - v^{n_k-1})|\nabla u^{n_k}|^2$$

goes to $\langle T, u \rangle$ using the strong convergence of u^{n_k} and Meyers' result, once again.

Therefore (24) becomes at the limit

$$\langle T - \mathscr{A}(\phi), u - \phi \rangle \geq 0.$$

We can take $\phi = u + h\psi$ for any $h > 0$ and $\psi \in C_c^\infty(\overline{\Omega})$, and this leads to:

$$\langle T - \mathscr{A}(u + h\psi), \psi \rangle \leq 0, \forall \psi \in C_c^\infty(\overline{\Omega}), \forall h > 0.$$

But $\left(\Phi'(|\nabla u + h\nabla\psi|)/|\nabla u + h\nabla\psi|\right)(\nabla u + h\nabla\psi)$ is a continuous function of h and goes to $\left(\Phi'(|\nabla u|)/|\nabla u|\right)\nabla u$ as h goes to zero, moreover it may be uniformly bounded (as soon as h is bounded) by a square integrable function and we may invoque Lebesgue's theorem to conclude that this limit as h goes to zero is also a strong limit in $L^2(\Omega)$. Therefore,

$$\langle \mathscr{A}(u + h\psi), \psi \rangle = \int_\Omega \frac{\Phi'(|\nabla u + h\nabla\psi|)}{|\nabla u + h\nabla\psi|}(\nabla u + h\nabla\psi)\nabla\psi$$

$$\to \int_\Omega \frac{\Phi'(|\nabla u|)}{|\nabla u|}\nabla u \nabla\psi = \langle \mathscr{A}(u), \psi \rangle$$

as h goes to zero and this shows that for all $\psi \in C_c^\infty(\overline{\Omega})$,

$$\langle T, \psi \rangle \leq \langle \mathscr{A}(u), \psi \rangle,$$

which means that $T = \mathscr{A}(u)$ and concludes the proof. □

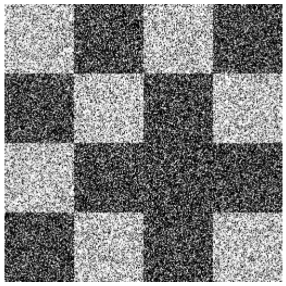

Fig. 1. Original image with noise of std. dev. 65

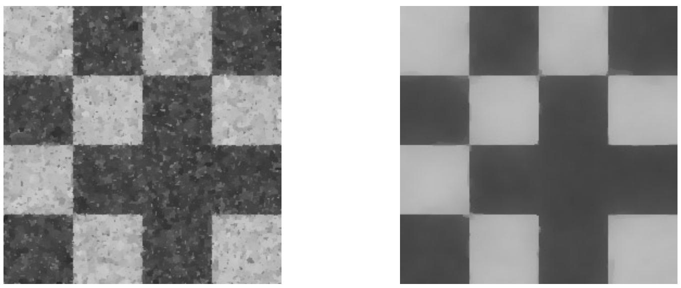

Fig. 2. Reconstructed images with $\sigma = 60$ and $\sigma = 67$

Remark 1. If A is not injective, the solutions of problem (17) may form a closed convex set (as Φ_ε is not uniformly convex). Then, u_n may not converge to one solution u of (17) but the proof shows that its limit points are solutions of the problem. If A *is* injective, then u_n has a unique limit point, therefore its limit.

Remark 2. In practice, in discrete images, the gradients are always bounded and it is for instance unnecessary to consider, in equation (16), the case $|x| \geq 1/\varepsilon$. Also, it may turn simpler to minimize a functional like $\Phi_\varepsilon(|\partial u/\partial x|)+\Phi_\varepsilon(|\partial u/\partial y|)$ rather than $\Phi_\varepsilon(|\nabla u|)$.

