On Elliptic Genera and Theta-Functions

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Introduction

The main purpose of this paper is to give a simple and unified new proof of the Witten rigidity theorems, which were conjectured by Witten and first proved by Taubes [T], Bott-Taubes [BT], Hirzebruch [H] and Krichever [Kr]. Our proof shows that the modular invariance, which is the intrinsic symmetry of elliptic genera, actually implies their rigidity. Some new properties of elliptic genera and their relationships with theta-functions are also discussed.

We remark that our proof makes essential uses of the new feature of loop groups and loop spaces, the modular invariance. We note that, with the help of the modular group, we can catch the topological information on loop space by simply working on finite dimensional manifold. By developing this idea further, in [Liu1] we have proved the rigidity of the Dirac operator on loop space twisted by higher level loop group representations, while the Witten rigidity theorems are the special cases of level 1. Many topological vanishing theorems are also derived in [Liu1] by refining the argument in this paper, especially an $\hat{\mathfrak{A}}$-vanishing theorem for loop space. In [Liu2] modular invariance is used again to establish a general miraculous cancellation formula, relating the Hirzebruch L-form to certain twisted $\hat{\mathfrak{A}}$-forms, which has many interesting topological results as consequences. These results were anounced in [Liu3].

Let X be a smooth compact spin manifold admitting a circle action and P be an elliptic operator on X commuting with the action. Then both the kernal and the cokernal of P are S^1-modules. The Lefschetz number, or the character-valued index of P at $g \in S^1$, is defined to be

$$F_P(g) = \mathrm{tr}|_g \mathrm{Ker} P - \mathrm{tr}|_g \mathrm{Coker} P \in R(S^1)$$

where $R(S^1)$ is the character ring of the S^1-modules. We say that P is rigid with respect to this S^1-action, if $F_P(g)$ is independent of g.

Example 0.1: the signature operator, denoted by d_s, is rigid. The reason is that both of its kernal and cokernal are subspaces of the deRham cohomology group $H^*(X, \mathbf{R})$ on which S^1 aways induces a trivial action.

Example 0.2: the Dirac operator D is rigid. This is a theorem of Atiyah and Hirzebruch [AH]. To help the reader gain some flavor of rigidity, we give the sketch of their proof. For simplicity let us restrict to the isolated fixed point case.

Let $g = e^{2\pi i t} \in S^1$ be a generator of the action group and $\{p\} \subset X$ be the set of fixed points. Let

$$TX|_p = E_1 \oplus \cdots \oplus E_k, \quad k = \frac{1}{2}\mathrm{dim} X$$

be the decomposition of the tangent bundle into sum of the S^1-invariant 2-planes when restricted to the fixed points. Assume that g acts on E_j by $e^{2\pi i m_j t}$. We call $\{m_j\} \subset \mathbf{Z}$ the exponents of TX at the fixed point p. Choose the orientations of E_j's compatibly with the orientation of X. Then a simple application of the Lefschetz fixed point formula in [AB] and [AS] gives us

$$F_D(g) = \sum_{\{p\}} \prod_{j=1}^{k} \frac{1}{z^{\frac{m_j}{2}} - z^{-\frac{m_j}{2}}},$$

where $z = e^{2\pi i t}$. On the other hand we know that $F_D(g)$ is the character of a finite dimensional virtual S^1-module, so

$$F_D(g) = \sum_{n=-N}^{N} a_n z^n.$$

Comparing the two expressions of $F_D(g)$, we easily see that $F_D(g)$ can be extended to a holomorphic function on $\mathbf{C} \cup \infty$, therefore is a constant. □

Around 1982, motivated by physics, Witten proved the rigidity of the twisted Dirac operator $D \otimes TX$ for compact homogeneous spin manifolds.

Using equivariant cobordism, Landweber and Stong were able to obtain a series of rigidity theorems for odd type semi-free circle actions. Their work, together with those of Ochanine and Chudnovskys, directly lead to the discovery of elliptic genus.

In [W], Witten introduced a series of elements in $K(X)[[q^{\frac{1}{2}}]]$ where $q = e^{2\pi i\tau}$, with τ in the upper half plane, is a parameter. Indices of the Dirac operator twisted by these elements are the signature, $\hat{\mathfrak{A}}$-genus or the Euler characteristic of the loop space of X. Surprisingly the elliptic genus of Landweber-Stong happens to be the index of one of these elliptic operators. Motivated by physics, Witten conjectured that these elliptic operators should be rigid. More precisely let

$$\begin{aligned}
\Theta'_q(TX) &= \otimes_{n=1}^{\infty}\Lambda_{q^n}(TX) \otimes_{m=1}^{\infty} S_{q^m}(TX), \\
\Theta_q(TX) &= \otimes_{n=1}^{\infty}\Lambda_{-q^{n-\frac{1}{2}}}(TX) \otimes_{m=1}^{\infty} S_{q^m}(TX), \\
\Theta_{-q}(TX) &= \otimes_{n=1}^{\infty}\Lambda_{q^{n-\frac{1}{2}}}(TX) \otimes_{m=1}^{\infty} S_{q^m}(TX).
\end{aligned}$$

Here for a vector bundle E,

$$\begin{aligned}
S_t(E) &= 1 + tE + t^2 S^2 E + \cdots, \\
\Lambda_t(E) &= 1 + tE + t^2 \Lambda^2 E + \cdots
\end{aligned}$$

are respectively the symmetric and wedge operations in $K(X)[[t]]$. Note that Θ_{-q} is obtained from Θ_q by replacing $q^{\frac{1}{2}}$ with $-q^{\frac{1}{2}}$.

Furthermore let V be a real vector bundle on X with structure group $\mathrm{Spin}(2l)$ and

$$\begin{aligned}
\Theta'_q(TX|V) &= \otimes_{n=1}^{\infty}\Lambda_{q^n}(V) \otimes_{m=1}^{\infty} S_{q^m}(TX), \\
\Theta_q(TX|V) &= \otimes_{n=1}^{\infty}\Lambda_{-q^{n-\frac{1}{2}}}(V) \otimes_{m=1}^{\infty} S_{q^m}(TX), \\
\Theta_{-q}(TX|V) &= \otimes_{n=1}^{\infty}\Lambda_{q^{n-\frac{1}{2}}}(V) \otimes_{m=1}^{\infty} S_{q^m}(TX), \\
\Theta^*_q(TX|V) &= \otimes_{n=1}^{\infty}\Lambda_{-q^n}(V) \otimes_{m=1}^{\infty} S_{q^m}(TX).
\end{aligned}$$

These Θ's are called the Witten elements. Let $p_1(\cdot)_{S^1}$ denote the first S^1-equivariant Pontrjagin class and $\triangle(V) = \triangle^-(V) \oplus \triangle^+(V)$ be the spinor bundle of V, then we have the

Witten rigidity theorem for spin manifolds:

a) For a spin manifold X, $d_s \otimes \Theta'_q(TX)$, $D \otimes \Theta_q(TX)$ and $D \otimes \Theta_{-q}(TX)$ are rigid.

b) If the action lifts to V and $p_1(V)_{S^1} = p_1(X)_{S^1}$, then $D \otimes \triangle(V) \otimes \Theta'_q(TX|V)$, $D \otimes (\triangle^+(V) - \triangle^-(V)) \otimes \Theta^*_q(TX|V)$, $D \otimes \Theta_q(TX|V)$ and $D \otimes \Theta_{-q}(TX|V)$ are rigid. $\Box$

Witten also made conjectures for almost complex manifolds. Let X be a smooth compact almost complex manifold of complex dimension k with an S^1-action. Let W be a complex vector bundle, i.e. a real vector bundle with complex structure, of complex dimension l, on X. Let $y = e^{2\pi i\alpha}$ be a complex number and

$$\begin{aligned}\Theta^\alpha_q(TX) &= \otimes_{n=0}^\infty \Lambda_{-y^{-1}q^n} T''X \otimes_{n=1}^\infty \Lambda_{-yq^n} T'X \otimes_{n=1}^\infty S_{q^n} T'X \otimes_{n=1}^\infty S_{q^n} T''X,\\ \Theta^\alpha_q(TX|W) &= \otimes_{n=0}^\infty \Lambda_{-y^{-1}q^n} W'' \otimes_{n=1}^\infty \Lambda_{-yq^n} W' \otimes_{n=1}^\infty S_{q^n} T'X \otimes_{n=1}^\infty S_{q^n} T''X\end{aligned}$$

where

$$TX \otimes \mathbf{C} = T'X \oplus T''X, \quad W \otimes \mathbf{C} = W' \oplus W''$$

are the decompositions of the complexified bundles. Here $T'X$ is the holomorphic part and $T''X$ denotes its complex dual. Also let $K = \det T'X$ and $L = \det W'$.

Assume that the S^1-action lifts to W and preserves the complex structures of X and W. Let $\bar\partial$ denote the anti-holomorphic differential operator and N be a positive integer, then we have the

Witten rigidity theorem for almost complex manifolds:

a) For an almost complex manifold X with $c_1(X) \equiv 0(mod N)$, the opreator $\bar\partial \otimes \Theta^\alpha_q(TX)$ is rigid for $y = e^{2\pi i\alpha}$ an N-th root of unity. If $c_1(X) = 0$, then $\bar\partial \otimes \Theta^\alpha_q(TX)$ is rigid for any complex number y.

b) If $p_1(X)_{S^1} = p_1(W)_{S^1}$, $w_2(X) = w_2(W)$ and $c_1(W) \equiv 0(mod N)$, then $\bar\partial \otimes (K^{-1} \otimes L)^{\frac{1}{2}} \otimes \Theta^\alpha_q(TX|W)$ is rigid for $y = e^{2\pi i\alpha}$ an N-th root of unity. If $c_1(W) = 0$, $\bar\partial \otimes (K^{-1} \otimes L)^{\frac{1}{2}} \otimes \Theta^\alpha_q(TX|W)$ is rigid for any complex number y. $\Box$

We actually have more rigidity theorems in this case, see the discussions in Section 2, especially Proposition 2.1.

These theorems can be viewed as the loop space analogues of the rigidity of the signature operator, the $\bar\partial$-operator and the Euler characteristic operator on finite dimensional manifolds. They were first proved by Taubes, then by Bott-Taubes [BT] for spin manifolds, by Hirzebruch [H] for almost complex manifolds with $c_1 \equiv 0(\mathrm{mod}N)$ and by Krichever [Kr] for almost comlex manifolds with $c_1 = 0$. The proof for $D \otimes (\triangle^+(V) - \triangle^-(V)) \otimes \Theta^*_q(TX|V)$

and $\bar{\partial} \otimes (K^{-1} \otimes L)^{\frac{1}{2}} \otimes \Theta_q^\alpha(TX|W)$ is only sketched by Witten and not given by the above authors. The rigidity of those elliptic operators in Proposition 2.1 is first discussed here.

