REMARKS ON NONABELIAN LOCALIZATION

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0. In this short note, we derive some nonabelian localization formulas in a very simple way. The method is to use Witten's idea of multiplying the integral by a Gaussian and then to complete square. The *localization procedure* we use is different from Witten's in that we do not need the Bismut localizing factor $e^{-tD\lambda}$ as in [W], at the same time we avoid the use of Fourier transform in [JK]. Along the way we will also derive Martin's formula which relates nonabelian symplectic reduction to the abelian case. The method is rather elementary and applicable to other situations which we hope to discuss on a later occasion.

We remark that all of the basic results in this note, which grows out of my trying to understand this circle of ideas, are due to Jeffrey-Kirwan, Martin and Witten. Our sole contribution is pointing out a different way to derive them. Other quite different approaches to such type results have also been obtained by Guillemin-Kalkman [GK], Martin [M] and Vergne [V]. The interested reader may compare these different methods and formulas.

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1. Let M be a compact symplectic manifolds with the Hamiltonian action of a compact Lie group K. Let k denote the Lie algebra of K and k^* its dual. Let

$$\mu_K: \ M \to k^*$$

be the corresponding moment map and assume 0 is a regular value of μ_K. Denote by $M_K = \mu^{-1}(0)/K$ the corresponding symplectic quotient. Although our method applies to a more general situation such as orbifolds, for simplicity we will assume the K-action on $\mu^{-1}(0)$ is free. All of the following cohomology groups have coefficients in complex numbers.

Let $H_K^*(M)$ be the equivariant cohomology group, that is the cohomology group of the complex $(\Omega_K^*(M), D)$ where

$$\Omega_K^*(M) = (S(k^*) \otimes \Omega^*(M))^K, \text{ and,}$$

$$D(\alpha\otimes f)(\phi)=f(\phi)(d\alpha-\sqrt{-1}i_{\bar{\phi}}\alpha).$$

Here $\alpha\otimes f\in\Omega_K^*(M)$, $\phi\in K$, $\bar{\phi}$ is the vector field on M generated by ϕ and $i_{\bar{\phi}}$ is the contraction operator. Choose a basis $\{e_j\}$ of k, let $\{e^j\}$ be the dual basis in k^*. Then $\phi=\sum_j\phi^je_j$ and μ_K can be written as $\sum_j\mu_je^j$. Let ω be the symplectic form on M and $\bar{\omega}=\omega+i\mu_K\in H_K^*(M)$ with $i=\sqrt{-1}$ be its equivariant extension.

Let ρ_K be the composite of the following maps in equivariant cohomology:

$$H_K^*(M)\to H_K^*(\mu^{-1}(0))\backsimeq H^*(M_K)$$

where the first map is induced by the inclusion map $j:\ \mu^{-1}(0))\hookrightarrow M$ and the second is the canonical isomorphism.

2. Let $\eta=\sum_I\eta_Ie^I$ where $I=(i_1,\cdots,i_l)$ with l the dimension of K is multi-index be an equivariant differential form. Here all of the i_j's are nonnegative integers. Following Witten [W], we consider the following integral

$$I(\varepsilon)=\int_k e^{-\varepsilon|\phi|^2}\int_M\eta(\phi)e^{\omega+i\mu_K(\phi)}d\phi$$

where $d\phi$ is the volume form of the Killing form on k.

Note that

$$|\phi|^2=\sum_j(\phi^j)^2,\ \eta(\phi)=\sum_I\eta_I\phi^I,\ \text{and}\ \mu_K(\phi)=\sum_j\mu_j\phi^j.$$

Completing square, we get

$$I(\varepsilon)=\int_k\int_M e^{-\varepsilon|\phi-i\mu_K/2\varepsilon|^2}e^{-|\mu_K|^2/4\varepsilon}\eta(\phi)e^{\omega}d\phi$$

where

$$|\phi-i\mu_K/2\varepsilon|^2=\sum_j(\phi^j-i\mu_j/2\varepsilon)^2,\ |\mu_K|^2=\sum_j\mu_j^2.$$

