HIRZEBRUCH χ_y GENERA OF THE HILBERT SCHEMES OF SURFACES BY LOCALIZATION FORMULA

KEFENG LIU, CATHERINE YAN, AND JIAN ZHOU

Abstract. We use the Atiyah-Bott-Berline-Vergne localization formula to calculate the Hirzebruch χ_y genus $\chi_y(S^{[n]})$, where $S^{[n]}$ is the Hilbert schemes of points of length n of a surface S. Combinatorial interpretation of the weights of the fixed points of the natural torus action on $(\mathbb{C}^2)^{[n]}$ is used.

1. Introduction

1.1. **Genera of complex manifolds.** Let $\Omega = \Omega^U \otimes \mathbb{Q}$ be the complex bordism ring with coefficients in $\mathbb{Q}$. Given a ring R, a complex genera with values in R is a ring homomorphism

$$\phi : \Omega \to R.$$

Recall that according to Milnor [11] and Novikov, two complex manifolds are complex cobordant if and only if they have the same Chern numbers. For a closed complex manifold M, Hirzebruch [10] defined its χ_y genus by

$$\chi_y(M) = \sum_{p,q \geq 0} y^p (-1)^q \dim H^q(M, \Lambda^p T^*M) = \sum_{p \geq 0} y^p \chi(M, \Lambda^p T^*M).$$

Recall that for any holomorphic vector bundle $V \to M$,

$$\chi(M, V) = \sum_{q \geq 0} (-1)^q \dim H^q(M, V).$$

This can be extended to Ω to obtain a complex genus with values in $\mathbb{C}[y]$. Dijkgraaf, Moore, Verlinde and Verlinde [4] introduced a complex genus as follows. For any vector bundle V over M, define the formal sums

$$\Lambda_q V = \bigoplus_{k \geq 0} q^k \Lambda^k V, \qquad S_q V = \bigoplus_{k \geq 0} q^k S^k V,$$

where Λ^k and S^k denote the k-th exterior and symmetric product respectively. Let $\dim_{\mathbb{C}} M = d$, set

$$E_{q,y} = y^{-\frac{d}{2}} \bigotimes_{n \geq 1} \left(\Lambda_{-yq^{n-1}} TM \otimes \Lambda_{-y^{-1}q^n} T^*M \otimes S_{q^n} TM \otimes S_{q^n} T^*M \right).$$

One can write

$$E_{q,y} = \bigoplus_{m,l} q^m y^l E_{m,l}.$$

It is easy to see that $E_{m,l}$ is a holomorphic vector bundle of finite rank. Define

$$\chi(M; q, y) = \sum_{m,l} q^m y^l \chi(M, E_{m,l}).$$

By Hirzebruch-Riemann-Roch theorem,

$$\chi(M; q, y) = \int_M \mathrm{ch}(E_{q,y})\tau(M),$$

where $\tau(M)$ is the Todd class of M. Hence $\chi(M; q, y)$ can be expressed in term of Chern numbers of M, therefore it is a complex cobordism invariant. By multiplicative properties of Λ_q and S_q, it is easy to see that $\chi(M; q, y)$ is indeed a genus. It is easy to see that

$$\chi(M; 0, y) = y^{-\frac{d}{2}}\chi_{-y}(M).$$

1.2. **A conjecture on Hilbert schemes of surfaces.** Let S be a smooth algebraic surface over $\mathbb{C}$, denote by $S^{[n]}$ the Holbert scheme of zero dimensional subscheme of length n. By a well-known result of Fogarty [7], $S^{[n]}$ is smooth and projective of dimension $2n$. Closely related to $S^{[n]}$ is the n-th symmetric product $S^{(n)}$ which is the quotient of the Cartesian product S^n by the natural action of permutation group on n objects. There is a morphism $\pi : S^{[n]} \to S^{(n)}$ which gives a crepant resolution of $S^{(n)}$. String theory on orbifolds motivates a definition of $\chi(S^{(n)}; q, y)$. By physical arguments, Dijkgraaf *et al* [4] showed that

$$\sum_{n\geq 0} p^n \chi(M^{(n)}; q, y) = \prod_{n>0, m\geq 0, l} \frac{1}{(1 - p^n q^m y^l)^{c(nm,l)}}$$

where $c(m, l)$ are given by

$$\chi(M; q, y) = \sum_{m\geq 0, l} c(m, l) q^m y^l.$$

They conjecture that for a K3 surface or an abelian surface S, $\chi(S^{(n)}, q, y) = \chi(S^{[n]}; q, y)$. Therefore, one is led to the following conjectured formula for algebraic surfaces:

$$\sum_{n\geq 0} p^n \chi(S^{[n]}; q, y) = \prod_{n>0, m\geq 0, l} \frac{1}{(1 - p^n q^m y^l)^{c(nm,l)}} \tag{1}$$