This paper is organized as follows. In Section 1 we prove the rigidity theorems for spin manifolds. In Section 2 we prove the rigidity theorems for almost complex manifolds. In Section 3 we discuss the virtual versions of the above elliptic genera. Technically the virtual versions allow us to unify the arguments in a very clean way and to degenerate elliptic genera to the singular fibers of some elliptic modular surfaces. In these sections a key role is played by the classical Jacobi theta-functions.

We construct elliptic genera of level 1 in Section 4, for both spin manifolds and almost complex manifolds with $c_1 \equiv 0(\mathrm{mod} N)$. This is a try to solve a problem of Landweber in [La1]. But these genera are not multiplicative. Some generalizations of the rigidity theorems and some general remarks about elliptic genera are also given in this section. Then in Section 5 we extend the arguments to the general fixed point case by verifying the transformation formulas needed in the above sections. In Section 6 we collect some results relating the universal elliptic genera considered by Landweber-Stong, Ochanine and Witten to some classical elliptic functions. We find that the characteristic series of these universal elliptic genera have been very well studied in pure elliptic function theory. Especially interesting is that the characteristic series of the three universal elliptic genera are exactly the three root functions in [DV]. We note that Bott-Taubes [BT] and Hirzebruch [HBT] have also briefly discussed this point from slightly different point of view. One can see from our discussion that the rigidity, the functional equations and many other properties of the three universal elliptic genera are actually the consequences of their theta-function expressions.

As in [BT] and [H], we also use the Atiyah-Bott-Segal-Singer Lefschetz fixed point formula, but replace the technical transfer argument by considering modular transformations. It is interesting to note that in our proof we do not need to consider any local behavior of the exponents of the fixed points which is essential to the argument in [BT], [H] and [Kr]. The topological conditions in the rigidity theorems are used to show that the elliptic operators in there all satisfy some modular properties under the actions of $SL_2(\mathbf{Z})$, which are much easier to verify. These modular properties are the essential reasons for the rigidity. This observation makes it possible for us to prove that, under some natural condition, the Dirac operator on loop space twisted by higher level loop group representations are also rigid. Note that all of the elliptic operators discussed here, except those generalizations in Section 5 which are some special higher level cases, are of level 1 (see [Liu1]). This brings the characters of affine Lie algebras into play. In some sense our proof gives some flavor of infinite dimensional geometry. More precisely we can say that the geometry of loop space should be some kind of 'modular geometry'. We expect an explanation of our proof from physics, especially from conformal field theory.

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1 The Proof of the Rigidity Theorem for Spin Manifolds

In this section, X is a compact smooth spin manifold of dimension $2k$ with an S^1-action and V is an equivariant spin bundle of rank $2l$ on it. We first consider the isolated fixed point case. Let

$$\begin{aligned}
\theta_3(v,\tau) &= c\cdot\prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}}e^{2\pi iv})\prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}}e^{-2\pi iv}) \\
\theta_2(v,\tau) &= c\cdot\prod_{n=1}^{\infty}(1-q^{n-\frac{1}{2}}e^{2\pi iv})\prod_{n=1}^{\infty}(1-q^{n-\frac{1}{2}}e^{-2\pi iv})
\end{aligned}$$

$$
\begin{aligned}
\theta_1(v,\tau) &= c\cdot q^{\frac{1}{8}} e^{\pi i v}\prod_{n=1}^{\infty}(1+q^n e^{2\pi i v})\prod_{n=0}^{\infty}(1+q^n e^{-2\pi i v}) \\
\theta(v,\tau) &= c\cdot q^{\frac{1}{8}} 2\sin\pi v\prod_{n=1}^{\infty}(1-q^n e^{2\pi i v})\prod_{n=1}^{\infty}(1-q^n e^{-2\pi i v})
\end{aligned}
$$

be the classical Jacobi theta-functions (see [Ch]), where $c=\prod_{n=1}^{\infty}(1-q^n)$ and $q=e^{2\pi i\tau}$ with $\tau\in\mathbf{H}$, the upper half plane. Let $\{m_j\}$, as in the introduction, be the exponents of TX and

$$
\begin{aligned}
F_{d_s}(t,\tau) &= i^{-k}\sum_p\prod_{j=1}^{k}\frac{\theta_1(m_jt,\tau)}{\theta(m_jt,\tau)}, \\
F_D(t,\tau) &= i^{-k}\sum_p\prod_{j=1}^{k}\frac{\theta_2(m_jt,\tau)}{\theta(m_jt,\tau)}, \\
F_{-D}(t,\tau) &= i^{-k}\sum_p\prod_{j=1}^{k}\frac{\theta_3(m_jt,\tau)}{\theta(m_jt,\tau)}.
\end{aligned}
$$

Then the Lefschetz fixed point formula of Atiyah-Bott-Segal-Singer tells us that

$$
\begin{aligned}
F_{d_s}(t,\tau) &= \text{the Lefschetz number of } d_s\otimes\Theta'_q(TX), \\
F_D(t,\tau) &= q^{-\frac{k}{8}}\cdot\text{the Lefschetz number of } D\otimes\Theta_q(TX), \\
F_{-D}(t,\tau) &= q^{-\frac{k}{8}}\cdot\text{the Lefschetz number of } D\otimes\Theta_{-q}(TX).
\end{aligned}
$$

Let

$$V|p=L_1\oplus\cdots\oplus L_l$$

be the corresponding equivariant decomposition of V restricted to p. We denote the exponents of V at p by $\{n_\nu\}$, i.e. g acts on L_ν by $e^{2\pi i n_\nu t}$. Let us write the Lefschetz numbers of $D\otimes\triangle(V)\otimes\Theta'_q(TX|V)$, $D\otimes\Theta_q(TX|V)$, $D\otimes\Theta_{-q}(TX|V)$ and $D\otimes(\triangle^+(V)-\triangle^-(V))\otimes\Theta^*_q(TX|V)$ as $q^{\frac{k-l}{8}}F^V_{d_s}(t,\tau)$, $q^{\frac{k}{8}}\cdot F^V_D(t,\tau)$, $q^{\frac{k}{8}}\cdot F^V_{-D}(t,\tau)$ and $q^{\frac{k-l}{8}}\cdot F^V_{D^*}(t,\tau)$ respectively. Then we have

$$
F^V_{d_s}(t,\tau) = i^{-k}\sum_p\frac{\prod_{\nu=1}^{l}\theta_1(n_\nu t,\tau)}{\prod_{m=1}^{k}\theta(m_jt,\tau)},
$$

$$\begin{aligned} F_D^V(t,\tau) &= i^{-k}\sum_p \frac{\prod_{\nu=1}^l \theta_2(n_\nu t,\tau)}{\prod_{j=1}^k \theta(m_jt,\tau)}, \\ F_{-D}^V(t,\tau) &= i^{-k}\sum_p \frac{\prod_{nu=1}^l \theta_3(n_\nu t,\tau)}{\prod_{j=1}^k \theta(m_jt,\tau)}, \\ F_{D^*}^V(t,\tau) &= i^{l-k}\sum_p \frac{\prod_{\nu=1}^l \theta(n_\nu t,\tau)}{\prod_{j=1}^k \theta(m_jt,\tau)}. \end{aligned}$$

Considered as functions of (t,τ), we can obviously extend these F's and F^V's to meromorphic functions on $\mathbf{C}\times\mathbf{H}$. The Witten rigidity theorems are equivalent to that these F's and F^V's are independent of t. First we have the following lemma,

Lemma 1.1: a) *$F_{d_s}(t,\tau)$, $F_D(t,\tau)$ and $F_{-D}(t,\tau)$ are invariant under the action*

$$U: \quad t \to t + a\tau + b$$

for $a, b \in 2\mathbf{Z}$.

b) If $p_1(V)_{S^1} = p_1(X)_{S^1}$, then $F_{d_s}^V(t,\tau)$, $F_D^V(t,\tau)$, $F_{-D}^V(t,\tau)$ and $F_{D^*}^V(t,\tau)$ are invariant under U.

Proof: We have the following transformation formulas of theta-functions (see [Ch])

$$\begin{aligned} \theta(t+1,\tau) = -\theta(t,\tau),\ \theta(t+\tau,\tau) = -q^{-\frac{1}{2}}e^{-2\pi it}\theta(t,\tau); \\ \theta_1(t+1,\tau) = -\theta_1(t,\tau),\ \theta_1(t+\tau,\tau) = q^{-\frac{1}{2}}e^{-2\pi it}\theta_1(t,\tau); \\ \theta_2(t+1,\tau) = \theta_2(t,\tau),\ \theta_2(t+\tau,\tau) = -q^{-\frac{1}{2}}e^{-2\pi it}\theta_2(t,\tau); \\ \theta_3(t+1,\tau) = \theta_3(t,\tau),\ \theta_3(t+\tau,\tau) = q^{-\frac{1}{2}}e^{-2\pi it}\theta_3(t,\tau). \end{aligned}$$

Using these formulas one can easily check that for $(a,b)\in(2\mathbf{Z})^2$,

$$\theta_\nu(m_j(t+a\tau+b),\tau) = e^{-\pi i m_j^2(a^2\tau+2at)}\theta_\nu(m_jt,\tau)$$

for $\theta_\nu = \theta, \theta_1, \theta_2, \theta_3$ which implies a) of the lemma. For b) one only needs to note that the condition $p_1(X)_{S^1} = p_1(V)_{S^1}$ implies that

$$\sum_j m_j^2 = \sum_\nu n_\nu^2$$

for every fixed point. In fact when localized at one fixed point p, $p_1(X)_{S^1}$ and $p_1(V)_{S^1}$ have expressions $\sum_j m_j^2 \cdot u^2$ and $\sum_\nu n_\nu^2 \cdot u^2$ respectively. Here u is the generator of the equivariant cohomology of p, $H_{S^1}^*(p, \mathbf{Z})$. □

This lemma tells us that for fixed τ these F's and F^V's are meromorphic functions on the torus $\mathbf{C}/2\mathbf{Z} \times 2\mathbf{Z}\tau$. Therefore to get the rigidity we only need to prove that they are holomorphic in t. We will actually prove that they are holomorphic in (t, τ) on $\mathbf{C} \times \mathbf{H}$.

