Math 115a: Selected Solutions for HW 2

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Exercise 1.4.10: Show that if

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then the span of $\{M_1, M_2, M_3\}$ is the set of all symmetric 2 matrices.

Solution: Let M be an arbitrary symmetric 2×2 matrix; we will denote

$$M = \begin{pmatrix} a & b \\ b & d \end{pmatrix}.$$

Via a rather superficial inspection, we see that

$$M = aM_1 + dM_2 + bM_3.$$

Since we've written an arbitrary symmetric matrix as a linear combination of the $M_i's$, we conclude that $\{M_1, M_2.M_3\}$ spans our space in question.

Exercise 1.4.14: Show that if S_1 and S_2 are arbitrary subsets of a vector space V, then $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$. (The sum of two subsets is defined in the exercises in Section 1.3).

Solution: In order to prove equality of two sets, we need to prove mutual inclusion.
$\subseteq$: Let $v \in \text{span}(S_1 \cup S_2)$. Then v can be written as a linear combination of vectors in $S_1 \cup S_2$, i.e.

$$v = \sum_i a_i x_i + \sum_j b_j y_j,$$

where $a_i, b_j \in \mathbb{F}$ and $x_i \in S_1,\ y_j \in S_2$. (We note that the two sums are finite, although we will not use this fact in this proof.) Since $\sum_i a_i x_i \in \text{span}(S_1)$, and $\sum_j b_j y_j \in \text{span}(S_2)$, we conclude that $v \in \text{span}(S_1) + \text{span}(S_2)$.

$\supseteq$: Let $v \in \text{span}(S_1) + \text{span}(S_2)$. Then by definition,

$$v = \sum_i a_i x_i + \sum_j b_j y_j,$$

where $a_i, b_j \in \mathbb{F}$ and $x_i \in S_1,\ y_j \in S_2$. This is clearly a linear combination of vectors from $S_1 \cup S_2$. Therefore $v \in \text{span}(S_1 \cup S_2)$.

Exercise 1.5.15: Let $S = \{u_1, u_2, ..., u_n\}$ be a finite set of vectors. Prove that S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in \text{span}(\{u_1, u_2, ..., u_k\})$ for some k $(1 \le k < n)$

Proof:
($\Rightarrow$) Suppose that S is linearly dependent. Then we need to prove that either $u_1 = 0$ or there exists some k such that $u_{k+1} \in \text{span}(\{u_1, u_2, ..., u_k\})$. If $u_1 = 0$, then we are done. So let us suppose that $u_1 \neq 0$. Then what we need to prove is that the second part of the statement must be true: there exists some k such that $u_{k+1} \in \text{span}(\{u_1, u_2, ..., u_k\})$. The way we approach this is to proceed via proof by contradiction. Suppose that there is *no such* k such that $u_{k+1} \in \text{span}(\{u_1, u_2, ..., u_k\})$. To rephrase, this means that for all k, $u_{k+1} \notin \text{span}(\{u_1, u_2, ..., u_k\})$. So we now need to use this assumption repeatedly, as follows: $u_2 \notin \text{span}(\{u_1\})$ implies that $\{u_1, u_2\}$ is a linearly independent set. Similarly, $u_3 \notin \text{span}(\{u_1, u_2\})$ implies that $\{u_1, u_2, u_3\}$ is a linearly independent set. We can continue in this fashion until we get the following final statement: $u_n \notin \text{span}(\{u_1, u_2, ..., u_{n-1}\})$ implies that $S = \{u_1, u_2, ..., u_n\}$ is a linearly dependent set. However, our initial assumption is that S is linearly dependent. This is our contradiction. Therefore our initial assumption is false; there must exist some k such that $u_{k+1} \in \text{span}(\{u_1, u_2, ..., u_k\})$.

