MATH 115A/3, Fall 2007, Midterm #1
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QUESTION	SCORE
[1]	
[2]	
[3]	
[4]	
[5]	
TOTAL	

- Justify your answers.
- Calculators are not allowed.
- This is a closed-book and closed-note exam.

[**1**] Consider the vector space $V = F^5$, over a field F. Show that

$$W = \Big\{(a_1, a_2, a_3, a_4, a_5) \in F^5 : \quad a_2 = a_3 = a_4 \text{ and } a_1 + a_5 = 0\Big\}$$

is a subspace of V. Find its dimension and give an example of a basis.

Justify your answers.

Solution: To show that W is a subspace:

• $0_V = (0, 0, 0, 0, 0) \in W$ since obviously it satisfies $a_2 = a_3 = a_4 (= 0)$ and $a_1 + a_5 = 0 + 0 = 0$.

• Let arbitrary $u = (a_1, a_2, a_3, a_4, a_5) = (a_1, a_2, a_2, a_2, -a_1) \in W$ and $v = (b_1, b_2, b_3, b_4, b_5) = (b_1, b_2, b_2, b_2, -b_1) \in W$, since $a_2 = a_3 = a_4$, $a_1 + a_5 = 0$, and $b_2 = b_3 = b_4$, $b_1 = b_5$.

Then $u + v = (a_1 + b_1, a_2 + b_2, a_2 + b_2, a_2 + b_2, -(a_1 + b_2)$ and clearly $u + v \in W$.

• Let $c \in F$ arbitrary and $u \in W$ as above. Then $cu = (ca_1, ca_2, ca_3, ca_4, ca_5) = (ca_1, ca_2, ca_2, ca_2, c(-a_1)) = (ca_1, ca_2, ca_2, ca_2, -ca_1) \in W$ obviously.

Thus W contains the zero element from V, and it is closed over the vector addition and scalar multiplication. Therefore, by a thm. from the course, W is a subspace of V.

Any vector $u \in W$ can be expressed as $u = (a_1, a_2, a_3, a_4, a_5) = (a_1, a_2, a_2, a_2, -a_1) = a_1(1, 0, 0, 0, -1) + a_2(0, 1, 1, 1, 0)$, with arbitrary $a_1 \in F$, $a_2 \in F$. Therefore the subset of W, $\beta = \{u_1 = (1, 0, 0, 0, -1), u_2 = (0, 1, 1, 1, 0)\}$ is a generator for W, or $Span(\beta) = W$.

In addition, the vectors u_1 and u_2 are linearly independent (can be easily verified), thus by the definition of a basis, β must be a basis for W and we deduce that dim(W)=2.

[**2**] Let $T : \mathcal{M}^{2\times 3}(F) \mapsto \mathcal{M}^{2\times 2}(F)$ be defined by

$$T\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix}.$$

Show that T is a linear transformation, and find bases for both $N(T)$ and $R(T)$. Then compute the nullity and rank of T, and verify the dimension theorem. Determine whether T is one-to-one or onto, using appropriate theorems.

Solution:

To show that T is linear: let arbitrary $c \in F$ and $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$, $A, B \in \mathcal{M}^{2\times 3}(F)$.

$$T(cA + B) = T\begin{pmatrix} ca_{11} + b_{11} & ca_{12} + b_{12} & ca_{13} + b_{13} \\ ca_{21} + b_{21} & ca_{22} + b_{22} & ca_{23} + b_{23} \end{pmatrix}$$

$$= \begin{pmatrix} 2(ca_{11} + b_{11}) - (ca_{12} + b_{12}) & (ca_{13} + b_{13}) + 2(ca_{12} + b_{12}) \\ 0 & 0 \end{pmatrix}$$

$$= c\begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2b_{11} - b_{12} & b_{13} + 2b_{12} \\ 0 & 0 \end{pmatrix} = cT(A) + T(B),$$

thus T is linear (no need to verify that $T(O) = O$).

Let $A \in N(T)$ as above, this means $T(A) = O$ or

$2a_{11} - a_{12} = 0,$

$a_{13} + 2a_{12} = 0$.

Therefore $a_{12} = 2a_{11}$, $a_{13} = -4a_{11}$. Thus $A \in N(T)$ iff $A = \begin{pmatrix} a_{11} & 2a_{11} & -4a_{11} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = a_{11}\begin{pmatrix} 1 & 2 & -4 \\ 0 & 0 & 0 \end{pmatrix} + a_{21}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + a_{22}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + a_{23}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Therefore the set $\left\{\begin{pmatrix} 1 & 2 & -4 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$ is a generator of $N(T)$. Moreover, the four matrices of this set are linearly independent (easy to see), thus this set forms a basis for $N(T)$ and dim N(T)=4.

A basis for $R(T)$ clearly is $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$ (lin. indep. and generator of $R(T)$). Thus dim R(T)=2.

We can verify Dim. Thm. which states dim(V)=dim(N(T))+dim(R(T)), or here 6=4+2, thus satisfied.

Finally, T is not 1-to-1 because dim N(T)=4 $\neq 0$. Also, T is not onto because dim R(T)=2 $\neq$ 4=dim $\mathcal{M}^{2\times 2}(F)$, thus $R(T) \subset \mathcal{M}^{2\times 2}(F)$ with strict inclusion.

[**3**] Let u, v and w be distinct vectors of a vector space V. Show that if $\{u, v, w\}$ is a basis for V, then $\{u + v + w, v + w, w\}$ is also a basis for V.