with $c(m, l)$ given by

$$\chi(M; q, y) = \sum_{m\geq 0, l} c(m, l) q^m y^l.$$

1.3. **Reduction to $\mathbb{CP}_2$ and $\mathbb{CP}_1 \times \mathbb{CP}_1$.** Following a recent paper by Ellingsrud, Göttsche and Lehn [5], write

$$H(S) = \sum_{n\geq 0} [S^{[n]}] p^n.$$

They have shown that the cobordism class $[H(S)] \in \Omega$ depends only on the complex cobordism class of $S \in \Omega$. They also have shown that if $[S] = a_1[S_1] + a_2[S_2]$ for some smooth algebraic surfaces S, S_1, S_2 and rational numbers a_a and a_2, then we have

$$H(S) = H(S_1)^{a_1} H(S_2)^{a_2}.$$

Milnor [11] showed that Ω is a polynomial algebra freely generated by the cobordism classes $[\mathbb{CP}_n]$ for positive integers n. Hence for any surface S,

$$[S] = a[\mathbb{CP}_2] + b[\mathbb{CP}_1 \times \mathbb{CP}_1]$$

for some rational number a and b. As a result, for any genus $\phi : \Omega \to R$, we have

$$\phi(H(S)) = \phi(H(\mathbb{CP}_2))^a \phi(H(\mathbb{CP}_1 \times \mathbb{CP}_1))^b. \tag{2}$$

In particular, to prove (1), it suffices to prove it for $\mathbb{CP}_2$ and $\mathbb{CP}_1 \times \mathbb{CP}_1$. As explained in Zhou [12], when $q = 0$, (1) reduces to

$$\chi_{-y}(H(S)) = \exp\left(\sum_{m \geq 0} \frac{\chi_{-y^m}(S)}{1-(yp)^m} \frac{p^m}{m} \right). \tag{3}$$

This formula has been proved by Göttsche-Soeregel [8] and Cheah [3] by different methods from algebraic geometry. Ellinsgrud *et al* gave a new proof using (2). They made use of a result of Carrell-Lieberman [2] on the Hodge numbers of Kähler manifolds which admit $\mathbb{C}^*$-action with isolated fixed points.

1.4. **Localization formula and combinatorics.** The motivation of this paper is to find a method to prove (3) which can be generalized to prove (1). The first named author proposed to use the Atiyah-Bott-Berline-Vergne localization theorem, this paper contains the details for the case of χ_y genera. The general case will appear in a separate publication.

Haiman [9] has used localization theorem for the natural torus action on the Hilbert schemes of $\mathbb{CP}_2$ to give a geometric interpreation of t, q-Catalan numbers. Our calculations in this paper shares some similar feature with his. The interpretation of the weights in terms of Young diagrams greatly simplies the notations used in the original version. A new feature is the use of limits which should correspond to the geometric limits under the torus action.

2. Preliminaries

2.1. **Localization of χ_y genus.** For a closed complex manifold M,

$$\chi_{-y}(M) = \int_M \mathrm{ch}\, \Lambda_{-y}(T^*M)\mathcal{T}(M) = \int_M \prod_{j=1}^{d} (1 - ye^{-x_j}) \frac{x_j}{1-e^{-x_j}},$$

where $\mathcal{T}(M)$ denotes the Todd class of M, $\{x_1, \cdots, x_d\}$ denote the formal Chern roots of TM. Assume that M admits a torus action with isolated fixed points $\{F_1, \cdots, F_m\}$, and at F_i, the weights of the action are $\{w_{i,1}, \cdots, w_{i,d}\}$. Then by Atiyah-Bott-Berline-Vergne localization theorem [1], we have

$$\chi_{-y}(M) = \sum_{i=1}^{m} \frac{\prod_{j=1}^{d}(1-ye^{-w_{i,j}})\frac{w_{i,j}}{1-e^{-w_{i,j}}}}{\prod_{j=1}^{d} w_{i,j}} = \sum_{i=1}^{m}\prod_{j=1}^{d} \frac{1-ye^{-w_{i,j}}}{1-e^{-w_{i,j}}}. \tag{4}$$

Denote by $\chi_{-y}(M)_{F_i}$ the contribution from F_i.