Considering k as R^l and changing variable $\phi^j\to\phi^j/\sqrt{\varepsilon}+i\mu_j/2\varepsilon$ and $x\to x$ where x denotes the variable in M, we get

$$I(\varepsilon)=\varepsilon^{-l/2}\int_M\int_k e^{-|\phi|^2}e^{-|\mu_K|^2/4\varepsilon}\eta'(\phi)e^{\omega}d\phi$$

where

$$\eta'(\phi)=\sum_I\eta_I(\phi/\sqrt{\varepsilon}+i\mu_K/2\varepsilon)^I.$$

Obviously when $\varepsilon \to 0$, the above integral is dominated by the term $e^{-|\mu_K|^2/4\varepsilon}$ and localizes to a neighborhood of $\mu_K^{-1}(0)$. In fact it is easy to see that for a small ball B_δ of radius δ around $0 \in k^*$,

$$\varepsilon^{-l/2} \int_k \int_{M-\mu_K^{-1}(B_\delta)} e^{-|\phi|^2} e^{-|\mu_K|^2/4\varepsilon} \eta'(\phi) e^{\omega} d\phi = O(e^{-\delta^2/4\varepsilon}).$$

In a standard way we can identify a small neighborhood of $\mu_K^{-1}(0)$ with $\mu_K^{-1}(0) \times B_\delta$, This gives us the following

Lemma 1:

$$I(\varepsilon) = a \int_k e^{-\varepsilon|\phi|^2} \int_{\mu_K^{-1}(0)\times B_\delta} \eta(\phi) e^{\omega + i\mu_K(\phi)} d\phi + O(e^{-\delta^2/4\varepsilon})$$

where $a = \pm 1$ *is determined by the compatibility of the orientation of* $\mu_K^{-1}(0) \times k^*$ *with that of* M.

Another method to prove the above localization to $\mu_K^{-1}(0)$ is to use stationary phase approximation with respect to ϕ. More precisely change variable $\phi \to \phi/\sqrt{\varepsilon}$, we get

$$I(\varepsilon) = \varepsilon^{-l/2} \int_k \int_M e^{\frac{i}{\sqrt{\varepsilon}}\mu_K(\phi)} e^{-|\phi|^2} \eta(\phi/\sqrt{\varepsilon}) e^{\omega} d\phi$$

which when ε goes to zero obviously localizes to $\mu_K = 0$.

3. To compute $I(\varepsilon)$ in a small neighborhood of $\mu_K^{-1}(0)$, it is standard to use the local symplectic model $\mu_K^{-1}(0) \times k^*$ where K acts on k^* by coadjoint action with symplectic form $\omega = \pi^*\omega_0 + d(\alpha, \theta)$. Here $\pi : \mu_K^{-1}(0) \to M_K$ is the projection, ω_0 is the reduced symplectic form on M_K, α is the coordinate function on k^* and θ is the k-valued connection 1-form of the principal bundle $\pi : \mu_K^{-1}(0) \to M_K$. Also (α, θ) denotes the obvious pairing. The corresponding moment map is given by $\mu(p, \alpha) = \alpha$.

Let $<,>$ denote the Killing form on K and $F = d\theta - \theta\Lambda\theta$ be the curvature of the connection θ. Write $F = \sum_{j=1}^l F^j e_j$ and $< F, F >= \sum_{j=1}^l F^j \Lambda F^j$. Then standard computation gives us

Lemma 2:

$$\int_k e^{-\varepsilon|\phi|^2} \int_{\mu_K^{-1}(0)\times B_\delta} \eta(\phi) e^{\omega + i\mu_K(\phi)} d\phi = (2\pi)^l V(K) \int_{M_K} \rho_K(\eta) e^{\varepsilon<F,F>} e^{\omega_0} + O(e^{-\delta^2/4\varepsilon})$$

where $V(K)$ *denotes the volume of* K.

In fact from the above local model, we have

$$\int_k e^{-\varepsilon|\phi|^2}\int_{\mu_K^{-1}(0)\times B_\delta}\eta(\phi)e^{\omega+i\mu_K(\phi)}d\phi$$
$$=\int_k e^{-\varepsilon|\phi|^2}\int_{\mu_K^{-1}(0)\times B_\delta}\eta(\phi)e^{\pi^*\omega_0+d(\alpha,\theta)}e^{i(\alpha,\phi)}d\phi.$$

