(I) Consider the polynomial function on the unit square defined by
\[ u(x, y) = 2^{4p}x^p(1 - x)^p y^p(1 - y)^p, \quad (x, y) \in [0, 1] \times [0, 1], \]
where \( p \) is a positive integer.

(a) Check that \( u = 0 \) on \( \partial \Omega \).

(b) Compute \(-\Delta u(x, y) + u(x, y)\) and denote the result by \( f(x, y) \).

(c) Consider the equation on the unit square \( \Omega = (0, 1) \times (0, 1) \):
\[ -\Delta u + u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega, \]
and use \( P_1 \) elements to approximate its solution. You can compare the numerical solution with the exact analytic solution (you can choose \( p \leq 10 \) or around 10; the accuracy of the approximation will depend on the choice of \( p \)).

(II) Use \( P_1 \) elements to approximate the solution of
\[ -\Delta u + u = \sin (2\pi(x + y)), \quad (x, y) \in \Omega = \text{unit square} \]
with the following boundary conditions:

Case (a) \( u = 0 \) for \((x, y) \in \partial \Omega\)

Case (b) \( u = 0 \) for \((x, y) \in \partial \Omega, x = 0, 1\)
\[ u_y = 0, \text{ for } (x, y) \in \partial \Omega, y = 0, 1. \]

Remarks
You can base the triangulations on a 10x10 grid.

- What you should turn in: the weak formulations, the linear systems, details about the discretizations, plots of the results, your computer program.
- Section 12.2 of the textbook discusses numerical integration or quadrature formulas, helpful to discretize the load vector, if needed.