2.2. **Partitions.** For any natural number n, recall the *set of partions of* n is

$$\mathcal{P}(n) = \{(b_0, b_1, \cdots, b_r) : b_0 \geq \cdots \geq b_{r-1} > b_r = 0, \quad \sum_{j=0}^{r} b_j = n\}.$$

Extend the definition to

$$\mathcal{P}(0) = \{(0)\}.$$

For a partition $P = (b_0, b_1, \cdots, b_r) \in \mathcal{P}(n)$, $n > 0$, set

$$r(P) = r.$$

Also set

$$r((0)) = 0.$$

There is a one-to-one correspondence between $\mathcal{P}(n)$ and the following set:

$$\{(N_1, N_2, \cdots) \in \mathbb{Z}^\infty : N_m \geq 0, \sum_m mN_m = n\},$$

given by setting $N_m(P)$ to be the number of m's in $P \in \mathcal{P}(n)$. Clearly,

$$r(P) = \sum_n N_m.$$

There is also a one-to-one correspondence between $\mathcal{P}(n)$ and the set of Young diagrams with n boxes. Since there is no danger of confusion, the Young diagram of a partion P will also be denoted by P. Denote by $B(P)$ the set of boxes in a Young diagram P. Notice that the transpose of a Young diagram is also a Young diagram, so one obtains an involution $t : \mathcal{P}(n) \to \mathcal{P}(n)$. Clearly $r(P)$ is the number of rows of the Young diagram of P, and $b_0 = r(P^t)$.

Given an box e in a Young diagram P, the number of boxes on the right of it is called its arm, while the number of boxes below it is called its leg. Denote by $a(e)$ and $l(e)$ the arm and the leg of e respectively. Notice that

$$\text{card}\{e \in B(P) : a(e) = 0\} = r(P). \tag{5}$$

Introduce the following notation: for $P \in \mathcal{P}(n)$,

$$I(P) = \{(i, j, s) | 1 \leq i \leq j, \;\; b_j \leq s \leq b_{j-1} - 1\}.$$

The set $I(P)$ is in one-to-one correspondence with the set of the boxes of the transpose of $Y(P)$. Indeed, $(i, j, s) \in I(P)$ corresponds to the box at $(s + 1, i)$. It is easy to see that $j - i$ is the arm of this box, while $b_{i-1} - s - 1$ is the leg.

2.3. **Fixed points and weights of the Hilbert schemes of $\mathbb{C}^2$.** There is a natural T^2-action on $\mathbb{C}^2$:

$$(\lambda, \mu) \cdot (x, y) = (\lambda x, \mu y),$$

for $\lambda, \mu \in \mathbb{C}^*$, $(x, y) \in \mathbb{C}^2$. The only fixed point of this action is $(0, 0)$. There is an induced action on $(\mathbb{C}^2)^{[n]}$. According to Ellingsrud and Strømme [6], the fixed points are in one-to-one correspondence with partitions $P = (b_0, b_1, \cdots, b_r)$ of n. Furthermore, in the representation ring of $(C^*)^2$,

$$T = \sum_{(i,j,s)\in I(P_l)} (\lambda^{i-j-1}\mu^{b_{i-1}-s-1} + \lambda^{j-i}\mu^{s-b_{i-1}}), \tag{6}$$

Here we have abused the notation: $\lambda^m\mu^n$ means the one-dimensional representation on which (λ, μ) acts as multiplication by $\lambda^m\mu^n$. With the notation in §2.2, we rewrite (6) as

$$T = \sum_{e\in B(P^t)} (\lambda^{-a(e)-1}\mu^{l(e)} + \lambda^{a(e)}\mu^{-l(e)-1)}). \tag{7}$$

Note that such combinatorial interpretation of the weights has appeared in Haiman [9]. Clearly, the above results still hold if λ and μ are two linearly independent weights on T^2.

3. Hirzebruch χ_y genera of Hilbert schemes of surfaces

3.1. **Weights of the torus action on the Hilbert schemes of $\mathbb{CP}_2$.** Consider the T^2-action on $\mathbb{CP}_2$ given by

$$(t_1, t_2) \cdot [z_0 : z_1 : z_2] = [z_0 : t_1 z_1 : t_2 z_2], \tag{8}$$

where $t_1, t_2 \in S^1$, $[z_0 : z_1 : z_2] \in \mathbb{CP}_2$. Clearly the fixed points are $x_0 = [1 : 0 : 0]$, $x_1 = [0 : 1 : 0]$ and $x_2 = [0 : 0 : 1]$. For $i = 0, 1, 2$, let

$$U_i = \{[z_0 : z_1 : z_2] \in \mathbb{CP}_2 : z_i \neq 0\}.$$

Then $U_i \cong \mathbb{C}^2$. Indeed, on U_0, the isomorphism is given by $x = \frac{z_1}{z_0}$, $y = \frac{z_2}{z_0}$. The action (8) corresponds on to the following action on $\mathbb{C}^2$:

$$(t_1, t_2) \cdot_0 (x, y) = (t_1 x, t_2 y). \tag{9}$$

Similarly, near x_1 and x_2, we get the following actions:

