MATH 115A/3, Spring 2005, Midterm #2
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NAME ____

STUDENT ID # ____________________________________

This is a closed-book and closed-note examination.
Calculators are not allowed.
Please show all your work.
Partial credit will be given to partial answers.
Use only the paper provided. You may write on the back if you need more space, but indicate this clearly on the front.
There are 6 questions of total 100 points.
Time: 1 hour.

QUESTION	SCORE
[1]	17
[2]	17
[3]	17
[4]	17
[5]	17
[6]	17
TOTAL	102

[1] (a) Let $T : V \to V$ be a linear operator on a vector space V, and let u be an eigenvector of T corresponding to the eigenvalue $\lambda \in F$. For any positive integer m, prove that u is an eigenvector of T^m corresponding to the eigenvalue λ^m.

Solution: We reason by mathematical induction.

For $m = 1$ this is obviously true, since $\lambda = \lambda^1$ is an eigenvalue of $T = T^1$ with u eigenvector. We have $u \neq 0_V$ and $T(u) = \lambda u$, by the definition of eigenvalues and eigenvectors.

For $m = 2$ (not necessarily needed, but as an example)

$$T^2(u) = T(T(u)) = T(\lambda u) = \lambda T(u) = \lambda\lambda u = \lambda^2 u,$$

since T is also linear; therefore λ^2 is an eigenvalue of T^2 and $u \neq 0_V$ is an associated eigenvector.

Assume the statement true for $m-1$, then $T^{m-1}(u) = \lambda^{m-1}u$, with $u \neq 0_V$.

We will prove it for m:

$$T^m(u) = T(T^{m-1}(u)) = T(\lambda^{m-1}u) = \lambda^{m-1}T(u) = \lambda^{m-1}\lambda u = \lambda^m u$$

(since T is also linear), with $u \neq 0_V$. Therefore λ^m is an eigenvalue of T^m and u is an associated eigenvector.

By the mathematical induction, the property is true for any positive integer m.

(b) Let $T : V \to V$ be now linear and invertible, and let $\lambda \in F$ be an eigenvalue of T. Show that $\lambda \neq 0_F$ and also show that λ^{-1} is an eigenvalue for the inverse T^{-1}.

Solution:

• Since T is invertible, then $N(T) = \{0_V\}$. Assume by contradiction that $\lambda = 0_F$ is eigenvalue of T; then there is a vector $u \in V$, $u \neq 0_V$ such that

$$T(u) = \lambda u = 0_F u = 0_V.$$

But this contradicts $N(T) = \{0_V\}$. In conclusion, $\lambda = 0_F$ cannot be an eigenvalue of T.

• If λ is eigenvalue of T, we know that $\lambda \neq 0_F$, and therefore λ^{-1} exists. We have that there is an eigenvector $u \in V$, $u \neq 0_V$ such that $T(u) = \lambda u$. Then $T^{-1}(T(u)) = T^{-1}(\lambda u)$. This implies $u = \lambda T^{-1}(u)$ since T^{-1} is linear, i.e. $\lambda^{-1}u = T^{-1}(u)$, and $u \neq 0_V$. In conclusion, λ^{-1} is eigenvalue of T^{-1}.

[2] Let $T : R^2 \to R^3$ be defined by $T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. Let β be the standard ordered basis for R^2 and $\gamma = \{(1,1,0), (0,1,1), (2,2,3)\}$. Find $[T]_\beta^\gamma$.

Solution:

Let $\beta = \{(1,0), (0,1)\}$, and $\gamma = \{(1,1,0), (0,1,1), (2,2,3)\} = \{f_1, f_2, f_3\}$.
$T(1,0) = (1,1,2) = -\frac{1}{3}(1,1,0) + \frac{2}{3}(1,1,2) = -\frac{1}{3}f_1 + 0f_2 + \frac{2}{3}f_3$,
$T(0,1) = (-1,0,1) = (0,1,1) - (1,1,0) = -f_1 + f_2 + 0f_3$.
Then

$$[T]_\beta^\gamma = \begin{pmatrix} -\frac{1}{3} & -1 \\ 0 & 1 \\ \frac{2}{3} & 0 \end{pmatrix}.$$

[3] Let V, W and Z be vector spaces, and let $T : V \to W$, and $U : W \to Z$ be linear.

(a) Prove that if UT is one-to-one, then T is one-to-one.

(b) Prove that if UT is onto, then U is onto.

Solution:

(a) Assume by contradiction that T is not one-to-one. Then there are two distinct vectors $x_1, x_2 \in V$, with $x_1 \neq x_2$ such that $T(x_1) = T(x_2)$. Then $U(T(x_1)) = U(T(x_2))$, or $UT(x_1) = UT(x_2)$, contradiction, since UT is one-to-one. Therefore, T is also one-to-one.

