**Solutions to selected exercises from the textbook**

**Exercise 2/2.1:** $T : R^3 \to R^2$ defined by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$.

To find $N(T)$ and a basis, let $a_1, a_2, a_3$ be s.t. $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3) = (0, 0)$. Then

$$\begin{cases} a_1 - a_2 = 0 \\ 2a_3 = 0 \end{cases}$$

or $a_1 = a_2$, $a_3 = 0$. Therefore $N(T) = \{(a, a, 0),\ a \in R\}$ and $\dim(N(T)) = 1$, a basis of $N(T)$ is given by $\{(1, 1, 0)\}$, therefore $nullity(T) = 1$. We deduce (by Thm. 2.4) that $T$ is not one-to-one, because $N(T) \neq \{0\}$.

We could have obtained this statement directly by the definition: note that there are distinct vectors $\vec{a}, \vec{b}$ in $R^3$ such that $T(\vec{a}) = T(\vec{b})$ with $\vec{a} \neq \vec{b}$. Indeed $T(2, 2, 1) = T(1, 1, 1) = (0, 2)$, therefore $T$ is not one-to-one.

By the Dimension Theorem, we obtained that $dim(R(T)) = 3 - dim(N(T)) = 2$. This implies that dim. of range of T coincides with dim. of $R^2$, i.e. $R(T) = R^2$, i.e. $T$ is onto.

This could have been done in the following way: let $(x, y) \in R^2$ be arbitrary, and find $(a_1, a_2, a_2)$, if any, such that $T(a_1, a_2, a_3) = (x, y)$. This would imply $a_1 - a_2 = x$, and $2a_3 = y$. We see that $(x, y)$ is always in the image of $T$, by $T(a, a - x, y/2) = (x, y)$. Again, we conclude that $T$ is onto and a basis of $R(T)$ is any basis of $R^2$, for instance of standard basis $\{(1, 0), (0, 1)\}$.

**Exercise 5/2.1:** $T : P_2(R) \to P_3(R)$, $T(f(x)) = xf(x) + f'(x)$.

Note that $T(a_0 + a_1x + a_2x^2) = a_0x + a_1x^2 + a_2x^3 + (a_1 + 2a_2x) = a_1 + (a_0 + 2a_2)x + a_1x^2 + a_2x^3$, with $f(x) = a_0 + a_1x + a_2x^2$.

$f(x) \in N(T)$ if $T(f(x)) = 0$ for any $x$, therefore if $a_1 = 0$, $a_0 + 2a_2 = 0$, $a_1 = 0$, $a_2 = 0$, i.e. $a_0 = a_1 = a_2 = 0$, therefore $N(T) = \{0\}$. We deduce that $T$ is one-to-one, by Thm. 2.4.

By the Dim. Thm, we deduce that $dim(R(T)) = dim(P_2(R)) - dim(N(T)) = dim(P_2(R)) = 3$. However, $dim(P_3(R)) = 4$, therefore $R(T) \neq P_3(R)$ and $T$ is not onto.

From $T(a_0 + a_1x + a_2x^2) = a_1 + (a_0 + 2a_2)x + a_1x^2 + a_2x^3 = a_1(1 + x^2) + a_0x + a_2(2x + x^3)$ we see that a basis for $R(T)$ is given by $\{1 + x^2, x, 2x + x^3\}$ (clearly we see that it is a generator of $\{T(a_0 + a_1x + a_2x^2)\}$ and is l.i.).

**Exercise 14/2.1:**

(a) Assume $T$ one-to-one. Let $S = \{v_1, v_2, ..., v_n\} \subset V$ be l.i., and let $S' = \{T(v_1), T(v_2), ..., T(v_n)\}$.

Assume $a_1T(v_1)+a_2T(v_2)+...+a_nT(v_n) = 0_W$ for some scalars $a_1, ..., a_n \in F$. Then, since $T$ is linear, we have $T(a_1v_1 + a_2v_2 + ...a_nv_n) = 0_W$. But $T$ is also one-to-one, i.e. $N(T) = \{x : T(x) = 0_W\} = \{0_V\}$, therefore $a_1v_1 + a_2v_2 + ...a_nv_n = 0_V$. But $S = \{v_1, v_2, ..., v_n\}$ is l.i., this implies that $a_1 = 0$, $a_2 = 0$, ..., $a_n = 0$. In conclusion, $S' = \{T(v_1), T(v_2), ..., T(v_n)\}$ is l.i.