$$(t_1, t_2) \cdot_1 (x, y) = (\frac{1}{t_1} x, \frac{t_2}{t_1} y), \tag{10}$$

$$(t_1, t_2) \cdot_2 (x, y) = (\frac{1}{t_2} x, \frac{t_1}{t_2} y). \tag{11}$$

There is an induced T^2-action on the Hilbert scheme $\mathbb{CP}_2^{[n]}$. We now recall the description of its fixed point set by Ellingsrud and Strømme [6]. If $Z \in \mathbb{CP}_2^{[n]}$ is a fixed point of this action, then the support of Z is contained in $\{x_0, x_1, x_2\}$. Hence we may write $Z = Z_0 \cup Z_1 \cup Z_2$, where Z_i is supported in P_i. Let n_i be the length of $\mathcal{O}_{Z_i}$, then Z_i can be regarded as a fixed point in $U_i^{[n_i]}$, hence it corresponds to a partition $P_i \in \mathcal{P}(n_i)$ by §2.3. Therefore the fixed point set on $\mathbb{CP}_2^{[n]}$ is in one-to-one correspondence with the following set:

$$\mathcal{F}(n) = \{(P_0, P_1, P_2) \in \mathcal{P}(n_0) \times \mathcal{P}(n_1) \times \mathcal{P}(n_2) : n_0 + n_1 + n_2 = n\}.$$

For each $(P_0, P_1, P_2) \in \mathcal{F}_n$, denote by F_{P_0,P_1,P_2} the corresponding fixed point. It is clear that a neighborhood of F_{P_0,P_1,P_2} in $\mathbb{CP}_2^{[n]}$ can be identified with the product of some neighborhoods of F_{P_i} in $U_i^{[n_i]}$. Hence we have a decomposition

$$T_{F_{P_0,P_1,P_2}} \mathbb{CP}_2^{[n]} = \bigoplus_{l=0}^{2} T_{F_{P_0}} U_{n_l}^{[n_l]}.$$

By §2.3 and (9) - (11), we have

$$T_{F_{P_0}} U_{n_l}^{[n_l]} = \sum_{e \in E(P_l^t)} (\lambda_l^{-a(e)-1} \mu_l^{l(e)} + \lambda_l^{a(e)} \mu_l^{-l(e)-1}). \tag{12}$$

where

$$\lambda_0 = t_1, \qquad \mu_0 = t_2, \tag{13}$$

$$\lambda_1 = 1/t_1, \qquad \mu_1 = t_2/t_1, \tag{14}$$

$$\lambda_2 = 1/t_2, \qquad \mu_2 = t_1/t_2. \tag{15}$$

3.2. **Application of the localization formula.** By (4) and (7), we have

$$\chi_{-y}(\mathbb{CP}_2^{[n]})_{F_{P_0,P_1,P_2}} = \chi_{-y}^{(0)}(P_0)\chi_{-y}^{(1)}(P_1)\chi_{-y}^{(2)}(P_2),$$

where

$$\chi_{-y}^{(l)}(P_l) = \prod_{e\in B(P_l^t)} \frac{1-y\lambda_l^{a(e)+1}\mu_l^{-l(e)}}{1-\lambda_l^{a(e)+1}\mu_l^{-l(e)}} \cdot \frac{1-y\lambda_l^{-a(e)}\mu_l^{l(e)+1}}{1-\lambda_l^{-a(e)}\mu_l^{l(e)+1}}. \tag{16}$$

Now we have

$$\begin{aligned}
\sum_{n\geq 0}\chi_{-y}(\mathbb{CP}_2^{[n]})q^n &= \sum_{n\geq 0}\sum_{(P_0,P_1,P_2)\in\mathcal{F}(n)} \chi_{-y}(\mathbb{CP}_2^{[n]})_{F_{P_0,P_1,P_2}}q^n \\
&= \sum_{n\geq 0}\sum_{(P_0,P_1,P_2)\in\mathcal{F}(n)} \chi_{-y}^{(0)}(P_0)\chi_{-y}^{(1)}(P_1)\chi_{-y}^{(2)}(P_2)q^n \\
&= \prod_{l=0}^{2}\left(\sum_{n_l\geq 0}\sum_{P_l\in\mathcal{P}(n_l)}\chi_{-y}^{(l)}(P_l)q^{n_l}\right).
\end{aligned}$$

I.e.,

$$\sum_{n\geq 0}\chi_{-y}(\mathbb{CP}_2^{[n]})q^n = \prod_{l=0}^{2}\left(\sum_{n_l\geq 0}\sum_{P_l\in\mathcal{P}(n_l)}\chi_{-y}^{(l)}(P_l)q^{n_l}\right). \tag{17}$$

