Math 115a: Selected Solutions for HW 1

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Exercise 1.2.2
Write the zero vector of $M_{3\times 4}(\mathbb{F})$.

Solution:

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Exercise 1.2.12
A real-valued function f defined on the real line is called an **even function** if $f(-t) = f(t)$ for each real number t. Prove that the set of even functions defined on the real line with the operations of addition and scalar multiplication defined in Example 3 is a vector space.

Proof:
Let W denote the space of all real-valued even functions on the real line. It is clear that $W \subset V$ where V is defined to be the vector space of all real-valued functions on the real line. By the virtue of V being a vector space, it suffices to prove that W is a subspace of V. First of all, it is clear that the zero function is even: $0_V(-x) = 0_V(x)$ for all $x \in \mathbb{R}$. Now let $c \in \mathbb{R}$, $f, g \in W$. Then $(cf + g)(-t) = cf(-t) + g(-t) = cf(t) + g(t) = (cf + g)(t)$, and therefore $cf + g \in W$. By Theorem 1.3, it follows that W is a subspace. Therefore W is a vector space. (Note: We could have done this proof by checking VS(1)-(8), but that's tedious.)

Exercise 1.2.18
Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. For $(a_1, a_2), (b_1, b_2) \in V$ and $c \in \mathbb{R}$, define

$$\begin{aligned} (a_1, a_2) + (b_1, b_2) &= (a_1 + 2b_1, a_2 + 3b_2) \text{ and} \\ c(a_1, a_2) &= (ca_1, ca_2). \end{aligned}$$

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Solution:

No- because VS(1) fails: Let $v = (1,0)$, $w = (0,1)$. Then

$$v + w = (1,0) + (0,1) = (1,3) \neq (2,1) = (0,1) + (1,0) = w + v$$

Exercise 1.2.19
Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. Define addition of elements of V coordinatewise, and for (a_1, a_2) in V and $c \in \mathbb{R}$, define

$$c(a_1, a_2) = \begin{cases} (0,0) & \text{if } c = 0 \\ \left(ca_1, \frac{a_2}{c}\right) & \text{if } c \neq 0. \end{cases}$$

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Solution:
No- because VS(8) fails: Let $c, d \in \mathbb{R}$ and $(a_1, a_2) \in V$. Then

$$\begin{aligned} (c+d) = (c+d)(a_1, a_2) &= \left((c+d)a_1, \frac{a_2}{c+d}\right) \\ &\neq \left((c+d)a_1, \frac{a_2}{c} + \frac{a_2}{d}\right) \\ &= \left(ca_1 + da_1, \frac{a_2}{c} + \frac{a_2}{d}\right) \\ &= c(a_1, a_2) + d(a_1, a_2). \end{aligned}$$

Exercise 1.3.20
Prove that if W is a subspace of a vector space V and $w_1, w_2, ..., w_n$ are in W, then $a_1w_1 + a_2w_2 + \cdots + a_nw_n \in$ W for any scalars $a_1, a_2, ..., a_n$.

Solution: Let $w_1, w_2, ..., w_n \in$ W and $a_1, a_2, ..., a_n \in F$. Since W is a subspace, $a_iw_i \in$ W, where $1 \leq i \leq n$, by the closure property in scalar multiplication. Then $a_1, w_1 + a_2w_2 \in$ W by closure in addition. Then $(a_1w_1 + a_2w_2) + a_3w_3 = a_1w_1 + a_2w_2 + a_3w_3 \in$ W again by closure in addition and associativity of addition in a vector space. We repeat this argument $n - 2$ more times and get our desired result.

□

Exercise 1.3.24
Show that $\mathbb{F}^n$ is the direct sum of the subspaces

$$W_1 = \{(a_1, a_2, ..., a_n) \in \mathbb{F}^n : a_n = 0\}$$

and

$$W_2 = \{(a_1, a_2, ..., a_n) \in \mathbb{F}^n : a_1 = a_2 = \cdots = a_{n-1} = 0\}.$$

Solution: In order to prove that $W_1 \oplus W_2 = \mathbb{F}^n$, we must prove the following two facts: $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = \mathbb{F}^n$.

i.) $W_1 \cap W_2 = \{0\}$: First of all, every subspace contains the zero vector. Since the intersection of two subspaces is another subspace, it follows that $\{0\} \subseteq W_1 \cap W_2$. We not need to establish the other inclusion. Let $v = (v_1, v_2, ..., v_n) \in W_1 \cap W_2$. Since $v \in W_1$, $v_n = 0$. Secondly, since $v \in W_2$, $v_1 = v_2 = \cdots = v_{n-1} = 0$. Therefore we conclude that $v = 0$, the zero vector. Since v was any arbitrary element in the intersection of W_1 and W_2, we conclude that $W_1 \cap W_2 \subseteq \{0\}$. Therefore both inclusions are establish and so $W_1 \cap W_2 = \{0\}$.

ii.) $W_1 + W_2 = \mathbb{F}^n$: By definition,

$$W_1 + W_2 = \{w_1 + w_2 \mid w_1 \in W_2, w_2 \in W_2\}.$$

Let $w = w_1 + w_2 \in W_1 + W_2$, where $w_1 \in W_1$ and $w_2 \in W_2$. Since $W_i \subseteq \mathbb{F}^n$, we conclude that $w_i \in \mathbb{F}^n$. By the closure property of addition, we conclude that $w \in \mathbb{F}^n$. Since w was arbitrarily chosen, we conclude that $W_1 + W_2 \subseteq \mathbb{F}^n$. Now let $v = (v_1, v_2, ..., v_n) \in \mathbb{F}^n$. We decompose v into two vectors:

$$v = (v_1, v_2, ..., v_{n-1}, 0) + (0, 0, ..., 0, v_n).$$

We see that this decomposition exactly shows that $v \in W_1 + W_2$. Since v was arbitrarily chosen we conclude that $W_1 + W_2 = \mathbb{F}^n$.

Therefore we can finally conclude that $W_1 \oplus W_2 = \mathbb{F}^n$

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