Solution: Clearly dim V=3. By a theorem from the course, it is sufficient to show that the set $\{u + v + w, v + w, w\}$ is linearly independent, since it contains exactly three elements and the dim. of $V = 3$.

Assume that

$$\alpha_1(u + v + w) + \alpha_2(v + w) + \alpha_3 w = 0_V \tag{1}$$

for some scalars α_1, α_2, $\alpha_3 \in F$.

Rearranging the terms, we deduce $\alpha_1 u + (\alpha_1 + \alpha_2)v + (\alpha_1 + \alpha_2 + \alpha_3)w = 0_V$. Since $\{u, v, w\}$ is linearly independent (being a basis of V), we deduce that

$\alpha_1 = 0_F$

$\alpha_1 + \alpha_2 = 0_F$

$\alpha_1 + \alpha_2 + \alpha_3 = 0_F$.

Therefore, from the first relation we get $\alpha_1 = 0$, using this in the second we get $0+\alpha_2 = 0$ thus $\alpha_2 = 0$. With $\alpha_1 = \alpha_2 = 0$ in the third relation, we get $0+0+\alpha_3 = 0$, therefore also $\alpha_3 = 0$.

In conclusion, since (1) implies $\alpha_1 = \alpha_2 = \alpha_3 = 0$, we deduce that $\{u+v+w, v+w, w\}$ is linearly independent, and this concludes the proof.

[**4**] Let V and W be vector spaces and $T : V \mapsto W$ be linear.

(a) Suppose that T is one-to-one and that $S = \{v_1, ..., v_k\}$ is a subset of V. Prove that S is linearly independent if and only if $T(S) = \{T(v_1), ..., T(v_k)\}$ is linearly independent.

(b) Suppose $\beta = \{v_1, ..., v_n\}$ is a basis for V and T is one-to-one and onto. Prove that $T(\beta) = \{T(v_1), ..., T(v_n)\}$ is a basis for W.

Solution:

(a) ($\Rightarrow$) Assume S linearly independent.

Let $a_1, ..., a_k \in F$ such that

$$a_1 T(v_1) + a_2 T(v_2) + ... + a_k T(v_k) = 0_W. \tag{2}$$

Since T is linear, this is equivalent with

$$T(a_1v_1 + a_2v_2 + ... + a_kv_k) = 0_W.$$

Moreover, since T is 1-to-1, we know that $N(T) = \{0\}$ (only 0_V goes to 0_W through T), thus we obtain $a_1v_1 + a_2v_2 + ... + a_kv_k = 0_V$. Now, since S is linearly independent, we must have $a_1 = a_2 = ... = a_k = 0$.

In conclusion, since (2) implies $a_1 = a_2 = ... = a_k = 0_F$, the set $T(S)$ must also be linearly independent.

($\Leftarrow$) Assume that $T(S)$ is lin. ind.

Let $a_1, ..., a_k \in F$ such that

$$a_1v_1 + a_2v_2 + ... + a_kv_k = 0_V. \tag{3}$$

Applying T to both sides, and using $T(0_V) = 0_W$, we get

$$T(a_1v_1 + a_2v_2 + ... + a_kv_k) = T(0_V) = 0_W,$$

or, by linearity

$$a_1T(v_1) + a_2T(v_2) + ... + a_kT(v_k) = 0_W.$$

Now, since $T(S)$ is linearly independent, we must have $a_1 = a_2 = ... = a_k = 0_F$.

In conclusion, since (3) implies $a_1 = a_2 = ... = a_k = 0_F$, the set S must also be linearly independent.

[**5**] (a) Construct a linear transformation $T : R^2 \mapsto R^3$ such that $T(1,1) = (1,0,2)$ and $T(2,3) = (1,-1,4)$. What is $T(8,11)$? Is T onto ? Explain.

(b) Is there a linear transformation $T : R^3 \mapsto R^2$ such that $T(1,0,3) = (1,1)$ and $T(-2,0,-6) = (2,1)$? Explain.

Solution:

(a) Note that $\{u_1 = (1,1), u_2 = (2,3)\}$ is linearly independent, therefore a basis of R^2 (easy to verify), thus such transformation exist and is unique.

Any vector $x = (x_1, x_2) = (3x_1 - 2x_2)(1,1) + (x_2 - x_1)(2,3)$.

Therefore we define T by $T(x) = (3x_1 - 2x_2)T(1,1) + (x_2 - x_1)T(2,3) = (3x_1 - 2x_2)(1,0,2) + (x_2 - x_1)(1,-1,4)$. We obtain

$T(x_1, x_2) = (2x_1 - x_2, x_1 - x_2, 2x_1)$. By a theorem from the course, T is linear (or this can be verified directly).

Thus $T(8,11) = (5,-3,16)$.

T cannot be onto: since by dim. thm. dim(N(T))+dim(R(T))=2, therefore $dim(R(T)) \leq 2 \neq 3$, thus $R(T) \neq R^3$ and T is not onto.

(b) Such T does not exist, because it would not satisfy the definition of a linear transformation: indeed,

$T(-2,0,-6) = T(-2(1,0,3)) = (2,1) \neq -2T(1,0,3) = (-2,-2)$ thus does not satisfy $T(cu) = cT(u)$ for scalar $c = -2$ and vector $(1,0,3)$.

Many students gave the incorrect answer: "T is not linear because $(1,0,3)$ and (-2,0,-6) are not lin. indep." (why incorrect ? the converse of Thm. 2.6 does not hold, or by other reasons: linear transformations are defined for any vectors of V, linear dep. or lin. indep.)