Math 273: **Homework #1** Assigned on October 9.

Due to: Teaching Assistant Eric Radke.

Due date: one week from the date of the assignment. Late homework is accepted.

[**1**] Compute the gradient $\nabla f(x)$ and Hessian $\nabla^2 f(x)$ of the function

$$f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2.$$

Show that $x^* = (1,1)^T$ is the only local minimizer of this function, and that the Hessian matrix at that point is positive definite.

[**2**] Let a be a given n-vector, and A be a given $n \times n$ symmetric matrix. Compute the gradient and Hessian of $f_1(x) = a^T x$ and $f_2(x) = x^T A x$.

[**3**] Suppose that f is a convex function. Show that the set of global minimizers of f is a convex set.

[**4**] Suppose that $\hat{f}(z) = f(x)$, where $x = Sz + s$ for some $S \in R^{n\times n}$ and $s \in R^n$. Show that

$$\nabla \hat{f}(z) = S^T \nabla f(x), \qquad \nabla^2 \hat{f}(z) = S^T \nabla^2 f(x) S.$$

[**5**] Computation of the Euler-Lagrange equation in the continuous case.

(a) Consider the minimization problem

$$\inf_u F(u) = \int_{x_0}^{x_1} L(x, u(x), u'(x))dx,$$

with $u(x_0) = u_0$, $u(x_1) = u_1$ given constants, and L a sufficiently smooth function. Obtain formally the Euler-Lagrange equation of the minimization problem that is satisfied by a smooth optimal u.

Hint: Consider test functions v, such that $v(x_0) = v(x_1) = 0$. Since u is a minimizer, we must have $F(u) \le F(u + \varepsilon v)$ for all such sufficiently smooth functions v and every real ϵ. Apply integration by parts to obtain the desired result. You should obtain a second-order differential equation.

(b) Let now $u(x,t)$ be a smooth solution of the time-dependent partial differential equation (PDE)

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} L_{u'}(x,u,u') - L_u(x,u,u'),$$

with $u(x,0) = u_0(x)$ on (x_0, x_1) and $u(x_0,t) = U_0$, $u(x_1,t) = U_1$ for $t \ge 0$. Show that the function $E(t) = F(u(\cdot,t))$ is decreasing in time, where $F(u) = \int_{x_0}^{x_1} L(x,u,u')dx$.

[Notes]

• Let Ω be an open and bounded subset of R^d, with Lipschitz-continuous (or sufficien tly smooth) boundary $\partial\Omega$. Let $\vec{n} = (n_1, n_2, ..., n_d)$ be the exterior unit normal to $\partial\Omega$. Recall the following fundamental Green's formula, or integration by parts formula: given two function s u, v (with u, v, and all their 1st order partial derivatives belonging to $L^2(\Omega)$, or $u,\ v \in H^1(\Omega)$), then

$$\int_\Omega u v_{x_i} dx = -\int_\Omega u_{x_i} v dx + \int_{\partial\Omega} u v n_i dS.$$