Examples of dual problems
What we know:
V, Y are two normed spaces, with V^* and Y^* their duals
$\mathcal{F} : V \mapsto \overline{\mathbb{R}}$, $F : V \mapsto \overline{\mathbb{R}}$, $G : Y \mapsto \overline{\mathbb{R}}$

We use the duality pairing notations:
• if $u^* \in V^*$ and $u \in V$, we write $\langle u^*, u\rangle = u^*(u)$.
• if $p^* \in Y^*$ and $p \in Y$, we write $\langle p^*, p\rangle = p^*(p)$.

Conjugate or polar of $F : V \mapsto \bar{\mathbb{R}}$ is $F^* : V^* \mapsto \bar{\mathbb{R}}$ defined by

$$F^*(u^*) = \sup_{u\in V} \Big\{ \langle u^*, u\rangle - F(u) \Big\}.$$

$\Lambda : V \mapsto Y$ is a linear and continuous operator with adjoint $\Lambda^* : Y^* \mapsto V^*$.

$$\text{primal problem: } (\mathcal{P}) \ \inf_V \mathcal{F}(u)$$

with $\mathcal{F}(u) = F(u) + G(\Lambda u)$

$$\text{dual problem: } (\mathcal{P})^* \ \sup_{p^*\in Y^*} -F^*(\Lambda^* p^*) - G^*(-p^*)$$

Extremality relation: if $\bar{u}$ solution of $(\mathcal{P})$ and $\bar{p}^*$ solution of $(\mathcal{P}^*)$, then these must satisfy:

$$F(\bar{u}) + F^*(\Lambda^*\bar{p}^*) - \langle \Lambda^*\bar{p}^*, \bar{u}\rangle = 0$$

$$G(\Lambda\bar{u}) + G^*(-\bar{p}^*) - \langle -\bar{p}^*, \Lambda\bar{u}\rangle = 0$$

Now we want to see examples of minimizations $(\mathcal{P})$ defined on Sobolev spaces and how to compute their duals.

Example: The Dirichlet Problem
Let $\Omega \subset \mathbb{R}^n$ be open, bounded and connected, $f \in L^2(\Omega)$.

$$-\triangle u = f \text{ in } \Omega,$$
$$u = 0 \text{ on } \partial\Omega.$$

Recall $H^1_0(\Omega) = \{v \in L^2(\Omega), D_i v \in L^2(\Omega), v = 0 \text{ on } \partial\Omega\}$

We know (exercise) that if we multiply the PDE by $v \in H^1_0(\Omega)$ and integrate by parts, we obtain the problem

$\Big\{$ Find $u \in H^1_0(\Omega)$ s.t. $a(u, v) = (f, v)$ for all $v \in H^1_0(\Omega)$ $\Big\}$
where $a(u, v) = \sum_{i=1}^n (D_i u, D_i v)$. $\Big\}$ and $(\cdot, \cdot)$ denotes the inner product in $L^2(\Omega)$.

We know (exercise) that this problem is equivalent with the minimization:

$$(\mathcal{P}) \quad \inf_{u\in H^1_0(\Omega)} \mathcal{F}(u),$$

with $\mathcal{F} : H^1_0(\Omega) \to \mathbb{R}$ defined by $\mathcal{F}(u) = \frac{1}{2}a(u, u) - (f, u)$.

From the Thm. from the course we can deduce that $(\mathcal{P})$ has a unique solution $\bar{u}$.

We want to compute its dual, which must also have a solution (Thm. from previous handout) and write down the extremality relations.