Another similar proof for (a) is: assume by contradiction that T is not one-to-one. Then $N(T) \neq \{0_V\}$, therefore there is $x \in N(T)$ with $x \neq 0_V$ and $T(x) = 0_W$. This implies $U(T(x)) = U(0_W) = 0_Z$, or $UT(x) = 0_Z$. Then $x \neq 0_V$ and $x \in N(UT)$, i.e. $N(UT) \neq \{0_V\}$, therefore UT is not one-to-one, contradiction.

(b) Let $y \in Z$ be arbitrary. Since $UT : V \to Z$ is onto, there is $x \in V$ such that $UT(x) = y$, or $U(T(x)) = y$. This implies that $y \in R(U)$, since $y = U(w)$ with $w = T(x)$. Therefore, U is onto.

[4] For the following linear transformation T, determine whether T is invertible and justify your answer:

$$T : \mathcal{M}_{2\times 2}(R) \to \mathcal{M}_{2\times 2}(R), \quad T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix}.$$

Solution: We first compute $N(T)$. A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in N(T)$ if

$$T(A) = \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $a + b = 0$, $a = 0$, $c = 0$, $c + d = 0$, or $a = b = c = d = 0$. Therefore $N(T) = \{O_{\mathcal{M}_{2\times 2}(R)}\}$. This implies that T is one-to-one. By the dimension theorem, $dimR(T) = dim(\mathcal{M}_{2\times 2}(R)) - dim(N(T)) = 4 - 0 = 4$, but 4 is also the dimension of the target space $\mathcal{M}_{2\times 2}(R)$. This shows that $R(T) = \mathcal{M}_{2\times 2}(R)$, and T is then onto.

In conclusion, T is one-to-one and onto, therefore T is invertible.

Another solution is to see if we can find T^{-1} directly: for any a', b', c', d', find a, b, c, d unique (if possible) such that

$$T(A) = \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}.$$

The system in a, b, c, d can be solved and has a unique solution: $a = b'$, $b = a' - b'$, $c = c'$, $d = d' - c'$. With these, we obtain that

$$T^{-1}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} b' & a' - b' \\ c' & d' - c' \end{pmatrix},$$

and it can be verified that $TT^{-1} = T^{-1}T = I$.

A third solution is to compute $[T]_\beta$, where β is the standard ordered basis of $\mathcal{M}_{2\times 2}(R)$, and to see if this 4x4 matrix $[T]_\beta$ is invertible.

Let $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ (see notations from the textbook), with $(E_{i,j})_{k,l} = 1$ if $(i,j) = (k,l)$ and $(E_{i,j})_{k,l} = 0$ if $(i,j) \neq (k,l)$, where $i, j, k, l = 1, 2$.

$T(E_{11}) = 1E_{11} + 1E_{12} + 0E_{21} + 0E_{22}$,
$T(E_{12}) = 1E_{11} + 0E_{12} + 0E_{21} + 0E_{22}$,
$T(E_{21}) = 0E_{11} + 0E_{12} + 1E_{21} + 1E_{22}$,
$T(E_{22}) = 0E_{11} + 0E_{12} + 0E_{21} + 1E_{22}$.

Then $[T]_\beta = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$.

The determinant of $[T]_\beta = -1 \neq 0$, therefore the 4x4 matrix $[T]_\beta$ is invertible. This implies that the transformation T is also invertible.

[**5**] Let V and W be n-dimensional vector spaces, and let $T : V \to W$ be a linear transformation. Suppose β is a basis for V. Prove that T is an isomorphism if and only if $T(\beta)$ is a basis for W.

(recall that $T : V \to W$ is an isomorphism if it is linear, one-to-one and onto).

Solution:

• Assume T is an isomorphism.

Let $\beta = \{v_1, v_2, ..., v_n\}$ basis of V, $T(\beta) = \{T(v_1), T(v_2), ..., T(v_n)\}$. We show that $T(\beta)$=l.i.

Indeed, let scalars $a_i \in F$ be such that

$$a_1 T(v_1) + ... + a_n T(v_n) = 0_W.$$

By linearity of T, we obtain $T(a_1v_1 + ... + a_nv_n) = 0_W$. This implies that $a_1v_1 + ... + a_nv_n \in N(T)$. Since T is one-to-one, then $N(T) = \{0_V\}$, therefore $a_1v_1 + ... + a_nv_n = 0_V$. But β=l.i. (because it is a basis), therefore $a_1 = a_2 = ... = a_n = 0_F$. Therefore $T(\beta)$=l.i., and since $T(\beta)$ contains exactly n distinct vectors, with $n = dim(W)$, then $T(\beta)$ is a basis of W.

Second proof: We know that T is linear and β is a basis, then $R(T) = Span\{T(\beta)\}$. But $R(T) = W$ since T is onto. Therefore $Span\{T(\beta)\} = W$, in other words $T(\beta)$ is a generator for W. But $T(\beta)$ has at most n distinct elements, and $dim(W) = n$, therefore $T(\beta)$ must have exactly n elements and is therefore also a basis of W.

