Solutions to selected exercises from Section 6.1

#11, Section 6.1, Page 337

By the definition of the norm, of an inner product, and Thm. 6.1, we have

$$\|x+y\|^2 + \|x-y\|^2 = < x+y, x+y > + < x-y, x-y >$$

$$= < x, x+y > + < y, x+y > + < x, x-y > - < y, x-y >$$

$$= < x,x > + < x,y > + < y,x > + < y,y > + < x,x > - < x,y >$$

$$- < y,x > + < y,y > = 2\|x\|^2 + 2\|y\|^2.$$

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It is sufficient to show that $N(T) = \{0_V\}$. Let $x \in N(T)$. Then $T(x) = 0_V$. But $\|T(x)\| = \|x\|$ and $\|T(x)\| = \|0_V\| = 0$, therefore $\|x\| = 0$. Property (b) from Thm. 6.2 or property (d) from Thm. 6.1. implies $x = 0_V$. Therefore $N(T) = \{0_V\}$, and T is one-to-one.

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Assume $< x,y >' = < T(x), T(y) >$ is an inner product on $V \times V$. We will show that T is one-to-one. Let $x \in N(T)$, then $T(x) = 0_W$. Therefore $< x,x >' = < T(x), T(x) > = < 0_W, 0_W > = 0$. Due to Thm. 6.1 (d) applied to $< \cdot, \cdot >'$, this implies that $x = 0_V$, and then $N(T) = \{0_W\}$, i.e. T is one-to-one.

Assume now that T is one-to-one, therefore $N(T) = \{0_W\}$. We need to show (a)-(d) from Definition Pages 329-330; for any $x, y, z \in V$, any $c \in F$:

(a) $< x+z, y >' = < T(x+z), T(y) > = < T(x) + T(z), T(y) >$
$= < T(x), T(y) > + < T(z), T(y) > = < x, y >' + < z, y >'$

(using the linearity of T and the fact that $< \cdot, \cdot >$ is an inner product on $W \times W$).

(b) $< cx, y >' = < T(cx), T(y) > = < cT(x), T(y) > = c < T(x), T(y) >$
$= c < x, y >'$ (again by the linearity of T and the fact that $< \cdot, \cdot >$ is an inner product on $W \times W$).

(c) $\overline{< x,y >'} = \overline{< T(x), T(y) >} = < T(y), T(x) > = < y, x >'$.

(d) Assume $x \neq 0_V$, then $T(x) \neq 0_W$ since T is one-to-one.
$< x,x >' = < T(x), T(x) > > 0$.

Therefore, $< \cdot, \cdot >'$ is an inner product on $V \times V$.

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Let $A = (a_{ij})_{1\le i,j\le n}$, $A^*_{i,j} = \overline{a_{ji}}$, and $x = (a_1, ..., a_n) \in F^n$ and $y = (b_1, ..., b_n) \in F^n$.

(a) $< x, Ay >=< x, \left(\sum_{j=1}^n a_{1j}y_j, \sum_{j=1}^n a_{2j}y_j, ..., \sum_{j=1}^n a_{nj}y_j \right) >$
$= \sum_{i=1}^n \left[x_i \overline{\sum_{j=1}^n a_{ij}y_j} \right] = \sum_{i=1}^n \left[x_i \sum_{j=1}^n \overline{a_{ij}}\ \overline{y_j} \right] = \sum_{i=1}^n \sum_{j=1}^n \left[x_i \overline{a_{ij}}\ \overline{y_j} \right]$
$= \sum_{i=1}^n \sum_{j=1}^n \left[x_j \overline{a_{ji}}\ \overline{y_i} \right] = \sum_{i=1}^n \left(\sum_{j=1}^n \left[x_j \overline{a_{ji}} \right] \right) \left(\overline{y_i} \right)$
$= \sum_{i=1}^n \left(\sum_{j=1}^n \left[x_j A^*_{ij} \right] \right) \left(\overline{y_i} \right) =< A^*x, y >.$

(b) From (a) we have $< x, Ay >=< A^*x, y >$, therefore $< A^*x, y >=< Bx, y >$ for any $x, y \in V$. Property (e) from Thm. 6.1. implies $A^*x = Bx$ for any $x \in V$. Choosing $x = e_1$,...,e_n, with e_i the elements of the standard basis, then $A^*e_j = Be_j$ implies that the $j - th$ column of A^* is equal with the $j - th$ column of B, for all $j = 1, 2, ..., n$. Therefore $A^* = B$.

(c) Let $\alpha = \{e_1, e_2, ..., e_n\}$ be the standard ordered basis of V, and $\beta = \{v_1, v_2, ..., v_n\}$ be an orthornomal basis of V. Assume $v_j = (v_{1j}, ..., v_{ij}, ..., v_{nj})$, for $j = 1, 2, ..., n$.

We have $Q = [I_V]_\beta^\alpha = (v_{ij})_{1\le i,j\le n}$ is the change-of-coordinate matrix from β to α. We also know that $Q^{-1} = [I_V]_\alpha^\beta$.

Assume $e_j = a_1v_1 + a_2v_2 + ... + a_nv_n$, with $a_i \in F$. Then $(a_1, a_2, ..., a_n)^t$ is the $j - th$ column of the matrix Q^{-1}.

Fix $j = 1, 2, ..., n$. For each $i = 1, 2, ..., n$, we have $< e_j, v_i >=< a_1v_1 + a_2v_2 + ... + a_nv_n, v_i >= a_1 < v_1, v_i > +a_2 < v_2, v_i > +... + a_n < v_n, v_i >= a_i < v_i, v_i >= a_i$, because β is an orthornomal basis.

But $a_i =< e_j, v_i >= \overline{v_{ji}}$, $i = 1, 2, ..., n$, therefore the $j - th$ column of Q^{-1} is $\overline{v_{ji}}$, $i = 1, 2, ..., n$, in other words $Q^{-1} = Q^*$.

(d) Assume $\beta = \{v_1, ..., v_n\}$ is an orthonormal basis of V. Let $[U]_\beta = (u_{ij})_{1\le i,j\le n}$ and $[T]_\beta = (t_{ij})_{1\le i,j\le n}$.

Then $U(v_j) = A^*v_j = u_{1j}v_1 + u_{2j}v_2 + ... + u_{nj}v_n$. Then $< A^*v_j, v_i >= u_{ij} < v_i, v_i >= u_{ij}$.

Similarly, $T(v_j) = Av_j = t_{1j}v_1 + t_{2j}v_2 + ... + t_{nj}v_n$. Then $< Av_j, v_i >= t_{ij} < v_i, v_i >= t_{ij}$. But using (a), $u_{ij} =< A^*v_j, v_i >=< v_j, Av_i >= \overline{< Av_i, v_j >} = \overline{t_{ji}}$, therefore $u_{ij} = \overline{t_{ji}}$, or $[U]_\beta = ([T]_\beta)^*$.

We have used the facts that $< v_i, v_j >= 0$ if $i \ne j$ and $< v_i, v_i >= 1$.