Since $d(\alpha,\theta)=(d\alpha,\theta)+(\alpha,d\theta)$, we get

$$\int_k e^{-\varepsilon|\phi|^2}\int_{\mu_K^{-1}(0)\times B_\delta}\eta(\phi)e^{\pi^*\omega_0+(\alpha,d\theta)}e^{(d\alpha,\theta)}e^{i(\alpha,\phi)}d\phi$$

in which the term $e^{(d\alpha,\theta)}$ gives $\Lambda\theta d\alpha$ where $\theta=\sum_j\theta^je_j$, $\Lambda\theta=\theta^1\Lambda\cdots\Lambda\theta^l$ with $l=\dim K$, and $d\alpha$ is the volume form on k^*. Inserting the term $\theta\Lambda\theta$ which does not affect the integral, because of the term $\Lambda\theta$, we then have

$$\int_{\mu_K^{-1}(0)}\int_k\int_{B_\delta}e^{-\varepsilon|\phi|^2}\eta(\phi)e^{\pi^*\omega_0+(\alpha,F)}e^{i(\alpha,\phi)}\Lambda\theta d\alpha d\phi.$$

Write $\eta(\phi)=\pi^*\rho_K(\eta)$, then the above integral becomes

$$\int_{\mu_K^{-1}(0)}\pi^*(\rho_K(\eta))e^{\pi^*\omega_0}\Lambda\theta\int_{B_\delta}\int_k e^{-\varepsilon|\phi|^2}e^{(\alpha,F)}e^{i(\alpha,\phi)}d\phi d\alpha.$$

Note that

$$\int_{B_\delta}\int_k e^{-\varepsilon|\phi|^2}e^{(\alpha,F)}e^{i(\alpha,\phi)}d\phi d\alpha=\int_{k^*}\int_k e^{-\varepsilon|\phi|^2}e^{(\alpha,F)}e^{i(\alpha,\phi)}d\phi d\alpha+O(e^{-\delta^2/4\varepsilon}).$$

It is easy to carry out the last integral which is equal to $(2\pi)^le^{\varepsilon<F,F>}$. Note that $\Lambda\theta$ is the volume form for the fiber of the principal bundle $\mu_K^{-1}(0)\to M_K$.

We remark that an explicit estimate of the δ in Lemma 2, in terms of the geometry of the moment map, has been given in [JK] and [V].

4. On the other hand, let $t\subset k$ be the Lie algebra of a maximal torus T of K. As an easy corollary of the Weyl integral formula and the fact that the Jacobian of the exponential map $\exp:\ k\to K$ is

$$J(\phi)=|\det\frac{1-e^{-\mathrm{ad}(\phi)}}{\mathrm{ad}(\phi)}|,$$

we have the following formula for an integrable adjoint invariant function $f(\phi)$ on k:

$$\int_k f(\phi)d\phi=C\int_t\nu(\psi)^2f(\psi)d\psi.$$

Here $\phi \in k$, $\psi \in t$, and $\nu(\psi) = \prod_{\alpha\in\triangle^+} \alpha(\psi)$ is the product of the positive roots with respect to t and $C = (2\pi)^{l-s}V(K)/|W|V(T)$ where $s = \dim T$, $|W|$, $V(\cdot)$ denote the order of the Weyl group and the volume with respect to the metric induced from the Killing form respectively. This formula can also be proved easily in a similar way to the derivation of the Weyl integral formula. By using the identification

$$\mathrm{Ad}:\ K/T \times_W t \equiv k,$$

we get

$$\int_k f(\phi)d\phi = \int_{k/Ad} f(\psi)\sigma(\psi)d\psi = \int_{t/W} f(\psi)\sigma(\psi)d\psi$$

where $k/Ad \backsimeq t/W$ is the Weyl chamber and

$$\sigma(\psi) = V(K/T)\cdot \text{the Jacobian of } \ i(\psi)$$

with $i(\psi):\ K/T \to k(\psi)$ the map to the adjoint orbit passing through ψ. This Jacobian is easy to compute which is equal to $|\det \operatorname{ad}(\psi)| = (2\pi)^{l-s}\nu(\psi)^2$ as required [JK]. Using this formula, we get

$$I(\varepsilon) = C\int_t e^{-\varepsilon|\psi|^2}\nu(\psi)^2\int_M \eta(\psi)e^{\omega+i\mu_T(\psi)}d\psi.$$

