Application of inner product space to optimization.

Question: find $a, b \in \mathbb{R}$ such that $\int_0^1 (t^4 - at - b)^2 \, dt$ is minimal.

Method 1: use calculus.

Method 2: the question can be interpreted as a question about distance in inner product space.

$V = \mathbb{P}_2(\mathbb{R})$ with inner product $\langle f(t), g(t) \rangle = \int_0^1 f(t)g(t) \, dt$.

Then the question is to minimize $\| t^4 - at - b \|^2$ for $a, b \in \mathbb{R}$.

Let $f(t) = t^2$ and $W = \{ at + b; a, b \in \mathbb{R} \}.

$W$ is a subspace of $V$ of dimension 2.

We want to find the element of $W$ that is closest to $f(t)$.

This element is simply the orthogonal projection of $f(t)$ on $W$.

![Diagram](image)

this is the element of $W$ that is closest to $f$.

Justification:

Theorem: $V$ inner product space, $W \subset V$ subspace, $x \in V$ and $p(x)$ the orthogonal projection of $x$ on $W$. Then $p(x)$ is the vector on $W$ that is closest to $x$, that is:

For all $y \in W$, $\| x - y \| > \| x - p(x) \|$. i.e. $p(x)$ minimizes $\| x - y \|$ for $y \in W$.

Proof: recall that $x - p(x) \in W^\perp$.

If $y \in W$, we have:

$$\| x - y \|^2 = \| x - p(x) + p(x) - y \|^2$$

$$= \| x - p(x) \|^2 + \| p(x) - y \|^2$$

by the Pythagorean theorem

$$\geq \| x - p(x) \|^2 \quad \square$$
Back to our problem: we need to find the orthogonal projection of $f(t) = t^2$ on $W = \{ at + b, \ a, b \in \mathbb{R} \}$.

We start by finding an orthogonal basis of $W$.

$w_1(t) = 1 \quad w_2(t) = t \quad$ bases of $W$, we apply Gram-Schmidt:

$v_1(t) = w_1(t) = 1$

$v_2(t) = w_2(t) - \langle w_2(t), v_1(t) \rangle v_1(t)
= t - \left( \int_0^1 x \, dx \right) v_1(t)
= t - \left( \left[ \frac{x^2}{2} \right]_0^1 \right) v_1(t)
= t - \frac{1}{2}

\Rightarrow \quad v_1(t), v_2(t) \quad \text{is an orthogonal basis of } W.

Now the projection of $f(t)$ on $W$ is:

$p(f(t)) = \frac{\langle f(t), v_1(t) \rangle}{\|v_1(t)\|^2} v_1(t) + \frac{\langle f(t), v_2(t) \rangle}{\|v_2(t)\|^2} v_2(t)$.

$\langle f(t), v_1(t) \rangle = \int_0^1 t^2 \, dt = \frac{1}{3}$
\[\|v_1(t)\|^2 = \int_0^1 1 \, dt = 1.\]

$\langle f(t), v_2(t) \rangle = \int_0^1 t^2 \left( t - \frac{1}{2} \right) \, dt = \int_0^1 \left( t^3 - \frac{1}{2} t^2 \right) \, dt$
\[= \left[ \frac{t^4}{4} - \frac{t^3}{6} \right]_0^1 = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}.\]

$\|v_2(t)\|^2 = \int_0^1 \left( t - \frac{1}{2} \right)^2 \, dt = \int_{-1/2}^{1/2} u^2 \, du = 2 \int_0^{1/2} u^2 \, du = 2 \cdot \frac{1}{3} \cdot \left( \frac{1}{2} \right)^3 = \frac{1}{12}$

$\Rightarrow \quad p(f(t)) = \frac{1}{3} + \frac{\frac{1}{12}}{\frac{1}{12}} \left( t - \frac{1}{2} \right) = t - \frac{1}{6}$

So $\int_0^1 (t^2 - a t - b)^2 \, dt$ is minimal for $a = 1, \ b = -\frac{1}{6}$.