4. Some remarks about the model

We treated a few examples, using the algorithm described above. Results are shown on Figs. 1–3. Here grey-level values range from 0 to 255, the dark squares have value 73 (on the original image) and the light squares have value 183. This model is excellent if the image to reconstruct is (almost) piecewise constant (cf Fig. 2). For a piecewise smooth, or affine image, staircase-like structures will tend to develop, and this is easy to explain. Actually, Figs. 4–6 show (in dimension one) the worst case you could imagine: here the "original" signal $\tilde{u}$ that we want to reconstruct is simply the function $y = x$, and the "noise" has turned it into a piecewise constant nondecreasing function. The standard deviation of this "noise" is approximately $\sigma \simeq 3.5$ (so that Fig. 5 should be the "correct" reconstruction).

Fig. 3. Original and reconstruction ($\sigma \simeq 30$)

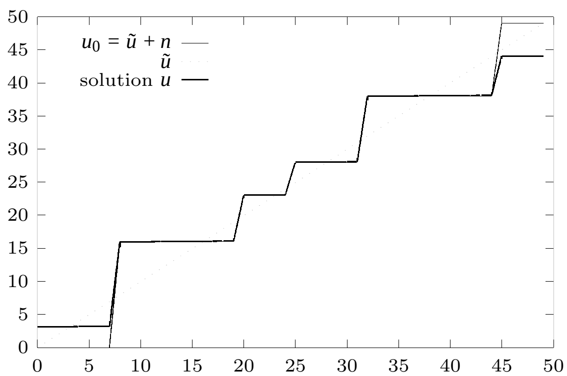

Fig. 4. $\sigma \simeq 2$

Then it is simple to imagine what happens: the total variation of the solution u, which one expects to be nondecreasing and piecewise constant, is just the difference $u(50) - u(0)$, and the easiest way to make it lower is to decrease $u(50)$ and increase $u(0)$. On the figures it is impossible to tell what the "best" reconstruction is (you probably would say u_0 is better). The same kind of effect appears on Fig. 7 (here the standard deviation of the noise is 5.5). Clearly, the model is not suited to the reconstruction of images that are not nearly piecewise constant. This defect has already been mentioned and analysed in [9]. Further in the paper we present a variant of this model that handles correctly signals such as the ones treated in Figs. 4–7. The results are very good, see Fig. 13. (The signal on Figs. 13 is a concatenation of the noisy signals of Figs. 4–6 and of Fig. 7.)

5. Inf-convolution of two convex potentials

The interest of denoising an image by minimizing a functional like $\int |\nabla u|$ rather that $\int |\nabla u|^p$, $p > 1$, is clear. Actually, a function $u \in BV$ may present discontinuities along $(N-1)$-dimensional surfaces, while $W^{1,p}$ functions may not: it is therefore theoretically impossible to reconstruct "edges" with functions in $W^{1,p}$.

