[1] Let $V$ be a Hilbert space and $L : V \to R$ be a linear form. Show that $L$ is bounded if and only if $L$ is continuous. Then directly state the corresponding result for a bilinear form $a : V \times V \to R$.

[2] Let $V$ be a Hilbert space and the (nonlinear) operator $A : V \to V$, satisfying

(i) there is $M \geq 0$ s.t. $\forall u,v \in V, \|Au - Av\| \leq M\|u - v\|$ 
(ii) there is $\alpha > 0$ s.t. $\forall u,v \in V, \langle Au - Av, u - v \rangle \geq \alpha\|u - v\|^2$

Show that the nonlinear equation $Au = f$ has a unique solution (for some $f \in V$), using the Banach fixed point theorem and the same technique for proving the Lax-Milgram theorem (introduce the function $g_\lambda$).

[3] Let $V$ be a complex Hilbert space, $a : V \times V \to C$ a sesquilinear form, $L : V \to C$ an anti-linear form. Assume that $a$ is Hermitian, thus $a(v,u) = \overline{a(u,v)}$, $\forall u,v \in V$, and that $a(v,v) \geq 0$. Consider the problems

(V) Find $u \in V$ s.t. $a(u,v) = L(v), \forall v \in V,$

(M) Find $u \in V$ s.t. $J(u) = \inf_{v \in V} J(v),$

with $J : V \to R$ defined by $J(v) = \frac{1}{2}a(v,v) - \text{Re}L(v)$.

Show that $u \in V$ is solution of (V) iff $u \in V$ is solution of (M).

**Note:** Over the complex space $C$, with the above notations, we say that $a : V \times V \to C$ is a sesquilinear form if $a$ is linear in the first argument (under the usual addition and scalar multiplication) and antilinear in the second argument (usual addition but $a(u,\lambda v) = \overline{\lambda}a(u,v)$). Note that if $\text{Re}a(v,v) \geq \alpha\|v\|^2$, together with $a$ and $L$ bounded, then the Lax-Milgram Lemma holds in the complex case with the same proof.