Converse: assume by contradiction that $T$ is not one-to-one. Then $N(T) \neq \{0_V\}$, therefore there is $v \in N(T)$ with $v \neq 0_V$, with $T(v) = 0_W$. But this is a contradiction, since $\{v\}$ is l.i., while $\{T(v) = 0_W\}$ is l.d. In conclusion, $T$ must be one-to-one.

(b) From left to right: directly by (a). From right to left: by Exercise 13 (was proved in class and at Midterm 1).

(c) By property (a), since $T$ is one-to-one and $\beta$ is l.i. (it is a basis), we deduce that $T(\beta)$ is l.i. Now since $T$ is also onto, and by the Dim. Thm. (also by $dim(N(T)) = 0$) we have that $dim(R(T)) = dim(W)$ and $dim(R(T)) = dim(V) - dim(N(T)) = n$. Therefore $dim(W) = dim(V) = n$ and $S(\beta)$ is therefore a basis of $W$, because $S(\beta)$ is l.i. and contains exactly $n$ distinct vectors.

**Exercise 17/2.1:**

(a) By the Dimension Thm., we have $dim(N(T))+dim(R(T)) = dim(V)$. If $dim(V) < dim(W)$, then $dim(R(T)) \leq dim(V) < dim(W)$, therefore $dim(R(T)) < dim(W)$. This shows that $R(T) \neq W$, i.e. $T$ is not onto.

(b) We apply again Dimension Thm: $dim(N(T))+dim(R(T)) = dim(V)$. We also know $dim(R(T)) \leq dimW$, therefore $dim(V) > dim(W) \geq dim(R(T)) = dim(V) - dim(N(T))$, i.e. $0 > -dim(N(T))$ or $dim(N(T)) > 0$. Therefore, $N(T) \neq \{0_V\}$, and by Thm. 2.4, $T$ is not one-to-one.

Note that these general properties (a) and (b) could have been applied to the linear transformations from Exercises 2 and 5 above.

**Exercise 15/2.2:**

(a) Clearly the zero transformation $T_0 : V \to W$ belongs to $S^0$, because $T_0(x) = 0_W$ ofr any $x \in V$, including any $x \in S$.

If $T_1, T_2 \in S^0$, and if $x \in S$, then $(T_1+T_2)(x) = T_1(x)+T_2(x) = 0_W+0_W = 0_W$, for any $x \in S$, therefore $T_1 + T_2 \in S^0$.

Similarly, if $T \in S^0$ and $c \in F$, then for any $x \in S$: $(cT)(x) = cT(x) = c0_W = 0_W$, therefore $(cT) \in S^0$. In conclusion, $S^0$ is a subspace.

(b) If $T \in S_2^0$, then $T(x) = 0$ for any $x \in S_2$; but $S_1 \subset S_2$, therefore $T(x) = 0$ for any $x \in S_1$, i.e. $T \in S_1^0$.

(c) If $T \in V_1^0 \cap V_2^0$, then $T \in V_1^0$ and $T \in V_2^0$. Therefore $T(x) = 0$ for any $x \in V_1$ and $T(x) = 0$ for any $x \in V_2$. This implies that for any $x = u + v \in V_1 + V_2$ with $u \in V_1$ and $v \in V_2$, then $T(x) = T(u + v) = T(u) + T(v) = 0 + 0 = 0$, therefore $T \in (V_1 + V_2)^0$. In other words, $V_1^0 \cap V_2^0 \subset (V_1 + V_2)^0$.

To show the other inclusion: we have $V_1 \subset V_1 + V_2$ (since $V_2$ is a subspace and $0_V \in V_2$), therefore from (b), $(V_1 + V_2)^0 \subset V_1^0$. Similarly, $V_2 \subset V_1 + V_2$ (since $V_1$ is a subspace and $0_V \in V_1$), therefore again from (b), $(V_1 + V_2)^0 \subset V_2^0$. These last two statements imply $(V_1 + V_2)^0 \subset V_1^0 \cap V_2^0$.

In conclusion: $(V_1 + V_2)^0 = V_1^0 \cap V_2^0$.

**Exercise 3/2.4:** By Thm. 2.19:

(a) not isomorphic, the dimensions are different: 3 and 4.

(b) isomorphic, the dimensions are the same = 4.

(c) isomorphic, the dimensions are the same = 4.

(d) not isomorphic, the dimensions are different: $dim(V) = 3$ and $dimW = 4$.

**Exercise 14/2.4:** Let $T(\begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix}) = (a, b, c)$. Finalize the problem by showing that $T$ is linear, one-to-one and onto.