**Linear independence**

Recall: A vector space $V$, $x_1, \ldots, x_n \in V$. We say that $x_1, \ldots, x_n$ are **linearly independent** if, for all $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$, if $\lambda_1 x_1 + \cdots + \lambda_n x_n = 0$, then $\lambda_1 = \cdots = \lambda_n = 0$.

We say that $x_1, \ldots, x_n$ are **linearly dependent** if they are not linearly independent, i.e., there exists $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ not all zero and $\lambda_1 x_1 + \cdots + \lambda_n x_n = 0$.

**Examples:**

1) $x_1$ is linearly independent if and only if $x_1 \neq 0$.

**Pf:** We prove the contrapositive: $x_1$ is linearly dependent if and only if $x_1 = 0$.

2) Assume $x_1$ is linearly dependent. Then there exists $\lambda_1 \neq 0$ such that $\lambda_1 x_1 = 0$. Since $\lambda_1 \neq 0$, $\lambda_1$ has an inverse $\frac{1}{\lambda_1}$ in $\mathbb{F}$.

$\frac{1}{\lambda_1} \lambda_1 x_1 = \frac{1}{\lambda_1} \cdot 0 = 0$. But $\frac{1}{\lambda_1} = 1$. Hence $x_1 = 0$.

3) Conversely, assume $x_1 = 0$. Then $1 \cdot x_1 = 0$, but $1 \neq 0$.

Hence $x_1$ is linearly dependent.

2) For which values of $a \in \mathbb{R}$ are the vectors $[\begin{smallmatrix} a \\ a \\ a \end{smallmatrix}]$, $[\begin{smallmatrix} a \\ 1 \\ a \end{smallmatrix}]$, $[\begin{smallmatrix} 1 \\ a \\ a \end{smallmatrix}]$ linearly independent?

We study the equation $\lambda_1 [\begin{smallmatrix} a \\ a \\ a \end{smallmatrix}] + \lambda_2 [\begin{smallmatrix} a \\ 1 \\ a \end{smallmatrix}] + \lambda_3 [\begin{smallmatrix} 1 \\ a \\ 1 \end{smallmatrix}] = 0$.

The vectors are linearly independent if it has $\lambda_1 = \lambda_2 = \lambda_3$ as the only solution.

- For $a = 1$:
  
  $[\begin{smallmatrix} a \\ a \\ a \end{smallmatrix}]$ is linearly dependent.

- For $a = -2$:
  
  $[\begin{smallmatrix} 1 \\ a \\ a \end{smallmatrix}]$ is linearly dependent.

- For $a \neq 1$ and $a \neq -2$:
  
  $[\begin{smallmatrix} 1 + a \\ a \\ 1 + a \end{smallmatrix}]$ is linearly independent otherwise.
3) In the vector space \( \mathbb{F}(\mathbb{R}, \mathbb{R}) \), the functions \((t \mapsto \cos(t))\) and \((t \mapsto \sin(t))\) are linearly independent.

**Proof:** Assume \( \lambda_1 \cos(t) + \lambda_2 \sin(t) = 0 \) for some \( \lambda_1, \lambda_2 \in \mathbb{R} \). This means, for all \( t \in \mathbb{R} \), \( \lambda_1 \cos(t) + \lambda_2 \sin(t) = 0 \).

For \( t = 0 \): \( \lambda_1 \cos(0) + \lambda_2 \sin(0) = 0 \), so \( \lambda_1 = 0 \).

For \( t = \pi/2 \): \( \lambda_1 \cos(\pi/2) + \lambda_2 \sin(\pi/2) = 0 \), so \( \lambda_2 = 0 \).

So \( \cos, \sin \) are linearly independent. \( \square \)

4) Sometimes, the values of the function are not so easy, and other ideas have to be used...

The functions \( f_1: \{ t \mapsto e^t \} \) and \( f_2: \{ t \mapsto e^{2t} \} \)

are linearly independent. Idea: \( f_2 \) grows "much faster" than \( f_1 \), towards \( \infty \). If \( f_1 \) and \( f_2 \) were linearly dependent, they would have the same behaviour at \( +\infty \).

More precisely: we have \( \lim_{t \to \infty} \frac{f_1(t)}{f_2(t)} = 0 \).

Assume \( \lambda_1 f_1 + \lambda_2 f_2 = 0 \) for some \( \lambda_1, \lambda_2 \in \mathbb{R} \).

Then for all \( t \in \mathbb{R} \), \( \lambda_1 e^t + \lambda_2 e^{2t} = 0 \).

Divide by \( e^{2t} \): \( \frac{\lambda_1}{e^{2t}} e^t + \frac{\lambda_2}{e^{2t}} = 0 \).