Notice that the left hand side does not depend on t_1 and t_2. As we will show below, on the right hand side we can take the limits for $t_1 \to 0$ then $t_2 \to 0$, or we can first let $t_1 \to \infty$ then $t_2 \to \infty$. So we have

$$\begin{aligned}
\sum_{n\geq 0}\chi_{-y}(\mathbb{CP}_2^{[n]})p^n &= \prod_{l=0}^{2}\left(\sum_{n_l\geq 0}\sum_{P_l\in\mathcal{P}(n_l)}\lim_{t_2\to 0}\lim_{t_1\to 0}\chi_{-y}^{(l)}(P_l)p^{n_l}\right) \\
&= \prod_{l=0}^{2}\left(\sum_{n_l\geq 0}\sum_{P_l\in\mathcal{P}(n_l)}\lim_{t_2\to \infty}\lim_{t_1\to \infty}\chi_{-y}^{(l)}(P_l)p^{n_l}\right).
\end{aligned}$$

3.3. **The $l = 0$ case.** ¿From (16) and (13), for any $P \in \mathcal{P}(n)$ we have

$$\chi_{-y}^{(0)}(P) = \prod_{e\in E(P^t)} \frac{1-yt_1^{a(e)+1}t_2^{-l(e)}}{1-t_1^{a(e)+1}t_2^{-l(e)}} \cdot \frac{1-yt_1^{-a(e)}t_2^{l(e)+1}}{1-t_1^{-a(e)}t_2^{l(e)+1}}.$$

Since $a(e) \geq 0$, it is straightforward to see that

$$\begin{aligned}
\lim_{t_1\to 0}\frac{1-yt_1^{a(e)+1}t_2^{-l(e)}}{1-t_1^{a(e)+1}t_2^{-l(e)}} &= 1, \\
\lim_{t_1\to \infty}\frac{1-yt_1^{a(e)+1}t_2^{-l(e)}}{1-t_1^{a(e)+1}t_2^{-l(e)}} &= y.
\end{aligned}$$

When $a(e) > 0$,

$$\lim_{t_1 \to 0} \frac{1 - y t_1^{-a(e)} t_2^{l(e)+1}}{1 - t_1^{-a(e)} t_2^{l(e)+1}} = y;$$

$$\lim_{t_1 \to \infty} \frac{1 - y t_1^{-a(e)} t_2^{l(e)+1}}{1 - t_1^{-a(e)} t_2^{l(e)+1}} = 1;$$

when $a(e) = 0$,

$$\frac{1 - y t_1^{-a(e)} t_2^{l(e)+1}}{1 - t_1^{-a(e)} t_2^{l(e)+1}} = \frac{1 - y t_2^{l(e)+1}}{1 - t_2^{l(e)+1}},$$

since $l(e) \geq 0$, we have

$$\lim_{t_2 \to 0} \lim_{t_1 \to 0} \frac{1 - y t_1^{-a(e)} t_2^{l(e)+1}}{1 - t_1^{-a(e)} t_2^{l(e)+1}} = 1;$$

$$\lim_{t_2 \to \infty} \lim_{t_1 \to \infty} \frac{1 - y t_1^{-a(e)} t_2^{l(e)+1}}{1 - t_1^{-a(e)} t_2^{l(e)+1}} = y.$$

Counting terms with y as the double limits and use (5), we get

$$\lim_{t_2 \to 0} \lim_{t_1 \to 0} \chi_{-y}^{(0)}(P) = y^{n - r(P^t)},$$

$$\lim_{t_2 \to \infty} \lim_{t_1 \to \infty} \chi_{-y}^{(0)}(P) = y^{n + r(P^t)},$$

for all $P \in \mathcal{P}(n)$. To summarize, we get

$$\sum_{P \in \mathcal{P}(n)} \lim_{t_1 \to 0} \lim_{t_0 \to 0} \chi_{-y}^{(1)}(P) = \sum_{P \in \mathcal{P}(n)} y^{n - r(P^t)} = \sum_{P \in \mathcal{P}(n)} y^{n - r(P)}, \tag{18}$$

$$\sum_{P \in \mathcal{P}(n)} \lim_{t_1 \to \infty} \lim_{t_0 \to \infty} \chi_{-y}^{(1)}(P) = \sum_{P \in \mathcal{P}(n)} y^{n + r(P^t)} = \sum_{P \in \mathcal{P}(n)} y^{n + r(P)}. \tag{19}$$

Remark 3.1. Furthermore, each term in the product which goes to 1 under the double limit $\lim_{t_2 \to 0} \lim_{t_1 \to 0}$ goes to y under the double limit $\lim_{t_2 \to \infty} \lim_{t_1 \to \infty}$, and vice versa. This still holds for $l = 1$ or 2.