$V = H_0^1(\Omega)$, $Y = L^2(\Omega)^n$, $\Lambda : V \mapsto Y$ is the gradient operator, $\Lambda u = Du$, for all $u \in V$.
$V^* = H_0^1(\Omega)^*$ (coincides with $H^{-1}(\Omega)$)
$Y^* = Y = L^2(\Omega)^n$

$$F(u) = -(f, u) = -\int_\Omega f(x)u(x)dx$$

$$G(p) = \frac{1}{2}\int_\Omega |p(x)|^2 dx$$

so problem $(\mathcal{P})$ is of the form $F(u) + G(\Lambda u)$
We need to compute F^* and G^*:
(we can view $f \in V \in V^*$ and $(f, u) = \langle f, \rangle u)$

$$F^*(u^*) = \sup_{u\in V}\Big\{\langle u^*, u\rangle - F(u)\Big\} = \sup_{u\in V}\Big\{\langle u^*, u\rangle + (f, u)\Big\}$$

$$= \sup_{u\in V}\langle u^* + f, u\rangle = \begin{cases} 0 \text{ if } u^* + f = 0 \\ +\infty \text{ otherwise} \end{cases}$$

In the current homework it is shown that if $G = \frac{1}{2}\|\cdot\|^2$, then $G^* = G$, thus we have

$$G^*(p^*) = \frac{1}{2}\int_\Omega |p^*(x)|^2 dx.$$

Therefore, using the dual formula from above:

$$\text{dual problem: } (\mathcal{P})^* \quad \sup_{p^*\in L^2(\Omega)^n}\Big[-F^*(\Lambda^* p^*) - G^*(-p^*)\Big]$$

or

$$\text{dual problem: } (\mathcal{P})^* \quad \sup_{p^*\in L^2(\Omega)^n,\ -\Lambda^* p^* = f}\Big\{-\frac{1}{2}\int_\Omega |p^*(x)|^2 dx\Big\}$$

We want to express the constraint $-\Lambda^2 p^* = f$, $\Lambda = \nabla$ for $u \in H_0^1(\Omega)$, $p^* \in L^2(\Omega)^n$:

$$\int_\Omega (\Lambda u \cdot p^*)dx = \int_\Omega \nabla u \cdot (p_1^*, ..., p_n^*)dx = \int_\Omega (u_{x_1}p_1^* + ... + u_{x_n}p_n^*)dx$$

$$= -\int_\Omega u\Big(\frac{\partial}{\partial x_1}p_1^* + ... + \frac{\partial}{\partial x_n}p_n^*\Big) + \int_\Omega up^* \cdot \vec{n} = -\int_\Omega u\Big(\frac{\partial}{\partial x_1}p_1^* + ... + \frac{\partial}{\partial x_n}p_n^*\Big)$$

$$= -\int_\Omega u \text{div} p^* dx = \int_\Omega u(-\text{ div}p^* dx = \int_\Omega u\Lambda^* p^*,$$

thus we deduce that $\Lambda^* = -$ div .
Then the constraint $-\Lambda^2 p^* = f \Leftrightarrow -(-\text{div}p^*) = f$ or div $p^* = f$.
In conclusion, the dual problem becomes

$$(\mathcal{P})^* \quad \sup_{p^*\in L^2(\Omega)^n,\ \text{div}p^* = f}\Big\{-\frac{1}{2}\int_\Omega |p^*(x)|^2 dx\Big\}.$$

Extremality relations: if $\bar{u}$ is the unique solution of $(\mathcal{P})$ and $\bar{p}^*$ a solution of $(\mathcal{P})^*$, we must have:

$$F(\bar{u}) + F^*(\Lambda^*\bar{p}^*) = \langle \Lambda^*\bar{p}^*, \bar{u}\rangle$$

$$G(\Lambda\bar{u}) + G^*(-\bar{p}^*) = -\langle \bar{p}^*, \Lambda\bar{u}\rangle$$

The second relation gives

$$\int_\Omega |\nabla\bar{u}|^2 dx + \int_\Omega |\bar{p}^*(x)|^2 dx = -2\int_\Omega \nabla\bar{u}(x)\cdot\bar{p}^*(x)dx,$$

or

$$\sum_{i=1}^n \int_\Omega \Big(\bar{u}_{x_i} - \bar{p}_i^*)^2 dx = 0$$

possible iff $\bar{p}^*(x) = -\nabla\bar{u}(x)$ a.e. $x \in \Omega$

Conclusion: the hypotheses of theorems from the course and handout hold; we know the existence and uniqueness of a solution $\bar{u}$ of $(\mathcal{P})$; we have the existence of a solution $\bar{p}^*$ of $(\mathcal{P})^*$ (this is also unique since the functional $p^* \mapsto \frac{1}{2}\int_\Omega |p^*(x)|^2 dx$ is strictly convex); we also must have $\inf(\mathcal{P}) = \sup(\mathcal{P})^*$ and the extremality conditions above must hold, giving us that $\bar{p}^*(x) = -\mathrm{grad}\bar{u}(x)$, a.e. in Ω.