Given

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}),$$

Define its modular transformation on $\mathbf{C} \times \mathbf{H}$ by

$$g(t, \tau) = (\frac{t}{c\tau + d}, \frac{a\tau + b}{c\tau + d}).$$

This defines a group action. Obviously two generators of $SL_2(\mathbf{Z})$,

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

act by

$$S(t, \tau) = (\frac{t}{\tau}, -\frac{1}{\tau}), \; T(t, \tau) = (t, \tau + 1).$$

Lemma 1.2: *a) Under the actions of S and T, we have, up to certaim multiples of 8-th root of unity,*

$$F_{d_s}(\tfrac{t}{\tau}, -\tfrac{1}{\tau}) = F_D(t, \tau), \; F_{d_s}(t, \tau + 1) = F_{d_s}(t, \tau);$$
$$F_{-D}(\tfrac{t}{\tau}, -\tfrac{1}{\tau}) = F_{-D}(t, \tau), \; F_D(t, \tau + 1) = F_{-D}(t, \tau).$$

b) If $p_1(X)_{S^1} = p_1(V)_{S^1}$,

$$F_{d_s}^V(\tfrac{t}{\tau}, -\tfrac{1}{\tau}) = \tau^{\frac{(l-k)}{2}} F_D^V(t, \tau), \; F_{d_s}^V(t, \tau + 1) = F_D^V(t, \tau);$$
$$F_{-D}^V(\tfrac{t}{\tau}, -\tfrac{1}{\tau}) = \tau^{\frac{l-k}{2}} F_{-D}^V(t, \tau), \; F_D^V(t, \tau + 1) = F_{-D}^V(t, \tau);$$
$$F_{D^*}^V(\tfrac{t}{\tau}, -\tfrac{1}{\tau}) = \tau^{\frac{(l-k)}{2}} F_{D^*}^V(t, \tau), \; F_{D^*}^V(t, \tau + 1) = F_{D^*}^V(t, \tau).$$

Proof: We have the following transformation formulas of the Jacobi theta-functions

$$\begin{aligned}
\theta(\frac{t}{\tau},-\frac{1}{\tau}) &= \frac{1}{i}\sqrt{\frac{\tau}{i}}e^{\frac{\pi i t^2}{\tau}}\theta(t,\tau),\ \theta(t,\tau+1)=e^{\frac{\pi i}{4}}\theta(t,\tau);\\
\theta_1(\frac{t}{\tau},-\frac{1}{\tau}) &= \sqrt{\frac{\tau}{i}}e^{\frac{\pi i t^2}{\tau}}\theta_2(t,\tau),\ \theta_1(t,\tau+1)=e^{\frac{\pi i}{4}}\theta_1(t,\tau);\\
\theta_2(\frac{t}{\tau},-\frac{1}{\tau}) &= \sqrt{\frac{\tau}{i}}e^{\frac{\pi i t^2}{\tau}}\theta_1(t,\tau),\ \theta_2(t,\tau+1)=\theta_3(t,\tau);\\
\theta_3(\frac{t}{\tau},-\frac{1}{\tau}) &= \sqrt{\frac{\tau}{i}}e^{\frac{\pi i t^2}{\tau}}\theta_3(t,\tau),\ \theta_3(t,\tau+1)=\theta_2(t,\tau).
\end{aligned}$$

See [Ch].

The lemma easily follows from these formulas. Note that for b) one needs the condition $\sum_j m_j^2 = \sum_\nu n_\nu^2$ for each fixed point which is a consequence of $p_1(V)_{S^1} = p_1(X)_{S^1}$ as discussed above. □

This lemma tells us that modulo some constants the two complex vector spaces spanned by the F's and by the F^V's are stable under the actions of $SL_2(\mathbf{Z})$. These give rise to projective representations of $SL_2(\mathbf{Z})$.

The following lemma is actually Proposition 6.1 in [BT] and is proved by an argument similar to the discussion of Example 2.

Lemma 1.3: *If X and V are spin, then all of the F's and F^V's above are holomorphic in (t,τ) for $t \in \mathbf{R}$ and $\tau \in \mathbf{H}$.* □

Proof: Let $z = e^{2\pi i t}$. In the following we view these F's and F^V's as meromorphic functions of two complex variables (z,q). Note that their possible poles on $|z| = 1$ are all independent of q.

By looking at each term in the summations of the fixed point formulas, one can easily see that, in the domain

$$\mathbf{D}_N: \quad |q|^{\frac{1}{N}} < |z| < |q|^{-\frac{1}{N}},\ 0<|q|<1$$

where $N = \max_{p,j}|m_j|$, these F's and F^V's have expansions of the form

$$q^{-\frac{a}{8}}\sum_{n\geq 0}^{\infty} b_n(z)q^n$$

where a is an integer and $\{b_n(z)\}$ are rational functions of z which can only have poles on the unit circle $|z| = 1 \subset \mathbf{D}_N$. On the other hand one can expand the Θ's in the introduction into formal power series of the form

$$\sum_{n\geq 0} R_n q^n$$

with $R_n \in K(X)$. Apply Lefschetz fixed point formula to each R_n, we get that, for $|z|=1$, each $b_n(z)$ is the Lefschetz number of an elliptic operator. This implies that

$$b_n(z) = \sum_{m=-N(n)}^{N(n)} a_{mn} z^m,$$

for $N(n)$ some positive integer depending on n. Since both sides are analytic functions of z, this equality holds for any $z \in \mathbf{C}$.

On the other hand, multiply these F's and F^V's by

$$f(z) = \prod_p \prod_{j=1}^{k} (1 - z^{m_j}),$$

we get holomorphic functions, therefore convergent power series expansions of the form

$$q^{-\frac{a}{8}} \sum_{n\geq 0}^{\infty} c_n(z) q^n$$

with $\{c_n(z)\}$ polynomial functions, in $\mathbf{D}_N$. Compare the above two expansions, one gets that for each n the equality

$$c_n(z) = f(z) \cdot b_n(z)$$

holds. So by the weierstrass preparation theorem or the weak Hilbert Nullstellensatz (see Chapter 1 of [GH]), we deduce that

$$q^{-\frac{a}{8}} \sum_{n\geq 0}^{\infty} b_n(z) q^n = q^{-\frac{a}{8}} \sum_{n\geq 0}^{\infty} \left(\frac{c_n(z)}{f(z)}\right) q^n$$

is holomorphic in $\mathbf{D}_N$. Obviously the domain in Lemma 1.3 is contained in $\mathbf{D}_N$. $\square$

For a different proof of this lemma, see [HBT], Appendix III. Also H. Miller informed me of a proof by M. Ando. Recently S. Ochanine told me that he has a different argument.

For the rigidity of the F's, this lemma is the only essential place where we need the spin condition on X, which ensures the existence of the Dirac operator D, therefore the existence of $D \otimes \Theta_q(TX)$ and $D \otimes \Theta_{-q}(TX)$, to prove that these F's do not have poles for $t \in \mathbf{R}$. While for the F^V's, the condition $p_1(V)_{S^1} = p_1(X)_{S^1}$ is crucially needed in Lemmas 1.1 and 1.2. But the spin condition on V is only essentially needed in Lemma 1.3.

Now we are ready to prove *the Witten rigidity theorem for spin manifolds:*

As easily seen from their expressions, the general polar divisors of the F's and F^V's in $\mathbf{C} \times \mathbf{H}$ are of the form $t = \frac{n(c\tau+d)}{l}$ with n, c, d, l integers and $(c, d) = 1$. We can always find integers a, b such that $ad - bc = 1$. Then take $g = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in SL_2(\mathbf{Z})$ which induces the action:

$$F(g(t,\tau)) = F(\frac{t}{-c\tau + a}, \frac{d\tau - b}{-c\tau + a}),$$

where F is one of the F's or F^V's. It is easy to see that, if $t = \frac{n(c\tau+d)}{l}$ is the polar divisor of $F(t, \tau)$, one polar divisor of $F(g(t, \tau))$ is given by

$$\frac{t}{-c\tau + a} = \frac{n}{l}(c\frac{d\tau - b}{-c\tau + a} + d)$$

which exactly gives $t = \frac{n}{l}$. This contradicts Lemma 1.2 and Lemma 1.3. Since by Lemma 1.2, up to some constant, $F(g(t, \tau))$ is still one of the F's or F^V's, and by Lemma 1.3 all of them are holomorphic for $t \in \mathbf{R}$.

This, together with the transformation formulas for general fixed point case in Section 5, proves the Witten rigidity theorem for spin manifolds. $\Box$

We remark that for the definition of $D \otimes \triangle(V) \otimes \Theta'_q(TX|V)$ we only need $w_2(X) = w_2(V)$, but for its rigidity we need the condition $w_2(X) = w_2(V) = 0$. The reason is that one needs this stronger condition to define its modular transformations $D \otimes \Theta_q(TX|V)$ and $D \otimes \Theta_{-q}(TX|V)$. For $D \otimes (\triangle^+(V) - \triangle^-(V)) \otimes \Theta_q^*(TX|V)$, the condition $w_2(X) = w_2(V)$ is enough, since it is invariant under the action of $SL_2(\mathbf{Z})$.

2 Almost Complex Manifolds

Let X be a compact smooth almost complex manifold with an S^1-action and W be a complex vector bundle on it with $w_2(X) = w_2(W)$. Note that this is equivalent to the condition $c_1(W) \equiv c_1(X) (\mathrm{mod} 2)$. Assume that the S^1-action lifts to W and preserves the complex structures of X and W. Let us consider the isolated fixed point case first.

Let $\{m_j\}$ be the exponents of $T'X$ and $\{n_\nu\}$ be the exponents of W'. Here $T'X$ and W' are as in the introduction. For convenience we take $\alpha = \frac{1}{N}$ and denote the Lefschetz numbers of $\bar{\partial} \otimes \Theta_q^\alpha(TX)$ and $\bar{\partial} \otimes (K^{-1} \otimes L)^{\frac{1}{2}} \otimes \Theta_q^\alpha(TX|W)$ by $F^\alpha(t, \tau)$ and $q^{\frac{k-l}{8}} F_W^\alpha(t, \tau)$ respectively, then

$$F^\alpha(t, \tau) = y^{-\frac{k}{2}} \sum_p \prod_{j=1}^k \frac{\theta(m_j t + \alpha, \tau)}{\theta(m_j t, \tau)},$$

$$F_W^\alpha(t, \tau) = y^{-\frac{l}{2}} \sum_p \frac{\prod_{\nu=1}^l \theta(n_\nu t + \alpha, \tau)}{\prod_{j=1}^k \theta(m_j t, \tau)}.$$

Recall that $\mathrm{dim} X = k$, $\mathrm{dim} W = l$ and $y = e^{2\pi i \alpha}$.

First note that, as in the spin case, $p_1(W)_{S^1} = p_1(X)_{S^1}$ implies the equality $\sum_\nu n_\nu^2 = \sum_j m_j^2$ for each fixed point.

Lemma 2.1: *$F^\alpha(t, \tau)$ and, if $p_1(X)_{S^1} = p_1(W)_{S^1}$ also $F_W^\alpha(t, \tau)$ are invariant under the action of*

$$t \to t + a\tau + b$$

for $a, b \in N\mathbf{Z}$.

Proof: Using the transformation formulas of the Jacobi theta-functions and the equality $\sum_j m_j^2 = \sum_\nu n_\nu^2$ for each fixed point, it is easy to prove that

$$\frac{\prod_{\nu=1}^l \theta(n_\nu(t + \tau) + \alpha, \tau)}{\prod_{j=1}^k \theta(m_j(t + \tau), \tau)} = y^{-\sum_\nu n_\nu} \frac{\prod_{\nu=1}^l \theta(n_\nu t + \alpha, \tau)}{\prod_{j=1}^k \theta(m_j t, \tau)}.$$

Since by the assumption, $y^N = 1$, one has $F^\alpha(t + N\tau, \tau) = F(t, \tau)$ and $F_W^\alpha(t + N\tau, \tau) = F_W^\alpha(t, \tau)$.

