Math 273: Take-home final exam

Due date: no later than Friday December 15, 2006

If you cannot find me, you can leave the final at my office MS 7620-D (under the door), or you can give it to Babette Dalton at MS 7619 (before 3pm) or at Mary Edwards at MS 6363. Do not send me the final by e-mail.

If you cannot meet the deadline due to special circumstances only, please let me know.

1. Theoretical questions

[1] Let V be a real vector space and $\mathcal{C} \subset V$ a convex subspace. Let $F : \mathcal{C} \to \overline{\mathbb{R}}$ be a convex function, thus for every $u, v \in \mathcal{C}$, we have $F(\lambda u + (1-\lambda)v) \leq \lambda F(u) + (1-\lambda)F(v), \forall \lambda \in [0,1]$, whenever the RHS is defined (the RHS is not defined when $F(u) = -F(v) = +\infty$ or $F(u) = -F(v) = -\infty$).

(a) If F is convex, show that for every $u_1, ..., u_n$ of points of V and for every family $\lambda_1, ..., \lambda_n$, $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$, we have

$$F(\sum_{i=1}^n \lambda_i u_i) \leq \sum_{i=1}^n \lambda_i F(u_i).$$

(b) If $F : V \to \overline{\mathbb{R}}$ is convex, show that the sections $\{u : \ F(u) \leq a\}$ and $\{u : \ F(u) < a\}$ are convex sets. By a counterexample, show that the converse is not true.

[2] Assume that V and V^* are normed vector spaces in duality. Let $F : V \to \overline{\mathbb{R}}$ and let $F^* : V^* \to \overline{\mathbb{R}}$ be the polar or conjugate of F. Show

(i) $F^*(0) = -inf_{u \in V} F(u)$.

(ii) $(\lambda F)^*(u^*) = \lambda F^*(\frac{1}{\lambda} u^*)$ for every $\lambda > 0$.

[3] Let $V = V^* = \mathbb{R}^n$. Let Q be a symmetric positive definite $n \times n$ matrix, $b \in \mathbb{R}^n$, and consider $f(x) := \frac{1}{2} < x, Qx > + < b, x >$, for all $x \in \mathbb{R}^n$. Find the conjugate f^* and deduce that, in particular, $\frac{1}{2}\| \cdot \|^2$ is its own conjugate.

[4] Let $F : V \to \mathbb{R}$ and F^* its polar. Then $u^* \in \partial F(u)$ if and only if $F(u) + F^*(u^*) = < u^*, u >$.

[5] Show that the polar F^* is convex.

[6] Let $F : \mathbb{R}^n \to \mathbb{R}$ be continuous and coercive with respect to the Euclidean norm $\|x\|$, for $x \in \mathbb{R}^n$. Then there is $x_* \in \mathbb{R}^n$ such that $F(x_*) = \inf_{x \in \mathbb{R}^n} F(x)$. If in addition F is strictly convex, then the minimizer is unique.

[7] Let $f \in \mathbb{R}^{N^2}$ be given, and let $u \in \mathbb{R}^{N^2}$ be an unknown minimizer of the functional (already seen before)

$$E(w) = \sum_{i,j=0}^{N-1} |\nabla w_{i,j}|^2 + \lambda \sum_{i,j=0}^{N-1} (w_{i,j} - f_{i,j})^2,$$

for $w \in \mathbb{R}^{N^2}$, where

$$\nabla w_{i,j} = \begin{pmatrix} (D_x w)_{i,j} \\ (D_y w)_{i,j} \end{pmatrix} = \begin{pmatrix} w_{i+1,j} - w_{i,j} \\ w_{i,j+1} - w_{i,j} \end{pmatrix},$$

for $(i,j) \in \{0, ..., N-1\}^2$ (we assume that all vectors f, u, w are periodic outside of their support).

(a) Find the adjoint operators D_x^* and D_y^* of D_x and D_y.

(b) Find a linear operator $B : \mathbb{R}^{N^2} \to \mathbb{R}^{N^2}$, a $c \in \mathbb{R}^{N^2}$, and $C(f)$, independent of w, such that for all $w \in \mathbb{R}^{N^2}$,

$$E(w) = < Bw, w > + < c, w > + C(f).$$

(c) Show that B is self-adjoint.

(d) Find the Gateaux differential of $E(w)$ in the direction v and thus give a necessary (and sufficient) condition for u to be a minimizer, by setting this differential to zero (as the zero functional).

2. Computational question

(*Edge-preserving denoising application*) Try to solve by any method of your choice the minimization of

$$F(u) = \sum_{1 \le i,j \le n} \left[\sqrt{1 + (u_{i+1,j} - u_{i,j})^2 + (u_{i,j+1} - u_{i,j})^2} + \lambda (u_{i,j} - f_{i,j})^2 \right],$$

where $f_{i,j}$ is given for $0 \le i,j \le n+1$, and with the boundary conditions $u_{i,j} = f_{i,j}$ if $i = 0$ or $i = n+1$ or $j = 0$ or $j = n+1$. Here $\lambda > 0$ is a tunning parameter. You can find a noisy image f on the class webpage, or you can add noise to an image using the matlab command "imnoise".

Test various values of the parameter λ and observe the properties of your implementation; choose λ that visually will give you a satisfactory result (noise is removed, while restored image u is sharp). Give your choice of the stopping criteria and also plot the value of the objective function versus steps. Plot the data f, your starting point and your final result, as 2D images (you can use your starting point the noisy image f, or a constant). Also, give the details of your numerical implementation and include your code.

- Those interested in solving additional computational questions (not required for the final grade) can contact the instructor.