Example: computation of the dual for a problem with constraint
(problem of elasto-plastic torsion)

Let $W = \{v \in H_0^1(\Omega) : \ |\mathrm{grad}v(x)| \leq 1 \ a.e.\}$, or
$W = \{v \in H_0^1(\Omega) : \ |\nabla v(x)| \leq 1, \ a.e.\}$ (using the notation grad $= \nabla$).
It is easy to show that this is a closed, convex subset of $V = H_0^1(\Omega)$.
The primal problem is

$$(\mathcal{P}) \quad \inf_{u\in W}\Big\{\frac{1}{2}\int_\Omega \Big[|\nabla u|^2 - 2fu\Big]dx\Big\},$$

where we can assume that $f \in L^\infty(\Omega)$ is given.

Based on a Thm. of Brezis and Stampacchia (not covered in class), there is a unique solution $\bar{u} \in W^{2,\alpha}(\Omega)$ of the problem, for all $1 \leq \alpha < \infty$.

We leave out the discussion on the existence of the solution; here we only want to compute the dual problem.

We set $V = H_0^1(\Omega)$, $V^* = H^{-1}(\Omega)$, $Y = Y^* = L^2(\Omega)^n$, $\Lambda = \nabla = gradient$.

$$F(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega f(x)u(x)dx$$

$$G(p) = \begin{cases} 0 \text{ if } |p(x)| \leq 1 \ a.e. \\ +\infty \text{ otherwise} \end{cases}$$

$$(\mathcal{P}) \quad \inf_u F(u) + G(\Lambda u)$$

$$F^*(u^*) = \sup_{v\in V}\Big(\langle u^*, v\rangle + (f,v) - \frac{1}{2}\int_\Omega |\nabla v|^2 dx\Big).$$

The sup problem has a unique solution $v = v(u^*)$ that must satisfy

$$(\nabla v, \nabla w) = \langle f + u^*, w\rangle \text{ for all } w \in V$$

from where we obtain that

$$F^*(u^*) = \frac{1}{2}\|f + u^*\|_*^2,$$

where $\|\cdot\|_*$ is the dual norm in V^*, dual to $\|u\| = \Big(\int_\Omega |\nabla u|^2 dx\Big)^{1/2}$.
We also have:
$G : L^2(\Omega)^n \mapsto \overline{\mathbb{R}}$, $G(p) = \begin{cases} 0 \text{ if } |p(x)| \le 1 \ a.e. \\ +\infty \text{ otherwise} \end{cases}$.
Then

$$G^*(p^*) = \sup_{p\in L^2(\Omega)^n}\Big(\langle p^*, p\rangle - G(p)\Big) = \sup_{p\in L^2(\Omega)^n}\Big(\int_\Omega p^*(x)\cdot p(x)dx - \begin{cases} 0 \text{ if } |p(x)| \le 1 \ a.e. \\ +\infty \text{ otherwise} \end{cases}\Big)$$

$$= \sup_{|p(x)|\le 1 \ a.e.} \int_\Omega p^*(x)\cdot p(x)dx = \int_\Omega |p^*(x)|dx.$$

Thus we obtain the dual problem

$$(\mathcal{P})^* \qquad \sup_{p^*\in L^2(\Omega)^n}\Big(-\frac{1}{2}\|\text{div} p^* - f\|_*^2 - \int_\Omega |p^*(x)|dx\Big)$$

which is an unconstrained problem.