Taking \( t \to \infty \), \( \frac{\lambda_1}{e^{2t}} \to 0 \).

Then \( \frac{\lambda_1}{\lambda_2} = 0 \) so \( \lambda_2 = 0 \).

Hence, \( f_1 \) and \( f_2 \) are linearly independent. \( \square \)
Generalization: assume $a_1 < \cdots < a_m$

let $f_i(t) = e^{a_i t}$.

Then $f_1, \ldots, f_m$ are linearly independent.

Proof: we proceed by induction on $m$.

- base case $m = 1$: since $f_1 \neq 0$, $f_1$ is linearly independent.

- inductive step: assume that $f_1, \ldots, f_m$ are linearly independent for some $m > 1$. We prove that $f_1, \ldots, f_{m+1}$ are linearly independent.

Assume that $\lambda_1 f_1 + \cdots + \lambda_m f_m + f_{m+1} = 0$ for some $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$.

Then for all $t \in \mathbb{R}$ we have: $\lambda_1 e^{a_1 t} + \cdots + \lambda_m e^{a_m t} + e^{a_{m+1} t} = 0$.

Divide by $e^{a_{m+1} t}$: $\lambda_1 e^{(a_1 - a_{m+1}) t} + \cdots + \lambda_m e^{(a_m - a_{m+1}) t} + 1 = 0$. \((*)\)

For $i < m$, we have $e^{(a_i - a_{m+1}) t} \to \infty$ since $a_i - a_{m+1} < 0$.

So taking limits when $t \to +\infty$ in \((*)\) gives $\lambda_m = 0$.

Hence $\lambda_1 f_1 + \cdots + \lambda_m f_m = 0$. By assumption, $f_1, \ldots, f_m$ are independent.

So $\lambda_1 = \cdots = \lambda_m = 0$. Thus we proved $\lambda_1 = \cdots = \lambda_m = 0$, and $f_1, \ldots, f_{m+1}$ are linearly independent.

5) Work over the field $F = \mathbb{Q}$.

We can view $\mathbb{R}$ as a $\mathbb{Q}$-vector space (usual addition and multiplication).

Then $1, \sqrt{2}$ are linearly independent over $\mathbb{Q}$, (but not over $\mathbb{R}$!)

Proof: assume $a + b \sqrt{2} = 0$ for $a, b \in \mathbb{Q}$.

if $b \neq 0$, then $\sqrt{2} = -\frac{a}{b} \in \mathbb{Q}$: contradiction!

so $b = 0$. Thus we also have $a = 0$, and $1, \sqrt{2}$ are linearly independent.
Bases and dimension:

Recall: a basis of \( V \) is a subset \( B \) of \( V \) that is both linearly independent and generates \( V \).

The dimension of \( V \) is the number of elements in \( B \).

(you have seen in lecture that this number does not depend on the choice of the basis \( B \)).

Examples:

1) Consider the vectors \( u_1 = (2, -3, 1) \), \( u_2 = (1, 4, -2) \), \( u_3 = (8, 12, -4) \), \( u_4 = (1, 37, -17) \) and \( u_5 = (-3, -5, 8) \).

Find a subset of \( \{u_1, \ldots, u_5\} \) that is a basis of \( \text{Span} \{u_1, \ldots, u_5\} \).

To solve this, we now reduce the matrix \([u_1, \ldots, u_5]\).

\[
\begin{bmatrix}
2 & 1 & -8 & 1 & -3 \\
-3 & 4 & 12 & 37 & -5 \\
1 & -2 & -4 & -17 & 8
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & -4 & -3 & 0 \\
0 & 1 & 0 & 7 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

The leading 1's of the row reduced form are in columns 1, 2 and 5.

\( \Rightarrow \{u_1, u_2, u_5\} \) is a basis.

The row-reduced form makes it easy to find the relation between the \( \{u_1, \ldots, u_5\} \).

For instance, we see that in the row-reduced form,

Column 4 = 7 Column 2 - 3 Column 1.

So \( u_4 = 7u_2 - 3u_1 \). Similarly, \( u_3 = -4u_1 \).
Remark: This is not the only correct answer.
For instance, \( \{ u_2, u_3, u_5 \} \) would also be correct.

2) Consider \( W = \{ (x_1, x_2, x_3, x_4, x_5) | x_1 + x_3 + x_4 + x_5 = 0 \text{ and } x_2 + 2x_3 - x_5 = 0 \} \).

Find a basis of \( W \) and its dimension.
(remark: \( W \) is a subspace of \( \mathbb{F}^5 \), exercise!)

We need to solve the system of equations:

\[
\begin{bmatrix}
1 & 0 & 1 & 1 & 1 \\
0 & 1 & 2 & 0 & -1
\end{bmatrix}
\]

is already now reduced.