The proofs of $F^\alpha(t + 1, \tau) = F^\alpha(t, \tau)$ and $F_W^\alpha(t + 1, \tau) = F_W^\alpha(t, \tau)$ are very simple. $\square$

Let us introduce two elements $\Theta_q^{\alpha(c\tau+d)}(TX)$ and $\Theta_q^{\alpha(c\tau+d)}(TX|W)$ which are the same as $\Theta_q^\alpha(TX)$ and $\Theta_q^\alpha(TX|W)$, but replacing the parameter α by

$\alpha(c\tau+d)$. Let

$$\begin{array}{rcl} F^\alpha(t,\tau)^c & = & \text{the Lefschetz number of } \bar\partial\otimes K^{c\alpha}\otimes\Theta_q^{\alpha(c\tau+d)}(TX); \\ F^\alpha_W(t,\tau)^c & = & q^{\frac{l-k}{8}}\cdot\text{the Lefschetz number of} \\ \bar\partial & & \otimes L^{c\alpha}\otimes(K^{-1}\otimes L)^{\frac{1}{2}}\otimes\Theta_q^{\alpha(c\tau+d)}(TX|W). \end{array}$$

Recall that $K=\det T'X,\ L=\det W'$. It is easy to show that

$$F^\alpha(t,\tau)^c=y^{-\frac{k}{2}}\sum_p e^{2\pi ic\alpha\sum_j m_jt}\prod_{j=1}^k\frac{\theta(m_jt+\alpha(c\tau+d),\tau)}{\theta(m_jt,\tau)},$$

$$F^\alpha_W(t,\tau)^c=y^{-\frac{l}{2}}\sum_p e^{2\pi ic\alpha\sum_\nu n_\nu t}\frac{\prod_{\nu=1}^l\theta(n_\nu t+\alpha(c\tau+d),\tau)}{\prod_{j=1}^k\theta(m_jt,\tau)}.$$

Lemma 2.2: *Under the action of* $g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbf{Z})$, *we have*

a)

$$F^\alpha(g(t,\tau))=e^{k\pi ic(c\tau+d)\alpha^2}\cdot F^\alpha(t,\tau)^c;$$

b) if $p_1(X)_{S^1}=p_1(W)_{S^1}$,

$$F^\alpha_W(g(t,\tau))=e^{l\pi ic(c\tau+d)\alpha^2}\cdot F^\alpha_W(t,\tau)^c.$$

Proof: Again by the transformation formulas of the theta-functions, we have

$$\begin{array}{rcl} \dfrac{\theta(\frac{m_jt}{c\tau+d}+\alpha,\frac{a\tau+b}{c\tau+d})}{\theta(\frac{m_jt}{c\tau+d},\frac{a\tau+b}{c\tau+d})} & = & e^{2m_j\pi ic\alpha t+\pi ic(c\tau+d)\alpha^2}\dfrac{\theta(m_jt+\alpha(c\tau+d),\tau)}{\theta(m_jt,\tau)}, \\ \theta(t+\tau,\tau) & = & q^{-\frac{1}{2}}e^{-2\pi it}\theta(t,\tau), \\ \theta(t+1,\tau) & = & -\theta(t,\tau). \end{array}$$

This obviously gives a) of the lemma.

With the condition $\sum_j m_j^2=\sum_\nu n_\nu^2$ for each fixed point, the case for $F^\alpha_W(t,\tau)$ is completely the same. □

Since $F^\alpha(t,\tau)^c$ and $F^\alpha_W(t,\tau)^c$ are the Lefschetz numbers of some elliptic operators, the same proof as that of Lemma 1.3 gives the following:

Lemma 2.3: *a) If $c_1(X) \equiv 0(modN)$, then $F^\alpha(t,\tau)^c$ is holomorphic in (t,τ) for $t \in \mathbf{R}$ and $\tau \in \mathbf{H}$.*

b) If $c_1(W) \equiv 0(modN)$ and $w_2(X) = w_2(W)$, then $F^\alpha_W(t,\tau)^c$ is holomorphic in the same domain. $\Box$

For the rigidity of $F^\alpha(t,\tau)$ this is the only essential place where we need the topological condition $c_1(X) \equiv 0(\mathrm{mod}N)$ which insures the existence of $K^{c\alpha}$, therefore the holomorphicity of $F^\alpha(t,\tau)^c$ for $t \in \mathbf{R}$. While for $F^\alpha_W(t,\tau)$ the condition on equivariant Pontrjagin classes are used in Lemmas 2.1 and 2.2. But the conditions on c_1 and w_2 are only essentially needed in Lemma 2.3.

It is easy to see that the factor $e^{n\pi ic(c\tau+d)\alpha^2}$, where $n = l$ or k, does not affect the argument of the proof of the rigidity theorems for spin manifolds, by the same method as in Section 1 we can prove the Witten rigidity theorems for almost complex manifolds from the above lemmas.

The first terms of the formal power series expansions of $F^\alpha(t,\tau)^c$ and $F^\alpha_W(t,\tau)^c$ in q are of the forms $\bar\partial \otimes K^{\frac{s}{N}}$ and $\bar\partial \otimes (K^{-1} \otimes L)^{\frac{1}{2}} \otimes L^{\frac{s}{N}}$ respectively with $-N < s < 1$. So under the topological condition in the rigidity theorem, we have the rigidity of

$$\bar\partial \otimes K^{\frac{s}{N}}, \quad \bar\partial \otimes (K^{-1} \otimes L)^{\frac{1}{2}} \otimes L^{\frac{s}{N}}$$

for $-N < s < 1$. The rigidity of $\bar\partial \otimes K^{\frac{s}{N}}$ for $-N < s < 1$ was first proved by Hattori ([Ha]).

Now let X and W be as above. Let us consider the following elements in $K(X)[[q^{\frac{1}{2}}]] \otimes \mathbf{C}$:

$$\begin{aligned}
P^\alpha_q(TX) &= \otimes_{n=0}^\infty \Lambda_{y^{-1}q^n} T''X \otimes_{n=1}^\infty \Lambda_{yq^n} T'X \otimes_{n=1}^\infty S_{q^n} T'X \otimes_{n=1}^\infty S_{q^n} T''X, \\
Q^\alpha_q(TX) &= \otimes_{n=1}^\infty \Lambda_{-y^{-1}q^{n-\frac{1}{2}}} T''X \otimes_{n=1}^\infty \Lambda_{-yq^{n-\frac{1}{2}}} T'X \otimes_{n=1}^\infty S_{q^n} T'X \otimes_{n=1}^\infty S_{q^n} T''X, \\
R^\alpha_q(TX) &= \otimes_{n=1}^\infty \Lambda_{y^{-1}q^{n-\frac{1}{2}}} T''X \otimes_{n=1}^\infty \Lambda_{yq^{n-\frac{1}{2}}} T'X \otimes_{n=1}^\infty S_{q^n} T'X \otimes_{n=1}^\infty S_{q^n} T''X;
\end{aligned}$$

and more generally let

$$\begin{aligned}
P^\alpha_q(TX|W) &= \otimes_{n=0}^\infty \Lambda_{y^{-1}q^n} W'' \otimes_{n=1}^\infty \Lambda_{yq^n} W' \otimes_{n=1}^\infty S_{q^n} T'X \otimes_{n=1}^\infty S_{q^n} T''X, \\
Q^\alpha_q(TX|W) &= \otimes_{n=1}^\infty \Lambda_{-y^{-1}q^{n-\frac{1}{2}}} W'' \otimes_{n=1}^\infty \Lambda_{-yq^{n-\frac{1}{2}}} W' \otimes_{n=1}^\infty S_{q^n} T'X \otimes_{n=1}^\infty S_{q^n} T''X, \\
R^\alpha_q(TX|W) &= \otimes_{n=1}^\infty \Lambda_{y^{-1}q^{n-\frac{1}{2}}} W'' \otimes_{n=1}^\infty \Lambda_{yq^{n-\frac{1}{2}}} W' \otimes_{n=1}^\infty S_{q^n} T'X \otimes_{n=1}^\infty S_{q^n} T''X.
\end{aligned}$$

Here still $y = e^{2\pi i\alpha}$ is a complex number.

The following proposition can be viewed as a combination of the rigidity theorems for spin manifolds and for almost complex manifolds.

Proposition 2.1: *a) If $w_2(X) = 0$ and $c_1(X) \equiv 0(mod N)$, let D be the Dirac operator on X, then $\bar{\partial} \otimes P_q^\alpha(TX)$, $D \otimes Q_q^\alpha(TX)$ and $D \otimes R_q^\alpha(TX)$ are rigid for y an N-th root of unity. If $c_1(X) = 0$, they are rigid for any complex number y.*

b) If $w_2(X) = w_2(W) = 0$, $c_1(W) \equiv 0(mod N)$ and $p_1(W)_{S^1} = p_1(X)_{S^1}$, then $\bar{\partial} \otimes (K^{-1} \otimes L)^{\frac{1}{2}} \otimes P_q^\alpha(TX|W)$, $D \otimes Q_q^\alpha(TX|W)$ and $D \otimes R_q^\alpha(TX|W)$ are rigid for y an N-th root of unity. If $c_1(W) = 0$, they are rigid for any complex number y.

Proof: We only sketch the proof of a). For convenience we still take $\alpha = \frac{1}{N}$. Let $P^\alpha(t,\tau)$, $Q^\alpha(t,\tau)$ and $R^\alpha(t,\tau)$ be the Lefschetz numbers of $\bar{\partial} \otimes P_q^\alpha(TX)$, $q^{\frac{k}{8}} D \otimes Q_q^\alpha(TX)$ and $q^{\frac{k}{8}} D \otimes R_q^\alpha(TX)$ respectively, then

$$\begin{aligned}
P^\alpha(t,\tau) &= y^{-\frac{k}{2}} \sum_p \prod_{j=1}^k \frac{\theta_1(m_j t + \alpha, \tau)}{\theta(m_j t, \tau)}, \\
Q^\alpha(t,\tau) &= \sum_p \prod_{j=1}^k \frac{\theta_2(m_j t + \alpha, \tau)}{\theta(m_j t, \tau)}, \\
R^\alpha(t,\tau) &= \sum_p \prod_{j=1}^k \frac{\theta_3(m_j t + \alpha, \tau)}{\theta(m_j t, \tau)}.
\end{aligned}$$

By using the transformation formulas of the theta-functions, one can easily check that, under the assumptions of the proposition, the action of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$, upto a complex constant, has the following effect:

$$P^\alpha(g(t,\tau)) = e^{k\pi i c(c\tau+d)\alpha^2} \cdot S^\alpha(t,\tau)^c$$

where $S^\alpha(t,\tau)^c$ is the Lefschetz number of one of the following three elliptic operators: $\bar{\partial} \otimes K^{c\alpha} \otimes P_q^{\alpha(c\tau+d)}(TX)$, $D \otimes K^{c\alpha} \otimes Q_q^{\alpha(c\tau+d)}(TX)$ and $D \otimes K^{c\alpha} \otimes R_q^{\alpha(c\tau+d)}(TX)$. Similarly for $Q^\alpha(t,\tau)$ and $R^\alpha(t,\tau)$. With this one can use the same arguments as in the spin case to get the rigidity. $\square$

If $c_1 = 0$, then K and L are trivial. Lemmas 2.1, 2.2 and 2.3 hold for any positive integer N. So $\bar{\partial} \otimes \Theta_q^\alpha(TX)$ and $\bar{\partial} \otimes (K^{-1} \otimes L)^{\frac{1}{2}} \otimes \Theta_q^\alpha(TX|W)$ and their modular transformations are rigid for $c_1 = 0$ manifolds or bundles and any positive integer N, or equivalently for any root of unity y. Since roots of unity are dense in the unit circle, we get that the rigidity theorem holds for any complex number y in the unit circle, therefore for any complex number, since F^α and F^α_W are analytic functions of y. We omit the discussion of the $c_1 = 0$ cases in Proposition 2.1.

