Math 115a: Selected Solutions for HW 9

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Exercise 6.3.6: Let T be a linear operator on an inner product space V. Let $U_1 = T + T^*$ and $U_2 = TT^*$. Prove that $U_1 = U_1^*$ and $U_2 = U_2^*$.

Solution:

$$U_1 = T + T^* = (T + T^*)^{**} = (T^* + T^{**})^* = (T^* + T)^* = U_1^*,$$

and

$$U_2 = TT^* = (TT^*)^{**} = (T^{**}T^*)^* = (TT^*)^* = U_2^*.$$

Exercise 6.3.12: Let V be an inner product space, and let T be a linear operator on V. Prove the following results. (a) $R(T^*)^\perp = N(T)$. (b) If V is finite-dimensional, then $R(T^*) = N(T)^\perp$.

Solution: (a) $v \in R(T^*)^\perp$ if and only if for all $w \in V$, $\langle v \mid T^*(w)\rangle = 0$. However this is true if and only if for all $w \in V$, $\langle T(v) \mid w\rangle = 0$, and this is true if and only if $T(v) = 0$, i.e. $v \in N(T)$. This completes our proof. (b) By exercise 6.2.13c, we can apply $\perp$ to both sides of part (a) and then we'd be done.

Exercise 6.4.11: Assume that T is a linear operator on a complex (not necessarily finite-dimensional) inner product space V with an adjoint T^*. Prove the following results:

(a) If T is self-adjoint, then $\langle T(x) \mid x\rangle$ is real for all $x \in V$.
(b) If T satisfies $\langle T(x) \mid x\rangle = 0$ for all $x \in V$, then $T = T_0$.
(c) If $\langle T(x) \mid x\rangle$ is real for all $x \in V$, then $T = T^*$.

Solution: (a) Suppose T is self-adjoint. Then

$$\begin{aligned}\langle T(x) \mid x\rangle &= \langle x \mid T(x)\rangle \\ &= \overline{\langle T(x) \mid x\rangle}\end{aligned}$$

and hence $\langle T(x) \mid x\rangle$ is real.

(b)

$$\begin{aligned} 0 &= \langle T(x+y) \mid x+y \rangle \\ &= \langle T(x) \mid x \rangle + \langle T(x) \mid y \rangle + \langle T(y) \mid x \rangle + \langle T(y) \mid y \rangle \\ &= \langle T(x) \mid y \rangle + \langle T(y) \mid x \rangle \end{aligned}$$

and therefore $\langle T(x) \mid y \rangle = -\langle T(y) \mid x \rangle$. Similarly,

$$\begin{aligned} 0 &= \langle T(x+iy) \mid x+iy \rangle \\ &= \langle T(x) \mid x \rangle - i\langle T(x) \mid y \rangle + i\langle T(y) \mid x \rangle + \langle T(y) \mid y \rangle \\ &= -i\langle T(x) \mid y \rangle + i\langle T(y) \mid x \rangle \end{aligned}$$

and therefore $\langle T(x) \mid y \rangle = \langle T(y) \mid x \rangle$. Combining these results we get $\langle T(x) \mid y \rangle = 0$ for all $x, y \in V$. This is enough to conclude that $T = T_0$.

Exercise 6.4.12: ($\rightarrow$) Let T be a normal operator on a finite-dimensional real inner product space V whose characteristic polynomial splits. Prove that V has an orthonormal basis of eigenvectors of T. Hence prove that T is self-adjoint.

Solution: The proof is near identical to that of Theorem 6.16. The only difference is that in Thm. 6.16 they needed to work over a complex inner product space because they needed their characteristic polynomial to split. We have that as an assumption. So the rest of the proof follows word for word. Then by Thm 6.17 it follows that T is self-adjoint.

Exercise 6.4.17a: Let T be a self-adjoint linear operator on an n-dimensional inner product space V. Then T is positive definite [semi-definite] if and only if all of its eigenvalues are positive [semi-positive].

Solution: Let λ be an eigenvalue of T, with corresponding eigenvector v. Then

$$\begin{aligned} \lambda\langle v \mid v \rangle &= \langle T(v) \mid v \rangle \\ &\geq 0 \end{aligned}$$

since v is an eigenvector, it must be nonzero. Therefore $||v||^2 > 0$ and so we can conclude that $\lambda \geq 0$. The same idea goes through for semi-definite. ($\leftarrow$) Suppose that all eigenvalues are positive [semi-positive]. Since T is self-adjoint, by Thm. 6.17, there exists an orthonormal basis of eigenvectors $\beta = \{\beta_1, \beta_2, ..., \beta_n\}$ with corresponding eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$, respectively. Let $x = a_1\beta_1 + \cdots + a_n\beta_n$

be arbitrary. Then

$$\begin{aligned}
\langle T(x) \mid x\rangle &= \langle T\big(\sum_{i=1}^{n} a_i\beta_i\big) \mid \sum_{j=1}^{n} a_j\beta_j\rangle \\
&= \langle\sum_{i=1}^{n} a_i\lambda_i\beta_i \mid \sum_{j=1}^{n} a_j\beta_j\rangle \\
&= \sum_{i,j=1}^{n} \lambda_i a_i\overline{a_j}\langle\beta_i \mid \beta_j\rangle \\
&= \sum_{i=1}^{n} \lambda_i|a_i|^2 \\
&\geq [>] \; 0.
\end{aligned}$$

This completes our proof.