Let μ_T denote the moment map of the maximal torus T acting on M, and M_T be the symplectic quotient $\mu_T^{-1}(0)/T$. For simplicity we also assume the T-action on $\mu_T^{-1}(0)$ is free. Let ρ_T be the corresponding map from $H_T^*(M)$ to $H^*(M_T)$, ω^0 the corresponding reduced symplectic form on M_T. Take $\eta \in H_K^*(M) \subset H_T^*(M)$ and apply the same arguments as the proofs of Lemmas 1 and 2 we get

$$I(\varepsilon) = b\cdot(2\pi)^s C\cdot V(T)\int_{M_T}\rho_T(\nu^2\eta)e^{\varepsilon<F^0,F^0>}e^{\omega^0} + O(e^{-\delta^2/4\varepsilon})$$

where F^0 is the curvature of the principal T-bundle $\mu_T^{-1}(0) \to M_T$ and $b = \pm 1$ is determined by the compatibility of the orientation of $\mu_T^{-1}(0)\times t^*$ with that of M.

Compare the two expressions of $I(\varepsilon)$ for k and for t, we obtain the following formula of Martin which, as pointed out in [GK] and the referee, also follows from [JK], Theorem 8.1:

Lemma 3:

$$\frac{c}{|W|}\int_{M_T}\rho_T(\nu^2\eta)e^{\omega^0} = \int_{M_K}\rho_K(\eta)e^{\omega_0}.$$

Here $c = \pm 1 = a{\cdot}b$ is determined by the compatibility of the orientation of $\mu_T^{-1}(0) \times t^*$ with that of $\mu_K^{-1}(0) \times k^*$.

Before we go further, let us first use a theorem in [BC] to derive a formula in [JK], Proposition 7.1. Let $f(x)$ be an L^1-integrable function in the Euclidean space R^k with inner product $<,>$ and

$$\phi(y) = \int_{R^k} f(x) e^{i<x,y>} dx$$

be its Fourier transform. Note that the definition of Fourier transform in [BC], which we use, is slightly different from the one in [JK], Sect. 3. We then have the following theorem from [BC] (Theorem 37 in pp. 65):

Lemma 4.

$$\lim_{\varepsilon \to 0} \int_{R^k} \phi(y) e^{-i<y,x>} e^{-\varepsilon <y,y>} dy = (2\pi)^k f(x)$$

almost everywhere.

If $f(x)$ is continuous, then the above limit is equal to $f(x)$ everywhere. This exactly fits our purposes. In fact from our localization proccess we know that

$$\int_{M_K} \rho_K(\eta) e^{\omega_0} = \lim_{\varepsilon \to 0} \int_k e^{-\varepsilon|\phi|^2} g(\phi) d\phi = \lim_{\varepsilon \to 0} \int_t e^{-\varepsilon|\psi|^2} h(\psi) d\psi$$

where

$$g(\phi) = a/[(2\pi)^l V(K)] \int_M \eta(\phi) e^{\omega + i\mu_K(\phi)}, \;\; h(\psi) = b/[(2\pi)^s |W| V(T)] \int_M \eta(\psi) \nu(\psi)^2 e^{\omega + i\mu_T(\psi)}$$

are smooth functions on k and t respectively. Let $G(x)$, $H(y)$ be the functions whose Fourier transforms are $g(\phi)$ and $h(\psi)$ respectively. From Lemma 6 below, it is easy to see that they are L^1-integrable. Our localization method and Lemma 4 immediately give us the following formula of [JK], Proposition 7.1:

Lemma 5

$$\int_{M_K} \rho_K(\eta) e^{\omega_0} = (2\pi)^l G(0) = (2\pi)^s H(0).$$

5. On the other hand if we note that $\nu(\psi)^2 \eta(\psi) e^{\omega + i\mu_T(\psi)} \in H_T^*(M)$, and first localize the expression

$$\int_M \nu(\psi)^2 \eta(\psi) e^{\omega + i\mu_T(\psi)}$$

to the fixed point sets $\{F\}$ of T on M by using the localization formula in [AB], [BV], we get

$$I(\varepsilon) = C\int_t e^{-\varepsilon|\psi|^2}\nu(\psi)^2\sum_{\{F\}} e^{i\mu_T(F)(\psi)}\int_F \frac{i_F^*(\eta(\psi)e^{\omega})}{e_F(\psi)}d\psi$$

where e_F is the equivariant Euler class of the normal bundle of F in M and i_F is the corresponding inclusion.