3 Virtual Elliptic Genera

In this section we consider the virtual versions of the elliptic operators in the Witten rigidity theorems. We will see that all of the Lefschetz numbers in this section satisfy transformation formulas of the same type. Also note that in Sections 1 and 2 we have to multiply the Lefschetz numbers by some 'anomaly factors', i.e. some rational powers of q, to get modularity. All these anomaly factors will disappear here. From the point of view of modularity, virtual elliptic genera are much more natural than their originals. But we can not see how to derive them directly from the geometry of loop spaces. A natural derivation may not need the regularization proceedure by zeta-function.

Let X and V be as before, i.e. X is a spin manifold with an S^1-action and V is an equivariant vector bundle with structure group $\mathrm{Spin}(2l)$. Assume $p_1(V)_{S^1} = p_1(X)_{S^1}$. Let

$$\begin{aligned}
\Theta'_q(TX)_v &= \otimes_{n=1}^\infty \Lambda_{q^n}(TX - \mathrm{dim}X) \otimes_{m=1}^\infty S_{q^m}(TX - \mathrm{dim}X),\\
\Theta_q(TX)_v &= \otimes_{n=1}^\infty \Lambda_{-q^{n-\frac{1}{2}}}(TX - \mathrm{dim}X) \otimes_{m=1}^\infty S_{q^m}(TX - \mathrm{dim}X),\\
\Theta_{-q}(TX)_v &= \otimes_{n=1}^\infty \Lambda_{q^{n-\frac{1}{2}}}(TX - \mathrm{dim}X) \otimes_{m=1}^\infty S_{q^m}(TX - \mathrm{dim}X);
\end{aligned}$$

and more generally let

$$\begin{aligned}
\Theta'_q(TX|V)_v &= \otimes_{n=1}^\infty \Lambda_{q^n}(V - \mathrm{dim}V) \otimes_{m=1}^\infty S_{q^m}(TX - \mathrm{dim}X),\\
\Theta_q(TX|V)_v &= \otimes_{n=1}^\infty \Lambda_{-q^{n-\frac{1}{2}}}(V - \mathrm{dim}V) \otimes_{m=1}^\infty S_{q^m}(TX - \mathrm{dim}X),\\
\Theta_{-q}(TX|V)_v &= \otimes_{n=1}^\infty \Lambda_{q^{n-\frac{1}{2}}}(V - \mathrm{dim}V) \otimes_{m=1}^\infty S_{q^m}(TX - \mathrm{dim}X),\\
\Theta^*_q(TX|V)_v &= \otimes_{n=1}^\infty \Lambda_{-q^n}(V - \mathrm{dim}V) \otimes_{m=1}^\infty S_{q^m}(TX - \mathrm{dim}X).
\end{aligned}$$

These elements are obtained by simply replacing TX and V in the Witten elements in the introduction by their virtual versions $(TX - \mathrm{dim}X)$ and $(V - \mathrm{dim}V)$ respectively.

Let us denote the Lefsctetz numbers of $2^{-k}d_s\otimes\Theta'_q(TX)_v$, $D\otimes\Theta_q(TX)_v$ and $D\otimes\Theta_{-q}(TX)_v$ by $F_{d_s}(t,\tau)_v$, $F_D(t,\tau)_v$ and $F_{-D}(t,\tau)_v$ respectively.

Similarly let us denote the Lefschetz numbers of $2^{-l}D\otimes\triangle(V)\otimes\Theta'_q(TX|V)_v$, $D\otimes\Theta_q(TX|V)_v$, $D\otimes\Theta_{-q}(TX|V)_v$ and $2^{-l}D\otimes(\triangle^+(V)-\triangle^-(V))\otimes\Theta^*_q(TX|V)_v$ by $F^V_{d_s}(t,\tau)_v$, $F^V_D(t,\tau)_v$, $F^V_{-D}(t,\tau)_v$ and $F^V_{D^*}(t,\tau)_v$ respectively. Then we have

$$\begin{aligned}
F_{d_s}(t,\tau)_v &= (2\pi i)^{-k}\sum_p\frac{\theta'(0,\tau)^k}{\theta_1(0,\tau)^k}\prod_{j=1}^k\frac{\theta_1(m_jt,\tau)}{\theta(m_jt,\tau)},\\
F_D(t,\tau)_v &= (2\pi i)^{-k}\sum_p\frac{\theta'(0,\tau)^k}{\theta_2(0,\tau)^k}\prod_{j=1}^k\frac{\theta_2(m_jt,\tau)}{\theta(m_jt,\tau)},\\
F_{-D}(t,\tau)_v &= (2\pi i)^{-k}\sum_p\frac{\theta'(0,\tau)^k}{\theta_3(0,\tau)^k}\prod_{j=1}^k\frac{\theta_3(m_jt,\tau)}{\theta(m_jt,\tau)}
\end{aligned}$$

where

$$\theta'(0,\tau)=\frac{\partial}{\partial t}\theta(t,\tau)|_{t=0},\ \ \theta_\mu(0,\tau)=\theta_\mu(t,\tau)|_{t=0};$$

and

$$\begin{aligned}
F^V_{d_s}(t,\tau)_v &= (2\pi i)^{-k}\sum_p\frac{\theta'(0,\tau)^k}{\theta_1(0,\tau)^l}\frac{\prod_{\nu=1}^l\theta_1(n_\nu t,\tau)}{\prod_{j=1}^k\theta(m_jt,\tau)},\\
F^V_D(t,\tau)_v &= (2\pi i)^{-k}\sum_p\frac{\theta'(0,\tau)^k}{\theta_2(0,\tau)^l}\frac{\prod_{\nu=1}^l\theta_2(n_\nu t,\tau)}{\prod_{j=1}^k\theta(m_jt,\tau)},\\
F^V_{-D}(t,\tau)_v &= (2\pi i)^{-k}\sum_p\frac{\theta'(0,\tau)^k}{\theta_3(0,\tau)^l}\frac{\prod_{\nu=1}^l\theta_3(n_\nu t,\tau)}{\prod_{j=1}^k\theta(m_jt,\tau)},\\
F^V_{D^*}(t,\tau)_v &= (2\pi i)^{l-k}\sum_p\theta'(0,\tau)^{k-l}\frac{\prod_{\nu=1}^l\theta(n_\nu t,\tau)}{\prod_{j=1}^k\theta(m_jt,\tau)}.
\end{aligned}$$

As can be easily seen, the F_v's and F^V_v's are exactly the normalizations at $t=0$ of the corresponding F's and F^V's.

The following lemma is a simple corollary of the transformation formulas of the theta-functions.

Lemma 3.1: *a) Under the actions of S and T, we have*

$$F_{d_s}(\tfrac{t}{\tau},-\tfrac{1}{\tau})_v=\tau^k F_D(t,\tau)_v,\ \ F_{d_s}(t,\tau+1)_v=F_{d_s}(t,\tau)_v;$$
$$F_{-D}(\tfrac{t}{\tau},-\tfrac{1}{\tau})_v=\tau^k F_{-D}(t,\tau)_v,\ \ F_D(t,\tau+1)_v=F_{-D}(t,\tau)_v.$$

b) If $p_1(X)_{S^1}=p_1(V)_{S^1}$, then

$$F^V_{d_s}(\tfrac{t}{\tau},-\tfrac{1}{\tau})_v=\tau^k F^V_D(t,\tau)_v,\ \ F^V_{d_s}(t,\tau+1)_v=F^V_{d_s}(t,\tau)_v;$$
$$F^V_{-D}(\tfrac{t}{\tau},-\tfrac{1}{\tau})_v=\tau^k F^V_{-D}(t,\tau)_v,\ \ F^V_D(t,\tau+1)=F^V_{-D}(t,\tau);$$
$$F^V_{D^*}(\tfrac{t}{\tau},-\tfrac{1}{\tau})_v=\tau^{k-l} F^V_{D^*}(t,\tau)_v,\ \ F^V_{D^*}(\tau,\tau+1)_v=F^V_{D^*}(t,\tau)_v.$$

□

Now we discuss the almost complex case. Let W and X be as in Section 2 and

$$\begin{aligned}\Theta^\alpha_q(TX)_v &= \otimes_{n=0}^\infty\Lambda_{-y^{-1}q^n}(T''X-\mathrm{dim}X)\otimes_{n=1}^\infty\Lambda_{-yq^n}(T'X-\mathrm{dim}X)\\ &\otimes_{n=1}^\infty S_{q^n}(T'X-\mathrm{dim}X)\otimes_{n=1}^\infty S_{q^n}(T''X-\mathrm{dim}X);\\ \Theta^\alpha_q(TX|W)_v &= \otimes_{n=0}^\infty\Lambda_{-y^{-1}q^n}(W''-\mathrm{dim}W)\otimes_{n=1}^\infty\Lambda_{-yq^n}(W'-\mathrm{dim}W)\\ &\otimes_{n=1}^\infty S_{q^n}(T'X-\mathrm{dim}X)\otimes_{n=1}^\infty S_{q^n}(T''X-\mathrm{dim}X).\end{aligned}$$

These two elements are obtained by simply replacing $T'X$, $T''X$ and W', W'' by their virtual versions in $\Theta^\alpha(TX)$ and $\Theta^\alpha(TX|W)$. Let $F^\alpha(t,\tau)_v$ and $F^\alpha_W(t,\tau)_v$ be the corresponding Lefschetz numbers of $\bar\partial\otimes\Theta^\alpha_q(TX)_v$ and $\bar\partial\otimes(K^{-1}\otimes L)^{\frac12}\otimes\Theta^\alpha_q(TX|W)_v$ respectively. Then we have

$$F^\alpha(t,\tau)_v=(2\pi i)^{-k}\sum_p\frac{\theta'(0,\tau)^k}{\theta(\alpha,\tau)^k}\prod_{j=1}^k\frac{\theta(m_jt+\alpha,\tau)}{\theta(m_jt,\tau)},$$

$$F^\alpha_W(t,\tau)_v=(\pi i)^{-k}\sum_p\frac{\theta'(0,\tau)^k}{\theta(\alpha,\tau)^l}\frac{\prod_{\nu=1}^l\theta(n_\nu t+\alpha,\tau)}{\prod_j\theta(m_jt,\tau)}.$$

It is easy to see that $F^\alpha(t,\tau)_v$ and $F^\alpha_W(t,\tau)_v$ are also the normalizations of $F^\alpha(t,\tau)$ and $F^\alpha_W(t,\tau)$ at $t=0$.

