HW#2, 269C, Spring 2004, Vese
Due on Friday, April 30

[1] Find the linear basis functions for the triangle $K$ with vertices at $(0,0)$, $(h,0)$ and $(0,h)$. Show that the corresponding element stiffness matrix is given by

$$
\begin{bmatrix}
1 & -\frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & 0 \\
-\frac{1}{2} & 0 & \frac{1}{2}
\end{bmatrix}
$$

Using this result show that the linear system (1.25) of Example 1.1. has the stated form (there is a typo in the textbook regarding the linear system).

[2] Let $\Omega = \{ x \in \mathbb{R}^2 : |x| \leq 1 \}$. Show that the function $v(x) = |x|^\alpha$ belongs to $H^1(\Omega)$ if $\alpha > 0$.

[3] Consider the problem with an inhomogeneous boundary condition,

$$
\begin{cases}
-\Delta u = f \text{ in } \Omega, \\
u = u_0 \text{ on } \Gamma = \partial\Omega,
\end{cases}
$$

where $f$ and $u_0$ are given. Show that this problem can be given the following equivalent variational formulations:

(V) Find $u \in V(u_0)$ such that $a(u,v) = (f,v)$, $\forall v \in H^1_0(\Omega)$,

(M) Find $u \in V(u_0)$ such that $F(u) \leq F(v)$, $\forall v \in V(u_0)$,

where $V(u_0) = \{ v \in H^1(\Omega) : v = u_0 \text{ on } \Gamma \}$.

Then formulate a finite element method and prove an error estimate.

[4] Let $\Omega$ be a bounded domain in the plane and let the boundary $\Gamma$ of $\Omega$ be divided into two parts $\Gamma_1$ and $\Gamma_2$. Give a variational formulation of the following problem:

$$
-\Delta u + u = f \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} = g \quad \text{on } \Gamma_1, \\
u = u_0 \quad \text{on } \Gamma_2,
$$

where $f$, $u_0$ and $g$ are given functions. Then formulate a FEM for this problem.