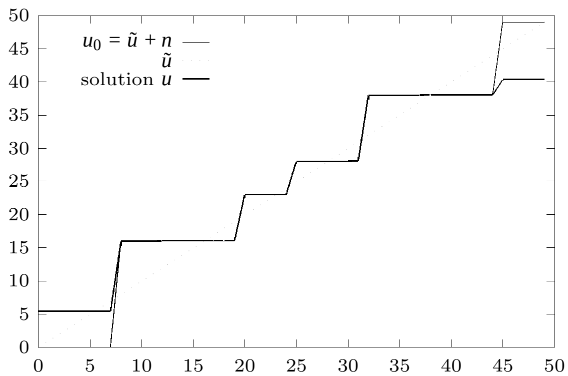

Fig. 5. $\sigma \simeq 3.5$

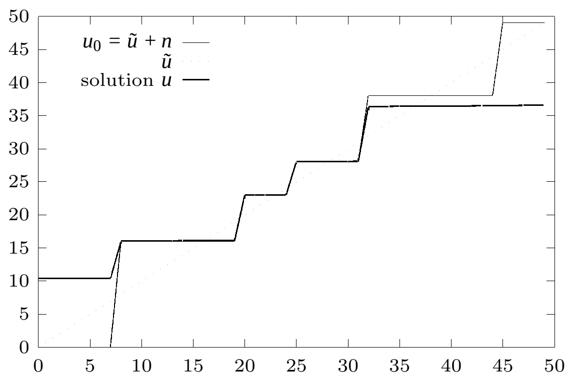

Fig. 6. $\sigma \simeq 5.8$

Many authors [2, 6, 12] have proposed to minimize non-convex functionals of the gradient of the image, and particularly functionals growing sublinearly at infinity (for instance, choosing $p < 1$). Their idea is to enhance the edges. This seems mathematically absurd in a continuous setting, however, these methods often give visually good results just because the discretization of the image automatically makes the problem finite-dimensional, in which case the convexity of the functional is no more necessary. Unstability is almost always the main drawback of this kind of approaches.

Here we want to present variants of another kind, based on the minimization of several convex functionals of the gradient. Suppose, for instance, that you want to build a low-diffusive filter, and that you process images in which appear mostly horizontal and vertical features. You might want to minimize the *minimum* of two "horizontal" and "vertical" functionals, say, (we assumed $N = 2$, $\Omega \subset \mathbb{R}^2$)

$$J_1(u) = \int_\Omega \left|\frac{\partial u}{\partial x}\right|^p, \; J_2(u) = \int_\Omega \left|\frac{\partial u}{\partial y}\right|^p. \tag{25}$$

(We will consider the cases $p = 2$ and $p = 1$.)

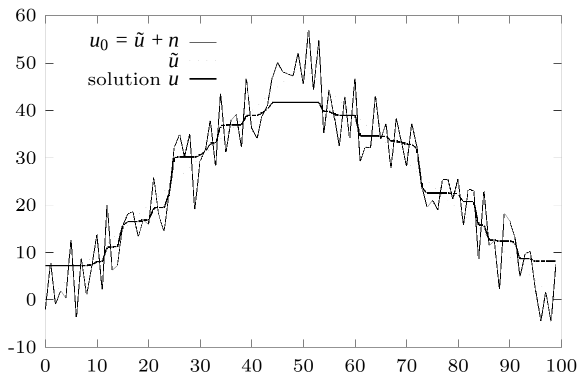

Fig. 7. Another noisy signal, with $\sigma \simeq 5.5$

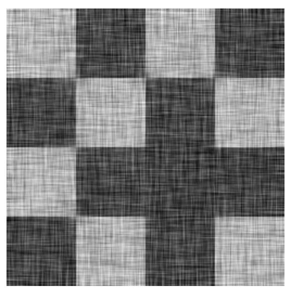

Fig. 8. A minimizer of (29) – with λ chosen in order to have $\sigma = 60$

Now the problem is the following: $u \mapsto J_1(u) \wedge J_2(u)$ is not a convex l.s.c. functional of u. We propose two different ways of dealing with this problem. One is to consider the *convex envelope* of $J_1 \wedge J_2$, i.e.,

$$(26)\quad \begin{aligned} J(u) &= \sup\{j(u) \ : \ j \text{ convex}, j \leq J_1, j \leq J_2\} \\ &= \inf\{\theta J_1(u_1) + (1-\theta)J_2(u_2) \ : \ \theta u_1 + (1-\theta)u_2 = u, \theta \in [0,1]\}, \end{aligned}$$

or the lower semicontinuous convex envelope of $J_1 \wedge J_2$, i.e.,

$$(27)\quad J(u) = (J_1 \wedge J_2)^{**} = (J_1^* \vee J_2^*)^*.$$

The second approach, that may turn to be simpler in some cases, is to consider the *inf-convolution* of J_1 and J_2, i.e.,

$$(28)\quad J(u) = J_1 \triangle J_2(u) = \inf\{J_1(u_1) + J_2(u_2) \ : \ u_1 + u_2 = u\},$$

or its semicontinuous envelope, $J(u) = (J_1(u)^* + J_2(u)^*)^*$.