Let $F^\alpha(t,\tau)^c_v$ and $F^\alpha_W(t,\tau)^c_v$ be the corresponding virtual versions of $F^\alpha(t,\tau)^c$ and $F^\alpha_W(t,\tau)^c$ in last section, then it is not difficult to see that

$$F^{\alpha}(t,\tau)_v^c = (2\pi i)^{-k}\sum_p e^{2\pi i c\alpha\sum m_j t}\frac{\theta'(0,\tau)^k}{\theta(\alpha(c\tau+d),\tau)^k}\prod_{j=1}^{k}\frac{\theta(m_j t+\alpha(c\tau+d),\tau)}{\theta(m_j t,\tau)},$$

$$F_W^{\alpha}(t,\tau)_v^c = (2\pi i)^{-k}\sum_p e^{2\pi i c\alpha\sum n_\nu t}\frac{\theta'(0,\tau)^k}{\theta(\alpha(c\tau+d),\tau)^l}\frac{\prod_{\nu=1}^{l}\theta(n_\nu t+\alpha(c\tau+d),\tau)}{\prod_{j=1}^{k}\theta(m_j t,\tau)}.$$

Lemma 3.2: *Under the action of* $g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL_2(\mathbf{Z})$, *we have*

a)

$$F^{\alpha}(g(t,\tau))_v = (c\tau+d)^k\cdot F^{\alpha}(t,\tau)_v^c;$$

b) if $p_1(X)_{S^1}=p_1(W)_{S^1}$, *then*

$$F_W^{\alpha}(g(t,\tau))_v = (c\tau+d)^k\cdot F_W^{\alpha}(t,\tau)_v^c.$$

Proof: One only needs to note that

$$\begin{aligned}\theta(\alpha,\frac{a\tau+b}{c\tau+d}) &= \chi(g)\frac{1}{\sqrt{c\tau+d}}e^{\pi i c\alpha^2(c\tau+d)}\theta(\alpha(c\tau+d),\tau),\\ \theta'(0,\frac{a\tau+b}{c\tau+d}) &= \chi(g)(c\tau+d)^{\frac{3}{2}}\theta'(0,\tau).\end{aligned}$$

Here $\chi(g)$ is a root of unity. The unwanted factors exactly cancel each other.
$\square$

It is interesting to note that here we do not need anomaly factors for the modularity. Lemmas 3.1 and 3.2 tell us that all of the functions considered here satisfy transformation formulas of the same type. All of them are actually Jacobi forms of weight k and index 0 over some modular groups. See [EZ].

Obviously Lemmas 1.3 and 2.3 still hold for the virtual versions. Therefore the method in Section 1 tells us that Lemmas 1.3 and 3.1 imply the rigidity theorems for spin manifolds; Lemmas 2.3 and 3.2 imply the rigidity theorems for almost complex manifolds. The virtual versions for the $c_1(X)=0$ case are contained in the almost complex case as we have seen in last section. we leave the discussions of the virtual versions of Proposition 2.1 to the reader.

4 Generalizations of Rigidity

In this section we first give two examples of elliptic genera of level 1. Here by level 1, without confusing with the level of loop group representations, we mean their invariance under the full modular group $SL_2(\mathbf{Z})$. Then we give some more general rigidity theorems by using theta-functions.

Example 4.1: This is an elliptic genus of level 1 for spin manifolds. Let X be a compact smooth spin manifold of dimension $2k$. Consider

$$D \otimes (\triangle(X) \otimes \Theta_q'(TX)_v + 2^k \Theta_q(TX)_v + 2^k \Theta_{-q}(TX)_v).$$

Its index can be computed by Atiyah-Singer index formula and is the integral over X of the following cohomology class:

$$(\prod_{j=1}^k 2x_j)(\frac{\theta'(0,\tau)^k}{\theta_1(0,\tau)^k}\prod_{j=1}^k \frac{\theta_1(x_j,\tau)}{\theta(x_j,\tau)} + \frac{\theta'(0,\tau)^k}{\theta_2(0,\tau)^k}\prod_{j=1}^k \frac{\theta_2(x_j,\tau)}{\theta(x_j,\tau)} + \frac{\theta'(0,\tau)^k}{\theta_3(0,\tau)^k}\prod_{j=1}^k \frac{\theta_3(x_j,\tau)}{\theta(x_j,\tau)})$$

where $\{\pm 2\pi i x_j\}$ are the formal Chern roots of $TX \otimes \mathbf{C}$. The transformation formulas of theta-functions immediately give its modular property under the action of $SL_2(\mathbf{Z})$.

Example 4.2: This is an elliptic genus of level 1 for almost complex manifold with $c_1 \equiv 0(\mathrm{mod} N)$. Let X be a compact almost complex manifold with $c_1(X) \equiv 0(\mathrm{mod} N)$. Let $\alpha = \frac{1}{N}$ and consider

$$\bar{\partial} \otimes (\sum_g K^{c\alpha} \otimes \Theta_q^{\alpha(c\tau+d)}(TX)_v)$$

where the summation is for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})/\Gamma_1(N)$. Let $\{2\pi i x_j\}$ be the formal Chern roots of TX'. Then the index of this operator is given by the integral of the following cohomology class over X:

$$(\prod_{j=1}^k 2ix_j)(\sum_g e^{2c\alpha\pi i \sum x_j} \frac{\theta'(0,\tau)^k}{\theta(\alpha(c\tau+d),\tau)^k} \prod_{j=1}^k \frac{\theta(x_j + \alpha(c\tau+d),\tau)}{\theta(x_j,\tau)}).$$

One can also check the invariance of this elliptic genus under the action of $SL_2(\mathbf{Z})$ by using the transformation formulas of theta-functions. In fact for both examples one only needs to verify their transformations under S and T. This is easy. Example 4.2 is more interesting for SU-manifolds.

An interesting problem is to study the elliptic cohomology associated to these two elliptic genera. Also we would like to know their characteristic series.

Our method actually gives more general rigidity theorems. Roughly speaking we can prove that, under a natural assumption, the Dirac operator on loop spaces twisted by higher level loop group representations of positive energy is rigid. The Witten rigidity theorems are the special cases of level 1 representations. We only sketch two generalizations of the Witten rigidity theorems here. These can be viewed as special cases of higher level rigidity theorems. The detailed proofs and other generalizations are discussed in [Liu1].

Let us first consider spin manifold case. Let X be a compact smooth spin manifold of dimension $2k$ with an S^1-action and V be an equivariant spin vector bundle of dimension $2l$ on it. We define the following elements in $K(X)[[q^{\frac{1}{2}}]]$:

$$\begin{aligned} A_q(V) &= \triangle(V) \otimes \otimes_{n=1}^{\infty} \Lambda_{q^n}(V - \dim V), \\ B_q(V) &= \otimes_{n=1}^{\infty} \Lambda_{-q^{n-\frac{1}{2}}}(V - \dim V), \\ C_q(V) &= \otimes_{n=1}^{\infty} \Lambda_{q^{n-\frac{1}{2}}}(V - \dim V). \end{aligned}$$

These three elements are the 'twisting' parts of $\Theta'_q(TX|V)_v$, $\Theta_q(TX|V)_v$ and $\Theta_{-q}(TX|V)_v$ respectively.

Generalization of the Witten rigidity theorem for spin manifolds: *For three non-negative integers a, b, c, assume that the S^1-action lifts to V and $(a+b+c)p_1(V)_{S^1} = p_1(X)_{S^1}$, then*

$$D \otimes \otimes_{m=1}^{\infty} S_{q^m}(TX - dimX) \otimes A_q(V)^{\otimes a} \otimes B_q(V)^{\otimes b} \otimes C_q(V)^{\otimes c}$$

is rigid. $\square$

In fact one can write down the Lefschetz number of this operator which we denote by $F^V_{abc}(t,\tau)$ in terms of the theta-functions:

$$F^V_{abc}(t,\tau) = 2^{al}(2\pi i)^{-k} \sum_p \left(\prod_{j=1}^{k} \frac{\theta'(0,\tau)}{\theta(m_j t,\tau)}\right) \cdot \left(\prod_{\nu=1}^{l} \frac{\theta_1(n_\nu t,\tau)^a \theta_2(n_\nu t,\tau)^b \theta_3(n_\nu t,\tau)^c}{\theta_1(0,\tau)^a \theta_2(0,\tau)^b \theta_3(0,\tau)^c}\right).$$

Using the condition $(a+b+c)p_1(V)_{S^1} = p_1(X)_{S^1}$ which implies the equality $(a+b+c)\sum_\nu n_\nu^2 = \sum_j m_j^2$ for each fixed point, one can prove that, under the action of S and T, we have

$$\begin{array}{rcl} S: \ F_{abc}^V(t,\tau) & \to & \tau^k F_{bac}^V(t,\tau), \\ T: \ F_{abc}^V(t,\tau) & \to & F_{acb}^V(t,\tau). \end{array}$$

By our proof of the Witten rigidity theorems in Section 1, one knows that this is actually enough for the rigidity.

It is interesting to note that, when two of the three integers a, b, c are zero, the above generalization follows from the Witten rigidity theorem in Section 1. In fact

$$A_q(\oplus_a V) = A_q(V)^{\otimes a}, \ p_1(X)_{S^1} = a \cdot p_1(V)_{S^1} = p_1(\oplus_a V)_{S^1}.$$

Similarly for $B_q(V), C_q(V)$. As an interesting corollary we get the following:

Corollary 5.1: *Assume that $p_1(X)_{S^1} = a \cdot p_1(V)_{S^1}$ for some non-negative integer a, then $D \otimes \triangle(V)^{\otimes r}$ is rigid for any non-negative integer $r \leq a$.* $\square$

If $p_1(X) = 0$, we can get another level 1 elliptic genus which is the index of

$$D \otimes \otimes_{m=1}^\infty S_{q^m}(TX - \dim X) \otimes A_q(TX) \otimes B_q(TX) \otimes C_q(TX).$$

One can get similar generalization of the rigidity theorem for almost complex manifolds by taking tensor product. Let X and W be as in Section 2 and define

$$D_q^\alpha(W) = \otimes_{n=0}^\infty \Lambda_{-y^{-1}q^n} W'' \otimes_{n=1}^\infty \Lambda_{-yq^n} W'$$

with $y = e^{2\pi i\alpha}$. Assume $c_1(W) \equiv 0(\text{mod}N)$ for some positive integer N and let $\alpha_i = \frac{i}{N}$. Then we have

Generalization of the Witten rigidity theorem for almost complex manifolds I: *Assume $c_1(W) \equiv 0(modN)$, $w_2(X) = w_2(W) = 0$ and $p_1(X)_{S^1} = mp_1(W)_{S^1}$ for some integer m. Then*

$$D \otimes L^{\frac{m}{2}} \otimes_{n=1}^\infty S_{q^n} T'X \otimes_{n=1}^\infty S_{q^n} T''X \otimes D_q^{\alpha_{i_1}}(W) \otimes \cdots \otimes D_q^{\alpha_{i_m}}(W)$$

is rigid. $\square$

Here $\{i_j\}$ is a set of integers with $0 \leq i_j < N$ and D is the Dirac operator on X which exists by the assumption $w_2(X) = 0$. In fact $D = \bar{\partial} \otimes K^{-\frac{1}{2}}$. If m is even we do not need the condition $w_2(W) = 0$.