3.4. **The $l = 2$ case.** Similarly, we have

$$\chi_{-y}^{(2)}(P) = \prod_{e \in B(P^t)} \frac{1 - y(\frac{1}{t_2})^{a(e)+1}(\frac{t_1}{t_2})^{-l(e)}}{1 - (\frac{1}{t_2})^{a(e)+1}(\frac{t_1}{t_2})^{-l(e)}} \cdot \frac{1 - y(\frac{1}{t_2})^{-a(e)}(\frac{t_1}{t_2})^{l(e)+1}}{1 - (\frac{1}{t_2})^{-a(e)}(\frac{t_1}{t_2})^{l(e)+1}}.$$

We have

$$\lim_{t_1 \to 0} \frac{1 - y(\frac{1}{t_2})^{a(e)+1}(\frac{t_1}{t_2})^{-l(e)}}{1 - (\frac{1}{t_2})^{a(e)+1}(\frac{t_1}{t_2})^{-l(e)}} = \begin{cases} y, & l(e) > 0, \\ \frac{t_2^{a(e)+1} - y}{t_2^{a(e)+1} - 1}, & l(e) = 0, \end{cases}$$

hence

$$\lim_{t_2 \to 0} \lim_{t_1 \to 0} \frac{1 - y(\frac{1}{t_2})^{a(e)+1}(\frac{t_1}{t_2})^{-l(e)}}{1 - (\frac{1}{t_2})^{a(e)+1}(\frac{t_1}{t_2})^{-l(e)}} = y,$$

and

$$\lim_{t_1 \to 0} \frac{1 - y(\frac{1}{t_2})^{-a(e)}(\frac{t_1}{t_2})^{l(e)+1}}{1 - (\frac{1}{t_2})^{-a(e)}(\frac{t_1}{t_2})^{l(e)+1}} = 1.$$

By (5) we have

$$\sum_{P\in\mathcal{P}(n)} \lim_{t_1\to 0}\lim_{t_0\to 0}\chi^{(2)}_{-y}(P) = \sum_{P\in\mathcal{P}(n)} y^n. \tag{20}$$

Similarly we have

$$\sum_{P\in\mathcal{P}(n)} \lim_{t_1\to \infty}\lim_{t_0\to \infty}\chi^{(2)}_{-y}(P) = \sum_{P\in\mathcal{P}(n)} y^n. \tag{21}$$

3.5. **The $l=1$ case.** For any $P\in\mathcal{P}(n)$, by (16) and (14), we have

$$\chi^{(0)}_{-y}(P) = \prod_{e\in B(P^t)} \frac{1-y(\frac{1}{t_1})^{a(e)+1}(\frac{t_2}{t_1})^{-l(e)}}{1-(\frac{1}{t_1})^{a(e)+1}(\frac{t_2}{t_1})^{-l(e)}}\cdot\frac{1-y(\frac{1}{t_1})^{-a(e)}(\frac{t_2}{t_1})^{l(e)+1}}{1-(\frac{1}{t_1})^{a(e)}(\frac{t_2}{t_1})^{l(e)+1}}$$

$$= \prod_{e\in B(P^t)} \frac{1-yt_1^{l(e)-a(e)-1}t_2^{-l(e)}}{1-t_1^{l(e)-a(e)-1}t_2^{-l(e)}}\cdot\frac{1-yt_1^{a(e)-l(e)-1}t_2^{l(e)+1}}{1-t_1^{a(e)-l(e)-1}t_2^{l(e)+1}}.$$

We have

$$\lim_{t_1\to 0}\frac{1-yt_1^{l(e)-a(e)-1}t_2^{-l(e)}}{1-t_1^{l(e)-a(e)-1}t_2^{-l(e)}} = \begin{cases} y, & a(e)+1>l(e),\\ \frac{t_2^{l(e)}-y}{t_2^{l(e)}-1}, & a(e)+1=l(e)(>0),\\ 1, & a(e)+1<l(e),\end{cases}$$

hence

$$\lim_{t_2\to 0}\lim_{t_1\to 0}\frac{1-yt_1^{l(e)-a(e)-1}t_2^{-l(e)}}{1-t_1^{l(e)-a(e)-1}t_2^{-l(e)}} = \begin{cases} y, & a(e)+1\geq l(e),\\ 1, & a(e)+1<l(e).\end{cases}$$

Similarly,

$$\lim_{t_1\to 0}\frac{1-yt_1^{a(e)-l(e)-1}t_2^{l(e)+1}}{1-t_1^{a(e)-l(e)-1}t_2^{l(e)+1}} = \begin{cases} 1, & a(e)>l(e)+1,\\ \frac{1-yt_2^{l(e)+1}}{1-t_2^{l(e)+1}}, & a(e)=l(e)+1,\\ y, & a(e)<l(e)+1,\end{cases}$$

and so

$$\lim_{t_1\to 0}\lim_{t_0\to 0}\frac{1-y(\frac{t_1}{t_0})^{a(e)}(\frac{t_2}{t_0})^{-l(e)-1}}{1-(\frac{t_1}{t_0})^{a(e)}(\frac{t_2}{t_0})^{-l(e)-1}} = \begin{cases} 1, & a(e)\geq l(e)+1,\\ y, & a(e)<l(e)+1.\end{cases}$$