Since the total sum in the ψ-integral is a polynomial in ψ, we can deform the contour of integral, switch the order of integral and sum to get

$$I(\varepsilon) = C\sum_{\{F\}}\int_F\int_{t+i\zeta} e^{-\varepsilon|\psi|^2}\nu(\psi)^2 e^{i\mu_T(F)(\psi)}\frac{i_F^*(\eta(\psi)e^{\omega})}{e_F(\psi)}d\psi$$

where $\zeta = (\zeta_1, \cdots, \zeta_s) \in t$ is a generic point such that there is no pole in the integrand.

Compare this formula with Lemmas 1 and 2, we thus have obtained

Lemma 6:

$$\int_{M_K}\rho_K(\eta)e^{\omega_0} = a\cdot(2\pi)^{-l}C/V(K)\sum_{\{F\}}\lim_{\varepsilon\to 0}\int_{t+i\zeta} e^{-\varepsilon|\psi|^2}\nu(\psi)^2 e^{i\mu_T(F)(\psi)}\int_F\frac{i_F^*(\eta(\psi)e^{\omega})}{e_F(\psi)}d\psi.$$

Here recall that $a = \pm 1$ and $(2\pi)^{-l}C/V(K) = 1/[(2\pi)^s|W|V(T)]$.

To compute the limit in Lemma 6, we only need to deal with terms like

$$\lim_{\varepsilon\to 0}\int_{t+i\zeta} e^{-\varepsilon|\psi|^2+i\mu_T(F)(\psi)}[\psi^J/\beta_I(\psi)]d\psi$$

where $\beta_I = \beta_{i_1}\cdots\beta_{i_N}$ are multiplication of certain linear functionals on t. More precisely

$$\mu_T(F)(\psi) = \sum_j \mu_j(F)\psi^j,\ |\psi|^2 = \sum_j(\psi^j)^2,\ \psi^J = (\psi^1)^{j_1}\cdots(\psi^s)^{j_s},\ \beta_j(\psi) = \sum_{i=1}^s a_{ij}\psi^i$$

where the $a'_{ij}s$ are integers. Change variables and recollect terms, we reduce the computation to

$$\lim_{\varepsilon\to 0}A(\varepsilon)\int_t[(\psi+i\zeta)^J/\beta_I(\psi+i\zeta)]e^{-i<\psi,\alpha>}e^{-\varepsilon|\psi|^2}d\psi$$

where

$$A(\varepsilon) = e^{\varepsilon|\zeta|^2-<\zeta,\alpha>}, \alpha = (2\varepsilon\zeta_1 - b_1, \cdots, 2\varepsilon\zeta_s - b_s)$$

with $b = (b_1, \cdots, b_s)$ and $b_j = \mu_j(F)$ some real numbers. Here we have used the notations

$$< \zeta, b >= \sum_j \zeta_j b_j, \ < \psi, \alpha >= \sum_j \psi^j (2\varepsilon\zeta_j - b_j).$$

Assume the Fourier transform of a function $h(x)$ is equal to $(\psi + i\zeta)^J/\beta_I(\psi + i\zeta)$, then a slight generalization of Lemma 4 following its proof in [BC] compute the above limit which is equal to

$$(2\pi)^s e^{-(\zeta,b)} h(-b).$$

Note that such functions like $h(x)$ are related to the Duistermaat-Heckman measure and have been studied in detail in [GLS].

For example if $s = 1$, by iteration we only need to evaluate

$$\lim_{\varepsilon\to 0} \int_{R+i\zeta} x^{-k} e^{-\varepsilon x^2 + ixb} dx$$

which is obviously 0 for $k \leq 0$. Here R denotes the real line. For $k > 0$, assume $\zeta > 0$. Since for $\mathrm{Im} z > 0$

$$z^{-k} = \frac{(-i)^k}{(k-1)!} \int_0^\infty s^{k-1} e^{izs} ds,$$

let $z = y + i\zeta$ we know that the Fourier transform of

$$h(x) = \begin{cases} \frac{(-i)^k}{(k-1)!} x^{k-1} e^{-x\zeta} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

is $(y + i\zeta)^{-k}$. So the above limit is

$$-2\pi \frac{i^k}{(k-1)!} b^{k-1}, \ \text{for } b < 0$$

and 0 for $b > 0$.

Other ways to compute these integrals are to use residues as in [K] or the method in [Wu]. The interested reader may try his own way for the computation.

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