**Summary of necessary and sufficient conditions for local minimizers**

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**Unconstrained problem** $\min_{x \in R^n} f(x)$

1st-order necessary conditions If $x_*$ is a local minimizer of $f$ and $f$ is continuously differentiable in an open neighborhood of $x_*$, then

- $\nabla f(x_*) = \vec{0}$.

2nd-order necessary conditions If $x_*$ is a local minimizer of $f$ and $\nabla^2 f$ is continuous in an open neighborhood of $x_*$, then

- $\nabla f(x_*) = \vec{0}$
- $\nabla^2 f(x_*)$ is positive semi-definite.

2nd-order sufficient conditions Suppose that $\nabla^2 f$ is continuous in an open neighborhood of $x_*$. If the following two conditions are satisfied, then $x_*$ is a strict local minimizer of $f$:

- $\nabla f(x_*) = \vec{0}$
- $\nabla^2 f(x_*)$ is positive definite.

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**Equality constrained problem** $\min_{x \in R^n} f(x),\ \text{subject to } c_i(x) = 0,\ i = 1, ..., m,\quad m \leq n$

Necessary conditions Assume $f$ and $c_i$ are twice continuously differentiable in an open neighborhood of $x_*$, that $\nabla c_i(x_*)$ are linearly independent vectors, and that $x_*$ is a local minimizer. Let $A(x_*) \in R^{m \times n}$ be the Jacobian of all constraints at $x_*$ (of full row rank), and $Z(x_*) \in R^{n \times (m-n)}$ be a basis matrix for the null space of $A(x_*)$. Then there is a Lagrange multiplier $\lambda^* \in R^m$ such that

- $c_i(x_*) = 0$ for $i = 1, ..., m$
- $\nabla_x \mathcal{L}(x_*, \lambda^*) = 0 \Leftrightarrow Z(x_*)^T \nabla f(x_*) = \vec{0} \Leftrightarrow \nabla f(x_*) = A(x_*)^T \lambda_*$
- $Z(x_*)^T \nabla^2_{xx} \mathcal{L}(x_*, \lambda^*) Z(x_*)$ is positive semi-definite

Sufficient conditions Assume $f$ and $c_i$ are twice continuously differentiable in an open neighborhood of $x_*$, and that $\nabla c_i(x_*)$ are linearly independent vectors at $x_*$. Let $A(x_*) \in R^{m \times n}$ be the Jacobian of all constraints at $x_*$ (of full row rank), and $Z(x_*) \in R^{n \times (m-n)}$ be a basis matrix for the null space of $A(x_*)$. If there is a $\lambda^* \in R^m$ such that the following conditions are satisfied at $(x_*, \lambda^*)$, then $x_*$ is a strict local minimizer:

- $c_i(x_*) = 0$ for $i = 1, ..., m$
- $\nabla_x \mathcal{L}(x_*, \lambda^*) = 0 \Leftrightarrow Z(x_*)^T \nabla f(x_*) = \vec{0} \Leftrightarrow \nabla f(x_*) = A(x_*)^T \lambda_*$
- $Z(x_*)^T \nabla^2_{xx} \mathcal{L}(x_*, \lambda^*) Z(x_*)$ is positive definite

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**Inequality constrained problem** $\min_{x \in R^n} f(x),\ \text{subject to } c_i(x) \geq 0,\ i = 1, ..., m,\quad m \leq n$

Necessary conditions Let $x_*$ be a local minimizer, such that the linear independence of $\nabla c_i(x_*)$ for all active constraints holds. Let $Z(x_*)$ be a basis matrix for the null space of the Jacobian of the active constraints at $x_*$. Then there is a Lagrange multiplier $\lambda^* \in R^m$ such that

- $c_i(x_*) \geq 0$
- $\nabla_x \mathcal{L}(x_*, \lambda^*) = 0 \Leftrightarrow Z(x_*)^T \nabla f(x_*) = 0$
- $\lambda_i^* \geq 0$
- ${\lambda^*}^T c(x_*) = 0$
- $Z(x_*)^T \nabla^2_{xx} \mathcal{L}(x_*, \lambda_*) Z(x_*)$ is positive semi-definite.

