This question corresponds to the 2D model problem. Find the linear basis functions for the triangle $K$ with vertices at $(0,0)$, $(h,0)$ and $(0,h)$. Show that the corresponding element stiffness matrix is given by

$$
\begin{bmatrix}
1 & -\frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & 0 \\
-\frac{1}{2} & 0 & \frac{1}{2}
\end{bmatrix}
$$

Using this result, show that the linear system (1.25) of Example 1.1. (textbook) has the stated form (pages 30-31) (there is a typo in the textbook regarding the values of the global stiffness matrix for this case). If you do not have the textbook, please let me know and I can copy the necessary page.

Consider the problem with an inhomogeneous boundary condition,

$$
\begin{cases}
-\Delta u = f & \text{in } \Omega, \\
u = u_0 & \text{on } \Gamma = \partial \Omega,
\end{cases}
$$

where $f$ and $u_0$ are given. Briefly show that this problem can be given the following equivalent formulations:

- $(V)$ Find $u \in V(u_0)$ such that $a(u,v) = (f,v), \forall v \in H_0^1(\Omega)$,
- $(M)$ Find $u \in V(u_0)$ such that $F(u) \leq F(v), \forall v \in V(u_0)$,

where $V(u_0) = \{v \in H^1(\Omega) : v = u_0 \text{ on } \Gamma\}$ and $F$ is defined in the usual way, $F(v) = \frac{1}{2}a(v,v) - (f,v)$.

Then formulate a finite element method and prove an error estimate (as in Thm. 1.1, page 24).

Recall: $H_0^1(\Omega) = \{v \in L^2(\Omega), \nabla v \in L^2(\Omega)^n, v = 0 \text{ on } \partial \Omega\}$, where $n$ is the spatial dimension.

(a) Give a weak variational formulation of the problem

$$
\frac{d^4u}{dx^4} = f \quad \text{for} \quad 0 < x < 1,
$$

$$
u(0) = u''(0) = u'(1) = u'''(1) = 0,
$$

and show that the assumptions of the Lax-Milgram Lemma are satisfied. Which boundary conditions are essential and which are natural?

(b) Solve the same problem with the following alternative boundary conditions:

$$
u(0) = -u''(0) + \gamma u'(0) = 0, \quad u(1) = u''(1) + \gamma u'(1) = 0,$$

where $\gamma$ is a positive constant.

\[ -\Delta u = f \quad \text{in } \Omega, \quad \gamma u + \frac{\partial u}{\partial n} = g \quad \text{on } \Gamma, \]

where \( \gamma \) is a constant. When are the assumptions of the Lax-Milgram Lemma satisfied?

[5] Consider the Neumann problem

\[ -\Delta u = f \quad \text{in } \Omega, \]
\[ \frac{\partial u}{\partial n} = g \quad \text{on } \Gamma = \partial \Omega, \]
\[ \int_{\Omega} u(x)dx = 0. \]

where \( f: \Omega \rightarrow R \) and \( g: \partial \Omega \rightarrow R \) satisfy the compatibility condition

\[ \int_{\Omega} f(x)dx + \int_{\partial \Omega} g(x)d\sigma(x) = 0. \]

(a) Why condition “\( \int_{\Omega} u(x)dx = 0 \)” was added here? Why do we need the compatibility condition?

(b) Give a weak variational formulation of the problem, and prove that the conditions of the Lax-Milgram Lemma are satisfied, under the necessary assumptions on \( f \) and \( g \) that you would specify.