Math 115a: Selected Solutions for HW 8

Paul Young

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Exercise 6.2.6: Let V be an inner product space, and let W be a finite dimensional subspace of V. If $x \notin W$, prove that there exists $y \in V$ such that $y \in W^\perp$, but $\langle x \mid y \rangle \neq 0$.

Solution: By Theorem 6.6, there exists unique vectors $u \in W$ and $y \in W^\perp$ such that $x = u + y$. Then

$$\begin{aligned} \langle x \mid y \rangle &= \langle u + y \mid y \rangle \\ &= \langle u \mid y \rangle + \langle y \mid y \rangle \\ &= ||y||^2 \\ &\geq 0. \end{aligned}$$

Since $x \notin W$, we conclude that $y \neq 0$ and hence $\langle x \mid y \rangle > 0$.

Exercise 6.2.11: Let W be a finite-dimensional subspace of an inner product space V. Prove that $AA^* = I$ if and only if the rows of A form an orthonormal basis for $\mathbb{C}^n$.

Solution: We first notice that

$$(AA^*)_{ij} = \sum_{k=1}^{n} A_{ik}\overline{A_{jk}} = \langle a_i \mid a_j \rangle.$$

With this observation this problem becomes trivial: $AA^* = I$ if and only if $(AA^*)_{ij} = \delta_{ij}$ if and only if $\langle a_i \mid a_j \rangle = \delta_{ij}$ if and only if the rows of A form an orthonormal basis for $\mathbb{C}^n$.

Exercise 6.2.18: Let $V = C([-1, 1])$. Suppose that W_e and W_o denote the subspaces of V consisting of the even and odd functions, respectively. Prove that $W_e^\perp = W_o$, where the inner product on V is defined by

$$\langle f \mid g \rangle = \int_{-1}^{1} f(t)g(t)dt.$$

Solution: Let $f \in W_e$ and $g \in W_0$. If we can show that $\langle f \mid g \rangle = 0$, then we are done.

$$\begin{aligned}
\langle f \mid g \rangle &= \int_{-1}^{1} f(t)g(t)dt \\
&\stackrel{*}{=} \int_{1}^{-1} f(-u)g(-u)d(-u) \\
&= \int_{1}^{-1} f(u)[-g(u)][-du] \\
&= \int_{1}^{-1} f(u)g(u)du \\
&= -\int_{-1}^{1} f(u)d(u)du \\
&= -\langle f \mid g \rangle,
\end{aligned}$$

where $*$ is from the change of variable $t \to -u$. If follows that $\langle f \mid g \rangle = 0$. This completes our proof.