($\Leftarrow$) Suppose that $u_1 = 0$ or $u_{k+1} \in \text{span}(\{u_1, u_2, ..., u_k\})$ for some k $(1 \le k < n)$. If $u_1 = 0$, then that means $0 \in S$, which immediately implies that S is linearly dependent (why?) So suppose that $u_1 \neq 0$. This means that there exists some k such that $u_1 = 0$ or $u_{k+1} \in \text{span}(\{u_1, u_2, ..., u_k\})$. Therefore $T = \{u_1, u_2, ..., u_{k+1}\}$ is a linearly dependent set. Since $T \subseteq S$, this implies that S is linearly dependent.

Exercise 1.6.12: Let u, v, w be distinct vectors of a vector space V. Show that if $\{u, v, w\}$ is a basis for V, then $\{u+v+w, v+w, w\}$ is also a basis for V.

Solution: Let $\{u, v, w\}$ be a basis for V. Since this is a three element set, we conclude that the dimension of V must be 3. Looking at $\{u+v+w, v+w, w\}$, we see that this is also a three element set. Therefore if we can prove that this set is either linearly independent <u>or</u> spans V, then we are done (make sure you understand why this is true). We will show that $\{u+v+w, v+w, w\}$ is a linearly independent set. Let $a_1, a_2, a_3 \in \mathbb{F}$ such that

$$a_1(u+v+w) + a_2(v+w) + a_3(w) = 0.$$

We will show that this implies that $a_1 = a_2 = a_3 = 0$, by rewriting the equality as:

$$\begin{aligned} 0 &= a_1(u+v+w) + a_2(v+w) + a_3(w) \\ &= (a_1)(u) + (a_1+a_2)(v) + (a_1+a_2+a_3)(w). \end{aligned}$$

Since $\{u, v, w\}$ is a basis for V, it is a linearly independent set. Therefore from the last equality, we can conclude that $a_1 = a_1 + a_2 = a_1 + a_2 + a_3 = 0$, and from here we can conclude immediately that $a_1 = a_2 = a_3 = 0$. Therefore we've proven that $\{u+v+w, v+w, w\}$ is a linearly independent set. Therefore it is a basis for V.

Exercise 1.6.20: Let V be a vector space having dimension n, and let S be a subset of V that generate V.

(a) Prove that there is a subset of S that is a basis for V.

(b) Prove that S contains at least n elements.

Solution:
(a): Let $\{\beta_1, \beta_2, ..., \beta_n\}$ be a basis for V. Since span(S)=V, each of the β_i's can be written as a finite linear combination of elements from S. More specifically,

$$\begin{aligned} \beta_1 &= \sum_{i \in I_1} a_{1,i} s_i \\ \beta_2 &= \sum_{i \in I_2} a_{2,i} s_i \\ &\vdots \\ \beta_n &= \sum_{i \in I_n} a_{n,i} s_i \end{aligned}$$

where all of the $a_{j,i}$'s are scalars, and I_n's are finite index sets (see the note at the end of the proof). Let us define the set

$$J = \bigcup_{j=1}^{n} I_j$$

be the finite union of all the index sets. Now consider the subset of the vector space

$$T = \bigcup_{j \in J} s_j.$$

Since T is a set made up of elements from S, $T \subseteq S$. Since J is a finite index set, T is also a finite set. Furthermore, we have constructed this set T that contains elements from S which "builds" each of the β_i's. Therefore

$$\begin{aligned} \{\beta_1, \beta_2, ..., \beta_n\} \subseteq \text{ span } (T) &\subseteq \text{ span } (S) = V \\ &\Rightarrow \\ V = \text{ span } (\{\beta_1, \beta_2, ..., \beta_n\}) &\subseteq \text{ span } (\text{span } (T)) = \text{span } (T) \subseteq V \\ &\Rightarrow \text{ span}(T) = V. \end{aligned}$$

Since T spans V, and it is a finite set, by the Replacement Theorem (1.10) we can find a subset of T–call it B–that is a basis for V. It is clear that B is a subset of S, as it is a subset of T. This finishes our proof.