HW #3, 269C, Vese
Due on Monday, May 17

1. (a) Give a weak variational formulation of the problem
\[
\frac{d^4 u}{dx^4} = f \quad \text{for} \quad 0 < x < 1,
\]
\[ u(0) = u''(0) = u'(1) = u'''(1) = 0, \]
and show that the conditions (i)-(iv) are satisfied. Which boundary conditions are essential and which are natural?

(b) Solve the same problem with the following alternative boundary conditions:
\[ u(0) = -u''(0) + \gamma u'(0) = 0, \quad u(1) = u''(1) + \gamma u'(1) = 0, \]
where \( \gamma \) is a positive constant.

2. Give a weak variational formulation of the Neumann problem
\[ -\Delta u + b(x)u = f \quad \text{in} \quad \Omega, \]
\[ \frac{\partial u}{\partial n} = g \quad \text{on} \quad \Gamma, \]
with the following assumptions on the functions \( b, f \) and \( g \):
\[ b \in L^\infty(\Omega), \quad b \geq b_0 > 0 \text{ a.e. on } \Omega, \quad f \in L^2(\Omega), \quad g \in L^2(\Gamma), \]
for some constant \( b_0 \). Check if the conditions (i)-(iv) are satisfied.

3. Give a weak variational formulation of the Robin’s problem
\[ -\Delta u = f \quad \text{in} \quad \Omega, \quad \gamma u + \frac{\partial u}{\partial n} = g \quad \text{on} \quad \Gamma, \]
where \( \gamma \) is a constant. When are conditions (i)-(iv) satisfied?

4. Consider the Neumann problem
\[ -\Delta u = f \quad \text{in} \quad \Omega, \]
\[ \frac{\partial u}{\partial n} = g \quad \text{on} \quad \Gamma, \]
\[ \int_\Omega u(x)dx = 0. \]

(a) Why condition “\( \int_\Omega u(x)dx = 0 \)” was added here?

(b) Give a variational formulation of the problem, and prove that the conditions (i)-(iv) are satisfied, under the “usual” assumptions on \( f \) and \( g \).
5. Consider the inhomogeneous Dirichlet problem

\[-\triangle u + b(x)u = f \text{ in } \Omega, \quad u = u_0 \text{ on } \Gamma,\]

where \( b \in L^\infty(\Omega), \ b(x) \geq 0 \text{ a.e. in } \Omega, \ f \in L^2(\Omega), \) and \( u_0|_\Gamma \) is the trace of a \( u_0 \in H^1(\Omega) \). To solve this problem by Lax-Milgram, modify it into a homogeneous Dirichlet problem, and analyze the resulting weak variational problem.