Notice that if J_1, for instance, is 1-homogeneous, then J_1^* is the characteristic function of some closed set V_1, and if J_2^* is non-negative on V_1 (for instance if $J_2(0) = 0$) then $J_1(u)^* + J_2(u)^* = J_1^* \vee J_2^*$ and both l.s.c. envelopes coincide.

Now consider J_1 and J_2 as defined by (25). The previous remark shows that for $p = 1$, both notions (inf-convolution and convex envelope) are equivalent (it

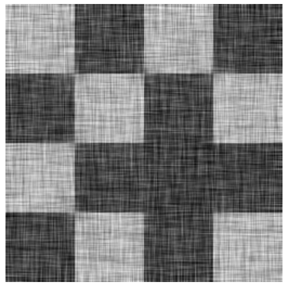

Fig. 9. A minimizer of (30)

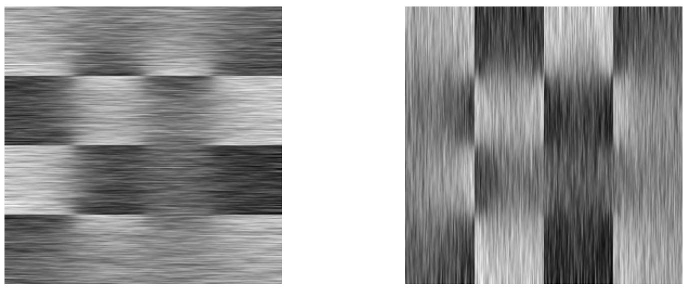

Fig. 10. Functions u_1 and u_2 found in the minimization of (30)

is straightforward to check also that in this case (26) and (28) give the same result). On the other hand, if $p = 2$, those formulas define distinct functionals that it is easy to compute in the case where Ω is a rectangle or the whole plane $\mathbb{R}^2$. Actually in both cases we may use the Fourier transform, and it is simple to check that the convex envelope of J_1 and J_2 is

$$J(u) = \inf_{\theta\in[0,1]} \int_{\hat{\Omega}} \frac{|\xi|^2|\eta|^2}{\theta|\xi|^2 + (1-\theta)|\eta|^2} |\hat{u}|^2,$$

whereas the inf-convolution is

$$J'(u) = J_1 \triangle J_2(u) = \int_{\hat{\Omega}} \frac{|\xi|^2|\eta|^2}{|\xi|^2 + |\eta|^2} |\hat{u}|^2$$

where $\hat{u}$ is the Fourier transform of u and $\hat{\Omega}$ the Fourier domain (i.e., $\mathbb{Z}^2$ or $\mathbb{R}^2$). Using this last formula and a FFT algorithm, it is quite simple to solve the following problem.

(29) $$\text{Minimize } J'(u) + \frac{\lambda}{2}\|u - u_0\|_2^2.$$

A result is shown on Fig. 8. The gradient descent flow of J' is associated to the pseudo-differential operator $(\Delta)^{-1}\partial_{xy}^2$.

For $p = 1$, we do not know how to compute the convex envelope of J_1 and J_2. However, we can solve numerically the following problem (using a method similar to the one described in Sect. 3).

Fig. 11. A minimizer of (31), with $\varepsilon = 10$ (left), and "first" and "second order" total variation minimization (right)

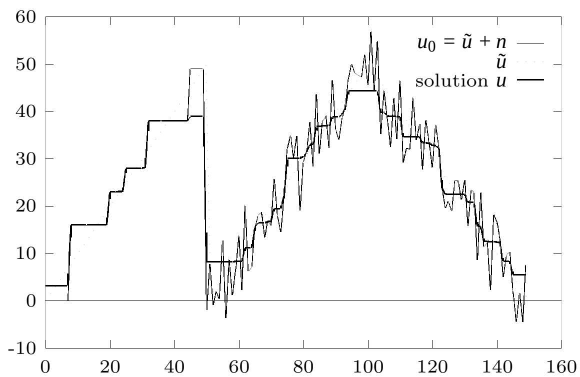

Fig. 12. Total variation minimization

$$\text{Minimize } J_1(u_1) + J_2(u_2) + \frac{\lambda}{2}\|u_1 + u_2 - u_0\|_2^2. \tag{30}$$

See Figs. 9–10 for an example.