Sufficient conditions Let $x_*$ be such that the linear independence of $\nabla c_i(x_*)$ for all active constraints holds. Let $Z(x_*)$ be a basis matrix for the null space of the Jacobian of the active constraints at $x_*$. If there is a Lagrange multiplier $\lambda^* \in R^m$ such that the following conditions are satisfied at $(x_*, \lambda^*)$, then $x_*$ is a strict local minimizer:

- $c_i(x_*) \geq 0$
- $\nabla_x \mathcal{L}(x_*, \lambda^*) = 0 \Leftrightarrow Z(x_*)^T \nabla f(x_*) = 0$
- $\lambda_i^* \geq 0$, and $\lambda_i^* > 0$ if $c_i(x_*) = 0$
- ${\lambda^*}^T c(x_*) = 0$
- $Z(x_*)^T \nabla_{xx}^2 \mathcal{L}(x_*, \lambda_*) Z(x_*)$ is positive definite.

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**Equality and inequality constrained problem** $\min_{x \in R^n} f(x)$, subject to $\begin{cases} c_i(x) = 0, & i \in \mathcal{E} \\ c_i(x) \geq 0, & i \in \mathcal{I} \end{cases}$

1st-order necessary conditions Let $\mathcal{A}(x) = \mathcal{E} \cup \{i \in \mathcal{I} : \ c_i(x) = 0\}$ be the set of all active constraints at a point $x$. Assume that at a point $x_*$, the active constraints gradients $\nabla c_i(x_*)$, $i \in \mathcal{A}(x_*)$ are linearly independent. Assume that $x_*$ is a local minimizer. Then there is a Lagrange multiplier $\lambda^* = (\lambda^*)_i$, $i \in \mathcal{E} \cup \mathcal{I}$, such that the following conditions are satisfied at $(x_*, \lambda^*)$:

- $\nabla_x \mathcal{L}(x_*, \lambda^*) = 0$
- $c_i(x_*) = 0$, for all $i \in \mathcal{E}$
- $c_i(x_*) \geq 0$, for all $i \in \mathcal{I}$
- $\lambda_i^* \geq 0$, for all $i \in \mathcal{I}$
- $\lambda_i^* c_i(x_*) = 0$, for all $i \in \mathcal{E} \cup \mathcal{I}$

(these conditions are called the *KKT conditions* from Karush-Kunn-Tucker conditions).

Note that for inactive constraints $c_i(x_*) > 0$ we have $\lambda_i^* = 0$, thus the condition on the Lagrangian function becomes

$$0 = \nabla_x \mathcal{L}(x_*, \lambda^*) = \nabla f(x_*) - \sum_{i \in \mathcal{A}(x_*)} \lambda_i^* \nabla c_i(x_*).$$

2nd-order necessary conditions Let $x_*$ be a local solution such that the linearity independence is satisfied at $x_*$. Let $\lambda^*$ such that the KKT conditions are satisfied at $(x_*, \lambda^*)$ and let $F(\lambda^*)$ be the set

$$w \in F(\lambda^*) \Leftrightarrow \begin{cases} \nabla c_i(x_*)^T w = 0, & i \in \mathcal{E}, \\ \nabla c_i(x_*)^T w = 0, & i \in \mathcal{A}(x_*) \cap \mathcal{I}, \ \lambda_i^* > 0 \\ \nabla c_i(x_*)^T w \geq 0, & i \in \mathcal{A}(x_*) \cap \mathcal{I}, \ \lambda_i^* = 0. \end{cases}$$

Then

- $w^T \nabla_{xx}^2 \mathcal{L}(x_*, \lambda^*) w \geq 0$, for all $w \in F(\lambda^*)$.

2nd-order sufficient conditions Suppose that at some feasible point $x_* \in R^n$ there is a Lagrange multiplier $\lambda^*$ such that the KKT conditions are satisfied. Then if the following condition is satisfied, then $x_*$ is a strict local minimizer:

- $w^T \nabla_{xx}^2 \mathcal{L}(x_*, \lambda^*) w > 0$ for all $w \in F(\lambda^*)$, $w \neq 0$.

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**Remarks:**

- If $\nabla_{xx}^2 \mathcal{L}(x_*, \lambda^*)$ positive definite, then $Z^T \nabla^2 f(x_*) Z$ is also positive definite (converse not true).
- "Negative definite" instead of "positive definite" for local maximizer.
- The Lagrange multiplier $\lambda_i^*$ is always 0 for inactive constraints.