For c, d two integers which are prime to each other, let

$$E_q^{c\alpha}(W) = \otimes_{n=0}^{\infty}\Lambda_{-y_c^{-1}q^n}W'' \otimes_{n=1}^{\infty} \Lambda_{-y_c q^n}W'$$

with $y_c = e^{2\pi i\alpha(c\tau+d)}$ and $\alpha = \frac{1}{N}$. Then

Generalization of the Witten rigidity theorem for almost complex manifolds II: *Under the same assumptions as in the generalization I*

$$D \otimes L^{\frac{m}{2}+\alpha\sum_{j=1}^{m} c_j} \otimes_{n=1}^{\infty} S_{q^n}T'X \otimes_{n=1}^{\infty} S_{q^n}T''X \otimes E_q^{c_1\alpha}(W) \otimes \cdots \otimes E_q^{c_m\alpha}(W)$$

is rigid. $\square$

Here $\{c_j, d_j\}$ are integers which satisfy $(c_j, d_j) = 1$. Once one writes down the Lefschetz numbers of the above elliptic operators, their rigidity immediately follows from the transformation formulas of theta-functions and the same arguments as in Section 1.

Let $F(t,\tau)$ be the Lefschetz number of one of the elliptic operators in the above sections. Our proof of the rigidity theorems can be summarized into the following two steps:

(1). *$F(t,\tau)$ is doubly periodic.* This needs the action group to be compact such that the exponents are integers. Here also comes into play of the condition on equivariant Pontrjagin classes.

(2). *Up to some factors independent of t, any modular transformation of $F(t,\tau)$ is still the Lefschetz number of some elliptic operators.* This needs the condition on equivariant Pontrjagin classes to cancel some unwanted factors and the spin conditions on manifolds and vector bundles to get the elliptic operators.

We call (2) the modular invariance. It naturally finds its *home* in the representation theory of loop groups and affine Lie algebras.

An interesting example is the non-spin manifold $\mathbf{C}P^2$ on which $F_{d_s}(t,\tau)$ is perfectly well-defined as the Lefschetz number of an elliptic operator. But $F_D(t,\tau)$ can only be defined formally, since the Dirac operator does not exist. It is easy to see that $F_D(t,\tau)$ has a polar divisor $t = 1$. Since

$$F_D(\frac{t}{\tau}, -\frac{1}{\tau}) = F_{d_s}(t,\tau),$$

our argument tells us that $F_{d_s}(t,\tau)$ should have $t = \tau$ as a polar divisor. This is the case as one can directly verify [BT].

Now we give some general remarks about elliptic genera. The indices of the three elliptic operators $d_s \otimes \Theta'_q(TX)_v$, $D \otimes \Theta_q(TX)_v$ and $D \otimes \Theta_{-q}(TX)_v$ are called the universal elliptic genera for spin manifolds. There are some equivalent characterizations of them by functional equations, by multiplicativity for spin fibrations and by rigidity for S^1-action in [Se] and [O]. See the appendix for the discussions of their functional equations.

(1) From the appendix, we see that the modular invariance is implicit in the functional equations of the three universal elliptic genera. So we can say that their rigidity is intrinsically contained in their functional equations. In fact once we write down the characteristic series of these three universal elliptic genera, their rigidity, functional equations and other properties are consequences.

(2) Since the characteristic series of the universal elliptic genera are ratios of theta-functions of the form

$$\frac{\theta_j(u,\tau)}{\theta(u,\tau)} \frac{\theta'(0,\tau)}{\theta_j(0,\tau)}$$

for $j = 1, 2, 3$. One would like to ask whether other ratios of these theta-functions, for example

$$\frac{\theta(u,\tau)}{\theta_j(u,\tau)} \frac{\theta_j(0,\tau)}{\theta'(0,\tau)},$$

also give characteristic series of some rigid elliptic genera. This is not true. The trouble is that one can not define $\triangle(X)^{-1}$ geometrically in $K_{S^1}(X)$, the equivariant K-group of X. Therefore one can not cancel poles on the unit circle. We do not pursue this point here. One can also see the non-rigidity by looking at the functional equations of these 'new' ratios. The logarithms of their genera do not satisfy the standard functional equation of elliptic genera,

$$y^2 = 1 - 2\delta x^2 + \varepsilon x^4$$

with δ, ε certain modular forms. But we note that they do satisfy some quartic equations of the form

$$y^2 = x^4 - 2\alpha x^2 + \beta$$

with α, β some non-trivial modular forms. See the appendix.

5 The General Fixed Point Case

In this section, we extend all of the above discussions to the general fixed point case. Obviously we only need to verify those transformation formulas used in the proofs of the rigidity theorems.

We only consider the spin case and the transformation formulas between $F_{d_s}(t,\tau)$ and $F_D(t,\tau)$. We leave the other cases, which are completely the same, to the interested reader.

Let $\{X_\alpha\}$ be the fixed submanifolds of the circle action and

$$TX|_{X_\alpha} = E_1 \oplus \cdots \oplus E_h \oplus TX_\alpha$$

be the equivariant decomposition with respect to the S^1-action. We denote the Chern root of E_γ by $2\pi i x_\gamma$ and the Chern roots of $TX_\alpha \otimes \mathbf{C}$ by $\{\pm 2\pi i y_j\}$. Assume that g acts on E_γ by $e^{2\pi i m_\gamma t}$. Then the Lefschetz number of $d_s \otimes \Theta'_q(TX)$ is

$$F_{d_s}(t,\tau) = \sum_{X_\alpha} (\prod_{j=1}^{k_\alpha} (2\pi i y_j F_1(y_j,\tau))(\prod_{\gamma=1}^{h} F_1(x_\gamma + m_\gamma t,\tau))[X_\alpha]$$

where

$$F_1(x,\tau) = i^{-1}\frac{\theta_1(x,\tau)}{\theta(x,\tau)}$$

and $2k_\alpha$ is the dimension of X_α. The Lefschetz number of $q^{-\frac{k}{8}} D \otimes \Theta_q(TX)$ is

$$F_D(t,\tau) = \sum_{X_\alpha} (\prod_{j=1}^{k_\alpha} (2\pi i y_j F_2(y_j,\tau))(\prod_{\gamma=1}^{h} (F_2(x_\gamma + m_\gamma t,\tau))[X_\alpha]$$

where

$$F_2(x,\tau) = i^{-1}\frac{\theta_2(x,\tau)}{\theta(x,\tau)}.$$

For simplicity we only check the action of S, the general case is only notationally more complicated. We have

$$F_{d_s}(\frac{t}{\tau}, -\frac{1}{\tau}) = \sum_{X_\alpha} (\prod_{j=1}^{k_\alpha} (2\pi i y_j F_1(y_j, -\frac{1}{\tau}))(\prod_{\gamma=1}^{h} F_1(x_\gamma + \frac{m_\gamma t}{\tau}, -\frac{1}{\tau}))[X_\alpha]$$

$$= (-i)^k \sum_{X_\alpha} (\prod_{j=1}^{k_\alpha} (2\pi i y_j F_2(\tau y_j, \tau))(\prod_{\gamma=1}^{h} F_2(\tau x_\gamma + m_\gamma t, \tau))[X_\alpha].$$

But

$$(\prod_{j=1}^{k_\alpha} (y_j F_2(\tau y_j, \tau))(\prod_{\gamma=1}^{h} F_2(\tau x_\gamma + m_\gamma t, \tau))[X_\alpha]$$

$$= (\prod_{j=1}^{k_\alpha} (y_j F_2(y_j, \tau))(\prod_{\gamma=1}^{h} F_2(x_\gamma + m_\gamma t, \tau))[X_\alpha]$$

which can be easily verified by looking at the k_α-th homogeneous terms of the polynomials in x's and y's on both sides. Here note that, for the F^V's, one needs to use the condition on equivariant Pontrjagin classes to cancel the unwanted factors. Therefore

$$F_{d_s}(\frac{t}{\tau}, -\frac{1}{\tau}) = (-i)^k F_D(t, \tau)$$

as in the isolated fixed point case. This transformation formula can also be used to complete the argument in [BT] to prove the rigidity of $D \otimes \Theta_q(TX)$ from the rigidity of $d_s \otimes \Theta'_q(TX)$ in general.

6 Elliptic Genera and Elliptic Modular Functions

In this section we collect some formulas relating the universal elliptic genera to classical elliptic functions. It is interesting to note that the characteristic series of the three universal elliptic genera discussed by Witten in [W] are exactly those 'root functions' discussed in detail in [DV].

The proofs of the formulas in this section can either be obtained by directly comparing polar divisors or be found in [DV]. We content ourselves with only giving formulas.

Let $\theta(v,\tau)$, $\theta_1(v,\tau)$, $\theta_2(v,\tau)$ and $\theta_3(v,\tau)$ be the Jacobi theta-functions as in Section 1. For convenience we will use the variable $u = \pi v$ and still write them as $\theta(u,\tau)$, $\theta_1(u,\tau)$, $\theta_2(u,\tau)$ and $\theta_3(u,\tau)$. The purpose of this change is to simplify the notations below, so that we need not to keep track of the factor π in the discussions.

Let Ω be the lattice generated by $(\pi, \pi\tau)$, Ω_1 by $(\pi, 2\pi\tau)$, Ω_2 by $(2\pi, \pi\tau)$ and Ω_3 by $(\pi - \pi\tau, \pi + \pi\tau)$. Let $\mathfrak{P}(\mathfrak{u}), \sigma(\mathfrak{u})$ and $\zeta(u)$ be the Weirstrass $\mathfrak{P}$-function, sigma-function and zeta-function associated to the lattice 2Ω. We start from the Weirstrass parametrization of the elliptic curve

$$\begin{aligned} \mathfrak{P}'(u)^2 &= 4\mathfrak{P}(\mathfrak{u})^3 - \mathfrak{g}_1\mathfrak{P} - \mathfrak{g}_3 \\ &= 4(\mathfrak{P}(\mathfrak{u}) - \mathfrak{e}_1)(\mathfrak{P}(\mathfrak{u}) - \mathfrak{e}_2)(\mathfrak{P}(\mathfrak{u}) - \mathfrak{e}_3). \end{aligned}$$

By looking at their poles and zeroes, one finds that $\mathfrak{P}(\mathfrak{u}) - \mathfrak{e}_\mathfrak{j}$ for $j = 1, 2, 3$ have well-defined square roots on the whole u-plane. Define $f_j(u)$ such that

$$f_j(u)^2 = \mathfrak{P}(\mathfrak{u}) - \mathfrak{e}_\mathfrak{j}.$$

Then each $f_j(u)$ is an elliptic function with period lattice $2\Omega_j$. $f_1(u)$, $f_2(u)$, $f_3(u)$ are called the root functions in [DV]. One obviously has

$$f_k^2(u) - f_j^2(u) = e_j - e_k.$$

Putting $\mathfrak{P}(\mathfrak{u}) = \mathfrak{f}_\mathfrak{j}(\mathfrak{u})^2 + \mathfrak{e}_\mathfrak{j}$ into the Weirstrass equation we get

$$(2f_l(u)f_l'(u))^2 = 4f_1^2(u)f_2^2(u)f_3^2(u)$$

or

$$f_l'(u) = -f_j(u)f_k(u)$$

with $l, j, k = 1, 2, 3$.

