Math 273: Homework #2, due on Wednesday, October 25

[**1**] Consider the minimization problem

$$\inf_u F(u) = \int_{x_0}^{x_1} L(x, u(x), u'(x), u''(x))dx,$$

with $u(x_0) = u_0$, $u(x_1) = u_1$, $u'(x_0) = U_0$, $u'(x_1) = U_1$ given, and L is a sufficiently smooth function. Obtain the Euler-Lagrange equation of the minimization problem that is satisfied by a smooth optimal u. Choose test functions v in $C^\infty[x_0, x_1]$ that satisfy $v(x_0) = v(x_1) = v'(x_0) = v'(x_1) = 0$, and proceed as before (you should obtain a fourth-order differential equation).

[**2**] Consider the 1D length functional

$$\text{Min}_u F(u) = \int_0^1 L(u'(x))dx, \text{ or } \text{Min}_u \int_0^1 \sqrt{1 + (u'(x))^2}dx,$$

with boundary conditions $u(0) = 0$, $u(1) = 1$.

(a) Find the exact solution of the problem.

(b) Show that the functional $u \mapsto F(u)$ is convex.

(c) Consider a discrete version of the problem: let

$$x_0 = 0 < x_1 < ... < x_n < x_{n+1} = 1$$

be equidistant points, with $x_{i+1} - x_i = h$. For $\vec{u} = (u_1, ..., u_n)$, consider $f(\vec{u}) = \sum_{i=0}^n \sqrt{1 + \left(\frac{u_{i+1}-u_i}{h}\right)^2}$, with the additional condition that $u_0 = 0$ and $u_{n+1} = 1$.

Choose an appropriate discretization integer n and numerically analyze the behavior of the gradient descent method with backtracking line search. Choose the initial starting point u^0 as a curve joining the points $(0, 0)$ and $(1, 1)$. Record the number of iterations and plot the error $u^k - u^*$, where u^* is the exact minimizer. You could also plot the curve given by $\vec{u}^k$ at some iterations.

Notes: Let Ω be an open and bounded subset of R^d, with Lipschitz-continuous (or sufficiently smooth) boundary $\partial\Omega$. Let $\vec{n} = (n_1, n_2, ..., n_d)$ be the exterior unit normal to $\partial\Omega$.

Recall the following fundamental Green's formula, or integration by parts formula: given two functions u, v (with u, v, and all their 1st order partial derivatives belonging to $L^2(\Omega)$), then

$$\int_\Omega u v_{x_i} dx = -\int_\Omega u_{x_i} v dx + \int_{\partial\Omega} u v n_i dS.$$