Set

$$\begin{aligned} s(P) &= \operatorname{card}\{e\in E(P^t): l(e)-1\leq a(e)\leq l(e)\}\\ &= \operatorname{card}\{e\in E(P^t): a(e)\leq l(e)\leq a(e)+1\},\end{aligned}$$

then it is clear that

$$\lim_{t_1\to 0}\lim_{t_0\to 0}\chi^{(0)}_{-y}(P) = y^{n-s(P)}.$$

To summarize, we have

$$\sum_{P\in\mathcal{P}(N)} \lim_{t_1\to 0}\lim_{t_0\to 0}\chi^{(0)}_{-y}(P) = \sum_{P\in\mathcal{P}(n)} y^{n-s(P)}. \tag{22}$$

Similarly we have

$$\sum_{P\in\mathcal{P}(n)} \lim_{t_1\to \infty}\lim_{t_0\to \infty}\chi^{(0)}_{-y}(P) = \sum_{P\in\mathcal{P}(N)} y^{n+s(P)}. \tag{23}$$

3.6. **Putting pieces together.** Now we have

$$\begin{aligned}
&\sum_{n\geq 0} p^n \chi_{-y}(\mathbb{CP}_2^{[n]}) \\
= &\left(\sum_{n\geq 0} p^n \sum_{P\in\mathcal{P}(n)} y^n\right) \cdot \left(\sum_{n\geq 0} p^n \sum_{P\in\mathcal{P}(n)} y^{n+r(P)}\right) \cdot \left(\sum_{n\geq 0} p^n \sum_{P\in\mathcal{P}(n)} y^{n-s(P)}\right) \\
= &\left(\sum_{n\geq 0} p^n \sum_{P\in\mathcal{P}(n)} y^n\right) \cdot \left(\sum_{n\geq 0} p^n \sum_{P\in\mathcal{P}(n)} y^{n-r(P)}\right) \cdot \left(\sum_{n\geq 0} p^n \sum_{P\in\mathcal{P}(n)} y^{n+s(P)}\right).
\end{aligned}$$

Lemma 3.1. *We have*

$$\sum_{P\in\mathcal{P}(n)} y^{r(P)} = \sum_{P\in\mathcal{P}(n)} y^{s(P)}.$$

Proof. For $n > 0$, write

$$r_n(y) = \sum_{P\in\mathcal{P}(n)} y^{r(P)}, \qquad s_n(y) = \sum_{P\in\mathcal{P}(n)} y^{s(P)}.$$

Also write $r_0(y) = s_0(y) = 1$. Suppose that we have shown that $r_j(y) = s_j(y)$ for $j = 1, \cdots, n$. ¿From the above formula we have

$$\sum_{n\geq 0}(py)^n r_n(y) \cdot \sum_{n\geq 0}(py)^n s_n(y^{-1}) = \sum_{n\geq 0}(py)^n r_n(y^{-1}) \cdot \sum_{n\geq 0}(py)^n s_n(y).$$

Comparing the coefficients of $(py)^{n+1}$ one gets

$$r_{n+1}(y) + \sum_{j=1}^{n} r_j(y)s_j(y^{-1}) + s_{n+1}(y^{-1}) = r_{n+1}(y^{-1}) + \sum_{j=1}^{n} r_j(y^{-1})s_j(y) + s_{n+1}(y).$$

By the induction hypothesis, one gets

$$r_{n+1}(y) + s_{n+1}(y^{-1}) = r_{n+1}(y^{-1}) + s_{n+1}(y).$$

Now $r_{n+1}(y)$ and s_{n+1} are polynomials in y, and r_{n+1} has no constant terms, so we must have $r_{n+1}(y) = s_{n+1}(y)$. The proof is complete. □

As a corallary, we have

$$\begin{aligned}
&\sum_{n\geq 0} p^n \chi_{-y}(\mathbb{CP}_2^{[n]}) \\
= &\left(\sum_{n\geq 0} p^n \sum_{P\in\mathcal{P}(n)} y^n\right) \cdot \left(\sum_{n\geq 0} p^n \sum_{P\in\mathcal{P}(n)} y^{n+r(P)}\right) \cdot \left(\sum_{n\geq 0} p^n \sum_{P\in\mathcal{P}(n)} y^{n-r(P)}\right) \\
= &\prod_{l=0}^{2}\left(\sum_{n\geq 0} p^n \sum_{P\in\mathcal{P}(n)} y^{n+(l-1)r(P)}\right).
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
& \exp\left(\sum_{n>0}\frac{p^n}{n}\frac{\chi_{-y^n}(\mathbb{CP}_2)}{1-(yp)^n}\right) = \exp\left(\sum_{n>0}\frac{p^n}{n}\frac{1+y^n+y^{2n}}{1-(yp)^n}\right) \\
= & \exp\left(\sum_{n>0}\frac{p^n}{n}(1+y^n+y^{2n})\sum_{m\geq 0}(yp)^{nm}\right) \\
= & \prod_{m\geq 0}\exp\left(\sum_{n>0}\frac{1}{n}((y^m p^{m+1})^n + (y^{m+1}p^{m+1})^n + (y^{m+2}p^{m+1})^n)\right) \\
= & \prod_{m\geq 0}\exp\left(-\ln(1-y^m p^{m+1}) - \ln(1-y^{m+1}p^{m+1}) - \ln(1-y^{m+2}p^{m+1})\right) \\
= & \prod_{m\geq 0}\frac{1}{(1-y^m p^{m+1})(1-y^{m+1}p^{m+1})(1-y^{m+2}p^{m+1})} \\
= & \prod_{m\geq 1}\frac{1}{(1-y^{m-1}p^m)(1-y^m q^m)(1-y^{m+1}p^m)}.
\end{aligned}$$

