Math 115a: Selected Solutions for HW 4

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Exercise 2.2.2bc: Let β and γ be the standard ordered bases for $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. For each linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, compute $[T]_\beta^\gamma$.

b) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(a_1, a_2, a_3) = (2a_1 + 3a_2 - a_3, a_1 + a_3)$
c) $T : \mathbb{R}^3 \to \mathbb{R}$ defined by $T(a_1, a_2, a_3) = 2a_1 + a_2 - 3a_3$.

Solution:
b)

$$\begin{aligned} T(1,0,0) &= (2,1) \\ T(0,1,0) &= (3,0) \\ T(0,0,1) &= (-1,1) \end{aligned}$$

and therefore

$$[T]_\beta^\gamma = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 0 & 1 \end{pmatrix}$$

c)

$$\begin{aligned} T(1,0,0) &= 2 \\ T(0,1,0) &= 1 \\ T(0,0,1) &= -3 \end{aligned}$$

and therefore

$$[T]_\beta^\gamma = \begin{pmatrix} 2 & 1 & -3 \end{pmatrix}.$$

Exercise 2.2.14: let $V = P(\mathbb{R})$, and for $j \geq 1$ define $T_j(f(x)) = f^{(j)}(x)$, where $f^{(j)}(x)$ is the jth derivative of $f(x)$. Prove that the set $\{T_1, T_2, ..., T_n\}$ is a linearly independent subset of $\mathcal{L}(V)$ for any positive integer n.

Solution: Let n be arbitrary. Let $a_1, a_2, ..., a_n$ be scalars such that

$$a_1T_1 + a_2T_2 + \cdots + a_nT_n = 0.$$

(We note that the above equality is between two linear transformations in $\mathcal{L}(V)$, where the RHS is the zero linear transformation.) This means that whatever we decide to evaluate on both sides should yield equal elements (in $P(\mathbb{R})$). So consider the polynomial x^n. On one hand,

$$\begin{aligned}(a_1T_1 + a_2T_2 + \cdots + a_nT_n)(x^n) &= a_1T_1(x^n) + a_2T_2(x^n) + \cdots a_nT_n(x^n)\\ &= a_1c_1x^{n-1} + a_2c_2x^{n-2} + \cdots + a_nc_n \in P(\mathbb{R}),\end{aligned}$$

where $c_j = \prod_{i=0}^{j-1}(n-i)$. We see that this is a polynomial of degree $n-1$. On the other hand, if we evaluate the right hand side, we get

$$0(x^n) = T_0(x^n) = 0 \in P(\mathbb{R}).$$

Therefore

$$a_1c_1x^{n-1} + a_2c_2x^{n-2} + \cdots + a_nc_n = 0$$

as polynomials. This implies $a_ic_i = 0$ for all i. However for all i, since $c_i \neq 0$ this implies that $a_i = 0$. Therefore we have proven linear independence. (Note: You could have also done this problem via induction on n, but that's another story.)

Exercise 2.2.16: Let V and W be vector spaces such that dim(V) = dim(W), and let T: V $\to$ W be linear. Show that there exist ordered bases β and γ for V and W, respectively, such that $[\mathrm{T}]_\beta^\gamma$ is a diagonal matrix.

Solution: First let dim(N(T))=k. By the dimension formula this implies that dim(R(T))=$n-k$. Let $\{\beta_1, \beta_2, ..., \beta_k\}$ be a basis for the subspace N(T). Now let $\gamma_{R(T)}$=$\{\gamma_{k+1}, \gamma_{k+2}, ..., \gamma_n\}$ be a basis for the subspace, R(T); we note that this set has $n-k$ elements. Since each γ_i is a member of R(T), there exists $\beta_i \in$ V such that T(β_i)=γ_i, where $k+1 \leq i \leq n$. Since $\{\gamma_{k+1}, ..., \gamma_n\}$ is linear independent, so is $\{\beta_{k+1}, ..., \beta_n\}$ (why?)

Before constructing β, let us first show that for $\{\beta_{k+1}, ..., \beta_n\}$ is linearly independent from all the vectors in $\{\beta_1, ..., \beta_n\}$: Suppose for the sake of contradiction that β_j not linearly independent from the vectors in $\{\beta_1, ..., \beta_n\}$. This means that for any $j \in \{1, ..., k\}$, we can write β_j as a linear combination of vectors in $\{\beta_1, ..., \beta_n\}$. So

$$\beta_j = \sum_{l=1}^{k} a_l^j \beta_l \text{ for scalars } a_l^j.$$

Applying T on both sides we get:

$$0 = \mathrm{T}(\beta_j) = \mathrm{T}\left(\sum_{l=1}^{k} a_l^j \beta_l\right) = \sum_{l=1}^{k} a_l^j \mathrm{T}(\beta_l) = \sum_{l=1}^{k} a_l^j \gamma_l$$

where the first equality comes from the fact that $\beta_j \in$ N(T) for $1 \leq j \leq k$, and the last equality comes from our definition of the β_l's, where $k+1 \leq l \leq n$. By linear independence of $\{\beta_1, ..., \beta_k\}$ [it's a basis for N(T)] we conclude that all of the coefficients a_l^j equal to zero. Therefore we have the following:

$$\beta_j = \sum_{l=1}^{k} a_l^j \beta_l = \sum_{l=1}^{k} 0 \cdot \beta_l = 0$$

which contradicts β_j being a member of $\{\beta_1, ..., \beta_k\}$, a linearly independent set. Therefore β_j is linearly independent from $\{\beta_1, ..., \beta_k\}$, for $k+1 \leq j \leq n$.

We are now in the position to construct the bases β and γ. First we construct β: define the ordered basis $\beta = \{\beta_1, \beta_2, ..., \beta_n\}$, simply by putting all the β's into one set, while adhering to the numerical order of the indices.

We now construct γ: Since $\gamma_{R(T)}$ is a linearly independent subset of W, we can extend this set to a basis via the Replacement Theorem. Therefore we construct the ordered basis $\gamma = \{\gamma_1, ..., \gamma_k, \gamma_{k+1}, \gamma_{k+2}, ..., \gamma_n\}$ where $\gamma_1, ..., \gamma_k$ were arbitrarily chosen from W to complete the basis. Then we have the following matrix:

$$[\mathrm{T}]_\beta^\gamma = \left(\begin{array}{c|c} A & B \\ \hline C & I \end{array}\right)$$

where A is the $k \times k$ zero matrix, B is the $(n-k) \times k$ zero matrix, C is the $k \times (n-k)$ zero matrix, and I is the $(n-k) \times (n-k)$ identity matrix. Clearly this matrix is diagonal.