Let us write $\theta_j = \theta_j(0, \tau)$ for short. Then one has

$$e_3 - e_2 = \theta_1^4, \; e_1 - e_3 = \theta_2^4, \; e_1 - e_2 = \theta_3^4.$$

See [DV], p 176.

We denote the characteristic series of $d_s \otimes \Theta_q'(TX)_v$, $D \otimes \Theta_q(TX)_v$ and $D \otimes \Theta_{-q}(TX)_v$ by $f_{d_s}(x)$, $f_D(x)$ and $f_{-D}(x)$ respectively. Recall the indices of these operators are the so-called universal elliptic genera. We have

$$f_{d_s}(u) = \frac{1}{2i}\frac{\theta_1(u,\tau)\theta'(0,\tau)}{\theta(u,\tau)\theta_1(0,\tau)},$$

$$f_D(u) = \frac{1}{2i}\frac{\theta_2(u,\tau)\theta'(0,\tau)}{\theta(u,\tau)\theta_2(0,\tau)},$$

$$f_{-D}(u)=\frac{1}{2i}\frac{\theta_3(u,\tau)\theta'(0,\tau)}{\theta(u,\tau)\theta_3(0,\tau)}.$$

The following are a series of relations between these three elliptic genera and classical elliptic functions:

(1)**The relation between elliptic genera and root functions:**

$$f_1(u)=2if_{d_s}(u),\ f_2(u)=2if_D(u),\ f_3(u)=2if_{-D}(u).$$

(2) **The functonal equations of elliptic genera:**

$$f'_{d_s}(u)^2=(f_{d_s}(u)^2-\frac{1}{4}\theta_3^4)(f_{d_s}(u)^2-\frac{1}{4}\theta_2^4),$$

$$f'_D(u)^2=(f_D(u)^2+\frac{1}{4}\theta_3^4)(f_D(u)^2+\frac{1}{4}\theta_1^4),$$

$$f'_{-D}(u)^2=(f_{-D}(u)^2+\frac{1}{4}\theta_2^4)(f_{-D}(u)^2-\frac{1}{4}\theta_1^4).$$

These formulas can be easily obtained by using the above derivative formulas of $f_j(u)$.

Let $g_*(x)$, where * denotes d_s, D or $-D$, be a function such that

$$g_*^{-1}(u)=\frac{1}{f_*(u)}.$$

$g_*(x)$ is called the logarithm of the elliptic genus associated to $f_*(x)$. In fact it is quite easy to show that

$$g_*(u)=\sum_{n=0}^{\infty}\frac{\varphi_*(CP^{2n+1})}{2n+1}u^{2n+1}$$

where φ_* is the genus associated to $uf_*(u)$.

¿From (2) one finds that $G_*(u)=g_*^{-1}(u)$ satisfy the following functional equations

$$G'_{d_s}(u)^2=(1-\frac{1}{4}\theta_3^4G_{d_s}(u)^2)(1-\frac{1}{4}\theta_2^4G_{d_s}(u)^2),$$

$$G'_D(u)^2=(1+\frac{1}{4}\theta_3^4G_D(u)^2)(1+\frac{1}{4}\theta_1^4G_D(u)^2),$$

$$G'_{-D}(u)^2=(1+\frac{1}{4}\theta_2^4G_{-D}(u)^2)(1-\frac{1}{4}\theta_1^4G_{-D}(u)^2).$$

As a simple fact in elementary function theory, one has

(3) **The logarithms of elliptic genera:**

$$g_{d_s}(x) = \int_0^x \frac{du}{\sqrt{(1-\frac{1}{4}\theta_3^4u^2)(1-\frac{1}{4}\theta_2^4u^2)}},$$
$$g_D(x) = \int_0^x \frac{du}{\sqrt{(1+\frac{1}{4}\theta_3^4u^2)(1+\frac{1}{4}\theta_1^4u^2)}},$$
$$g_{-D}(x) = \int_0^x \frac{du}{\sqrt{(1+\frac{1}{4}\theta_2^4u^2)(1-\frac{1}{4}\theta_1^4u^2)}}.$$

Note that the quartic equation in the square root inside the integral of $g_*(x)$ is exactly the functional equation of $g_*^{-1}(u) = \frac{1}{f_*(u)}$. One can compare these with the standard equation of elliptic genera

$$y^2 = 1 - 2\delta x^2 + \varepsilon x^4$$

as given by Ochanine to get the following formulas:

$$\text{For } d_s \otimes \Theta_q'(TX)_v: \quad \delta = \frac{1}{8}(\theta_2^4+\theta_3^4),\ \varepsilon = \frac{1}{16}\theta_2^4\theta_3^4;$$
$$\text{For } D \otimes \Theta_q(TX)_v: \quad \delta = -\frac{1}{8}(\theta_1^4+\theta_3^4),\ \varepsilon = \frac{1}{16}\theta_1^4\theta_3^4;$$
$$\text{For } D \otimes \Theta_{-q}(TX)_v: \quad \delta = \frac{1}{8}(\theta_1^4-\theta_2^4),\ \varepsilon = -\frac{1}{16}\theta_1^4\theta_2^4.$$

These expressions of δ's and ε's are used in [Liu2] to derive a general miraculous cancellation formula.

In the following equations we will just use $f_j(u)$ for convenience. One can easily get the corresponding formulas for $f_*(u)$ by plugging in the factor $2i$.

(4) **Formal group laws of elliptic genera:**

$$\begin{aligned} f_l(u+v) &= \frac{f_l(u)f_j(v)f_k(v) - f_l(v)f_j(u)f_k(u)}{f_l(v)^2 - f_l(u)^2} \\ &= \frac{f_l(v)f_l'(u) - f_l(u)f_l'(v)}{f_l(v)^2 - f_l(u)^2} \end{aligned}$$

where ljk is any permutation of 123. For the proof of this formula, see [DV], p 65 where the above formulas are derived from the formula for $\mathfrak{P}(\mathfrak{u}+\mathfrak{v})$.

(5) **Elliptic genera and the Jacobi elliptic modular functions:**

Let $z = \theta_3^2 u$, then

$$cs(z) = \theta_3^{-2} f_1(u),\ ns(z) = \theta_3^{-2} f_2(u),\ ds(z) = \theta_3^{-2} f_3(u)$$

are three of the Jacobi elliptic modular functions. One has the following relations

$$sn(z) = \frac{1}{ns(z)},\ cn(z) = \frac{cs(z)}{ns(z)},\ dn(z) = \frac{ds(z)}{ns(z)}.$$

We refer the detailed discussions of the Jacobi elliptic modular functions to [DV] or [Ch]. See [DV] p 64-69 for the functional equations of these functions.

(6) **Elliptic genera and sigma-functions:**

$$f_j(u) = \frac{\sigma_j(u)}{\sigma(u)} \text{ for } j = 1, 2, 3$$

where

$$\sigma_j(u) = \frac{e^{\eta_j u}\sigma(u+\omega_j)}{\sigma(\omega_j)}$$

with

$$\omega_1 = \pi,\ \omega_2 = \pi\tau,\ \omega_3 = \pi(1+\tau)$$

and

$$\eta_j = \zeta(u+2\omega_j) - \zeta(u).$$

Actually the above e_j, for $j = 1, 2, 3$, is equal to $\mathfrak{P}(\omega_\mathrm{j})$.

There are more formulas relating the universal elliptic genera to elliptic functions. We would like to discuss the details in a subsequent paper.

Appendix: The modular transformation of theta-functions

There are many diffrent proofs for the modular transformation formulas of the four Jacobi theta-functions in literature. Most of them use Poisson summation formula or the heat equation satisfied by the theta-functions. Here we want derive these transformation formulas directly from the Fourier transform. This proof can also be found in the text books about elliptic functions.

We first look at $\theta_3(v,\tau)$ which has the following infinite sum expression,

$$\theta_3(v,\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i\tau n^2 + 2\pi i n v}.$$

We will prove

$$\theta_3(v, -\frac{1}{\tau}) = \sqrt{\frac{\tau}{i}} \sum_{n=-\infty}^{\infty} e^{\pi i \tau (n+v)^2}$$

which, from replacing v by $\frac{v}{\tau}$, gives

$$\theta_3(\frac{v}{\tau}, -\frac{1}{\tau}) = \sqrt{\frac{\tau}{i}} e^{\frac{\pi i v^2}{\tau}} \theta_3(v,\tau).$$

Assume v is real, τ is purely imaginary, we compute the Fourier expansion

$$\sum_{n=-\infty}^{\infty} e^{\pi i\tau(n+v)^2} = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k v}.$$

By definition, we have

$$\begin{aligned}
c_k &= \int_0^1 \sum_{n=-\infty}^{\infty} e^{\pi i \tau (n+v)^2 - 2\pi i k v} dv \\
&= \int_{-\infty}^{\infty} e^{\pi i \tau v^2 - 2\pi i k v} dv \\
&= e^{-\frac{\pi i k^2}{\tau}} \int_{-\infty}^{\infty} e^{\pi i \tau (v - \frac{k}{\tau})^2} dv.
\end{aligned}$$

Setting $\tau = iy$ with $y > 0$ and

$$\begin{aligned}
s &= \sqrt{\frac{\tau}{i}}(v - \frac{k}{\tau}) \\
&= \sqrt{y}(v + \frac{ik}{y}).
\end{aligned}$$

Then

$$\begin{aligned}
c_k &= e^{-\frac{\pi k^2}{y}} \sqrt{\frac{1}{y}} \int_{-\infty + \frac{ik}{\sqrt{y}}}^{\infty + \frac{ik}{\sqrt{y}}} e^{-\pi s^2} ds \\
&= e^{-\frac{\pi k^2}{y}} \frac{1}{\sqrt{y}}.
\end{aligned}$$

Then anlytic continuation finishes the proof. For other theta-functions, we use the relations

$$\begin{aligned}\theta_2(v,\tau) &= \theta_3(v+\frac{1}{2},\tau),\\ \theta_1(v,\tau) &= \theta_3(v+\frac{\tau}{2},\tau)e^{\frac{\pi i\tau}{4}}e^{\pi iv},\\ \theta(v,\tau) &= \theta_3(v+\frac{1}{2}+\frac{\tau}{2},\tau)e^{\frac{\pi i\tau}{4}}e^{\pi iv}(-i).\end{aligned}$$

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