• Assume $T(\beta)$ is a basis for W.

This implies that T is onto, since $W = Span(T(\beta)) \subset R(T) \subset W$, therefore $R(T) = W$. By the Dimension Thm., we deduce that $dim(N(T)) = dim(V) - dim(R(T)) = dim(V) - dim(W) = 0$, therefore $N(T) = \{0_V\}$. This implies that T is also one-to-one.

In conclusion, T is linear, one-to-one and onto, therefore T is an isomorphism.

Second proof: assume that $v \in N(T)$. Then $T(v) = 0_W$. Since β is a basis of V, then v can be expressed as a linear combination of v_i: $v = a_1v_1 + a_2v_2 + ... + a_nv_n$ for some scalars $a_i \in F$. Since T is linear, we have $0_W = T(v) = T(a_1v_1 + a_2v_2 + ... + a_nv_n) = a_1T(v_1) + a_2T(v_2) + ... + a_nT(v_n)$. But $T(\beta) = \{T(v_1), ..., T(v_n)\}$ is l.i. since it is a basis, therefore $a_1 = a_2 = ... = a_n = 0$. In this case, $v = a_1v_1 + a_2v_2 + ... + a_nv_n = 0_V$. This implies that $N(T) = \{0_V\}$, then T is one-to-one. Since the dimensions of V and W are the same, by the Dim. Thm., this implies that T is also onto. Therefore, T is invertible. T linear and invertible, implies T is an isomorphism.

Third proof: let $T(v_i) = w_i$, $i = 1, 2, ..., n$. Both $\beta = \{v_1, ..., v_n\}$ and $T(\beta) = \{w_1, ..., w_n\}$ are ordered bases of V and W respectively, with $T(v_i) = w_i$. This shows that the matrix $[T]_\beta^{T(\beta)} = I_n$ is the identity matrix, which is an invertible matrix. Therefore, the operator T is also invertible.

[6] Find the eigenvalues of $T : V \to V$ and an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix, where $T : R^2 \to R^2$, $V = R^2$,

$$T(a, b) = (-2a + 3b, -10a + 9b)$$

is linear.

Solution: Let $\gamma = \{(1, 0), (0, 1)\}$ be the standard ordered basis of R^2, and we compute $[T]_\gamma$.

$T(1, 0) = (-2, -10)$, $T(0, 1) = (3, 9)$. Then $A = [T]_\gamma = \begin{pmatrix} -2 & 3 \\ -10 & 9 \end{pmatrix}$.

We compute the eigenvalues of A: $det(A - \lambda I) = det \begin{pmatrix} -2-\lambda & 3 \\ -10 & 9-\lambda \end{pmatrix} = (-2-\lambda)(9-\lambda) + 30 = \lambda^2 - 7\lambda + 12 = 0$ iff $\lambda_1 = 4$, $\lambda_2 = 3$. These are also the eigenvalues of the operator T.

To compute the eigenvectors of A:

For $\lambda_1 = 4$: if $A(x_1\ x_2)^t = \lambda_1(x_1\ x_2)^t$ iff $x_2 = 2x_1$, or $(x_1, x_2) = (1, 2)$.

For $\lambda_2 = 3$: if $A(x_1\ x_2)^t = \lambda_2(x_1\ x_2)^t$, iff $3x_2 = 5x_1$, or $(x_1, x_2) = (3, 5)$.

Here $V = F^2 = R^2$, therefore $u_1 = (1, 2)$ and $u_2 = (3, 5)$ are also the eigenvectors of T.

Indeed, $T(u_1) = T(1, 2) = (4, 8) = 4(1, 2) = \lambda_1 u_1$, and
$T(u_2) = T(3, 5) = (9, 15) = 3(3, 5) = \lambda_2 u_2$.

Note that $\beta = \{u_1, u_2\}$=l.i. (always true if these are eigenvectors corresponding to distinct eigenvalues), therefore β is a basis of R^2 of eigenvectors of T, and moreover $[T]_\beta = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$ is a diagonal matrix.

Mistakes to avoid in the future:

- The following quantities and notions: $[T]_\beta$, $[T]_\beta^\gamma$, the Dimension Thm., or $det([T]_\beta - \lambda I)$ can be used only for finite-dimensional spaces V and W.
- The matrix $[T]_\beta = [T]_\beta^\beta$ can be used only if $T : V \to V$ (and not for $T : V \to W$).
- When T is a linear transformation from one space of matrices into another space of matrices: $T(A) = B$ with $A, B = matrices$, then to see if T is invertible we do not look at the invertibility of B. The matrix $[T]_\beta$ does not coincide with B. We do not compute B^{-1} to find the inverse of the operator T^{-1}. $[T]_\beta^{-1}$ can be used to see if T is invertible.
- Eigenvectors are always vectors different from the zero vector.
- Any element in a basis cannot be the zero vector.