We have 3 free variables: \( x_3, x_4, x_5 \).
2 leading variables: \( x_1, x_2 \).

Solution:

\[
\begin{align*}
x_1 &= -s - t - u \\
x_2 &= -2s + u \\
x_3 &= s \\
x_4 &= t \\
x_5 &= u
\end{align*}
\]

or \( (x_1, x_2, x_3, x_4, x_5) = (-s - t - u, -2s + u, s, t, u) \)

\[= s(-1, -2, 1, 0, 0) + t(-1, 0, 0, 1, 0) + u(-1, 1, 0, 1, 0).\]

So a basis of \( W \) is \( \{ (-1, -2, 1, 0, 0), (-1, 0, 0, 1, 0), (-1, 1, 0, 1, 0) \} \).

And \( \dim W = 3 \).
Around Grassmann's formula:

If $V$ is finite dimensional vector space and $W_1, W_2$ are subspaces, then:

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

**Analogy:** If $S$ is a finite set, and $T_1, T_2$ are subsets, then

$$\text{Card}(T_1 \cup T_2) = \text{Card}(T_1) + \text{Card}(T_2) - \text{Card}(T_1 \cap T_2).$$

**Illustration:** $V = \mathbb{R}^3$, $W_1, W_2$ planes intersecting at a line.

$W_1 + W_2 = \mathbb{R}^3 \Rightarrow \dim(W_1 + W_2) = 3$

$W_1 \cap W_2$ is a line $\Rightarrow \dim(W_1 \cap W_2) = 1$

$\dim(W_1) = \dim(W_2) = 2$

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) \Rightarrow 3 = 2 + 2 - 1 \checkmark$$

In particular, if $V = W_1 \oplus W_2$, then we have

$$\dim(V) = \dim(W_1) + \dim(W_2).$$

A useful converse:

**Prop.** Assume that $W_1 \cap W_2 = \{0\}$ and $\dim(W_1) + \dim(W_2) = \dim(V)$.

Then $V = W_1 \oplus W_2$.

**Pf.** We need to show that $V = W_1 + W_2$. We have $W_1 + W_2 \subseteq V$.

Furthermore by Grassmann's formula, we have

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) = \dim(V).$$

So $W_1 + W_2 = V \square$
Interest: you do not need to check both \( W_1 \oplus W_2 = V \) and \( W_1 \cap W_2 = \{ 0 \} \) to prove \( V = W_1 \oplus W_2 \) if you know the dimensions of \( W_1, W_2 \) and \( V \).

**Example:** \( V = M_n(R) \)

\[
W_1 = \{ \text{symmetric matrices} \} = \{ M \in M_n(R) \mid b_M = M \} \\
W_2 = \{ \text{anti-symmetric matrices} \} = \{ M \in M_n(R) \mid b_M = -M \}.
\]

We proved that \( V = W_1 \oplus W_2 \)

**New proof:** \( W_1 \cap W_2 = \{ 0 \} \) proved as before.

Instead of proving \( W_1 + W_2 = V \), we compute the dimensions.

A basis of \( V \) is given by \( E_{ij} \), \( i, j \in \{ 1, \ldots, n \} \)

where \( E_{ij} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \) so \( \dim(V) = n^2 \).

If \( M = (a_{ij}) \in W_1 \), we have \( a_{ij} = a_{ji} \) for all \( i, j \).

\[
M = \sum_{i,j} a_{ij} E_{ij} = \sum_{i=1}^n a_{ii} E_{ii} + \sum_{1 \leq i < j \leq n} a_{ij} (E_{ij} + E_{ji})
\]

A basis of \( W_1 \) is \( \{ E_{ii}, i = 1, \ldots, n \} \cup \{ E_{ij} + E_{ji}, 1 \leq i < j \leq n \} \).

\( \Rightarrow \) \( \dim(W_1) = \frac{n(n+1)}{2} \).

Similarly, if \( M = (a_{ij}) \in W_2 \), we have \( a_{ii} = 0 \) for all \( i \)

and \( a_{ij} = -a_{ji} \) for \( i \neq j \).

\[
M = \sum_{1 \leq i < j \leq n} a_{ij} (E_{ij} - E_{ji})
\]

A basis of \( W_2 \) is \( \{ E_{ij} - E_{ji}, 1 \leq i < j \leq n \} \).

\( \Rightarrow \) \( \dim(W_2) = \frac{n(n-1)}{2} \).

We have \( \dim(W_1) + \dim(W_2) = n^2 = \dim(V) \) and \( W_1 \cap W_2 = \{ 0 \} \).

\( \Rightarrow \) \( V = W_1 \oplus W_2 \).