The same method could be used with any other kind of convex functions. For instance, one might want to minimize a combination of the total variation and the integral of the squared norm of the gradient. In this case the convex envelope is equal to the inf-convolution and we get, if $J_1(x) = \int_\Omega |\nabla u|$ and $J_2(x) = (1/2\varepsilon)\int_\Omega |\nabla u|^2$,

$$J_1 \triangle J_2(u) = \frac{1}{2\varepsilon}\int_{|\nabla u|<\varepsilon} |\nabla u|^2 + \int_{|\nabla u|\geq\varepsilon} |\nabla u| - \frac{\varepsilon}{2}, \tag{31}$$

which is straightforward to minimize with the method shown in Sect. 3 – see Fig. 11, left.

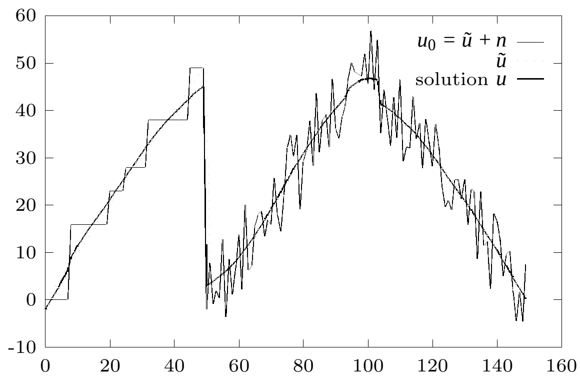

Fig. 13. "First and second order variation" minimization

Or, one could try to minimize this way the inf-convolution of a first order functional (like the total variation) and a second-order functional (like the total variation of the gradient, or first derivative). Let us have a particular look at this last idea. It may be written as the following minimization problem:

$$\min_{u_1,u_2} \int_\Omega |\nabla u_1| + \alpha |d^2 u_2| + \lambda |u_1 + u_2 - u_0|^2$$
$$= \min_{u,v} \int_\Omega |\nabla u - \nabla v| + \alpha |\nabla(\nabla v)| + \lambda |u - u_0|^2$$

if we let $u = u_1 + u_2$ and $v = u_2$. Here u (and u_1) $\in \mathrm{BV}(\Omega)$ and $v \in W^{1,1}$, with $\nabla v \in BV(\Omega, \mathbb{R}^N)$. The interpretation is very simple. In some sense we "first" approximate locally the gradient of the function u_0 by ∇v, that has itself a (very) low total variation (we have to choose $\alpha >> 1$). Then, we find u as an approximation of u_0 such that $u - v$ has a low total variation. As a consequence we do not get any more an almost piecewise constant result. In dimension one, the improvement is remarkable, see Fig. 13 (compare with Fig. 12). Notice that in this case the problem may simply be rewritten

$$\min_{u,w} \int_\Omega |u' - w| + \alpha |w'| + \lambda |u - u_0|^2$$

if we let $w = v'$. This is related to standard signal reconstruction methods, introduced in [4]. In dimension two, a result is shown on Fig. 11, right.

The interesting point here is that the functional we minimize

$$J(u) = (J_1^* + J_2^*)^* = (J_1 \wedge J_2)^{**},$$

with $J_1(u) = \int_\Omega |\nabla u|$ and $J_2(u) = \alpha \cdot \int_\Omega |d^2 u|$, is a convex homogeneous functional that is low when either ∇u or $d^2 u$ are low (and zero when u is affine). This idea could certainly be used in many other settings, when a signal or an image presents various types of characteristic features that need to be preserved and reconstructed.

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