For $l = 0, 1, 2,$

$$\begin{aligned}
& \prod_{m\geq 1}\frac{1}{1-y^{m+l-1}p^m} = \prod_{m\geq 1}\sum_{N_m\geq 0}(y^{m+l-1}p^m)^{N_m} \\
= & \sum_{n\geq 0}\sum_{\sum mN_m=n} y^{\sum(mN_m+(l-1)N_m)}p^{\sum mN_m} = \sum_{n\geq 0}\sum_{\sum mN_m=n} y^{n+(l-1)\sum N_m}p^n \\
= & \sum_{n\geq 0}\sum_{P\in\mathcal{P}(n)} y^{n+(l-1)r(P)}p^n,
\end{aligned}$$

and so

$$\sum_{n\geq 0}\chi_{-y}(\mathbb{CP}_2^{[n]})p^n = \exp\left(\sum_{n>0}\frac{p^n}{n}\frac{\chi_{-y^n}(\mathbb{CP}_2)}{1-(yp)^n}\right).$$

3.7. **The case $\mathbb{CP}_1\times\mathbb{CP}_1$ case.** Consider the following T^2-action on $\mathbb{CP}_1\times\mathbb{CP}_1$:

$$(t_1,t_2)\cdot([z_0:z_1],[w_0:w_1]) = ([z_0:t_1z_1],[w_0:t_2w_1]),$$

where $t_1, t_2\in\mathbb{C}^*$, $[z_0:z_1],[w_0:w_1]\in\mathbb{CP}_1$. It has four fixed points:

$$\begin{aligned}
P_{00} &= ([1:0],[1:0]), & P_{01} &= ([1:0],[0:1]), \\
P_{10} &= ([0:1],[1:0]), & P_{11} &= ([0:1],[0:1]).
\end{aligned}$$

Similar to the discussion in §3.1, the fixed point set of the induced action on the n-th Hilbert scheme of $\mathbb{CP}_1\times\mathbb{CP}_1$ is in one-to-one correspondence with the following set

$$\{(P_{00},P_{01},P_{10},P_{11})\in\mathcal{P}(n_{00})\times\mathcal{P}(n_{01})\times\mathcal{P}(n_{10})\times\mathcal{P}(n_{11}) : n_{00}+n_{01}+n_{10}+n_{11}=n\}.$$

Furthermore the tangent space at the fixed point corresponding to the quadruple of partitions $(P_{00},P_{01},P_{10},P_{11})$ has the following decomposition:

$$T = T_{00}\oplus T_{01}\oplus T_{10}\oplus T_{11},$$

where in the representation ring of T^2,

$$T_{ab} = \sum_{e\in E(P_{ab})} (\lambda_{ab}^{-a(e)-1}\mu_{ab}^{l(e)} + \lambda_{ab}^{a(e)}\mu_{ab}^{-l(e)-1}),$$

where

$$\lambda_{ab} = t_1^{(-1)^a}, \qquad \mu_{ab} = t_2^{(-1)^b},.$$

As in 3.2 we get

$$\sum_{n\geq 0} \chi_{-y}((\mathbb{CP}_1\times\mathbb{CP}_1)^{[n]})q^n = \prod_{a,b=0}^{1}\left(\sum_{n_{ab}\geq 0}\sum_{P_{ab}\in\mathcal{P}(n_{ab})} \chi_{-y}^{(ab)}(P_{ab})q^{n_{ab}}\right),$$

where

$$\chi_{-y}^{(ab)}(P_{ab}) = \prod_{e\in E(P^t_{ab})} \frac{1-y\lambda_{ab}^{-a(e)-1}\mu_l^{l(e)}}{1-\lambda_{ab}^{-a(e)-1}\mu_l^{l(e)}}\cdot\frac{1-y\lambda_{ab}^{a(e)}\mu_{ab}^{-l(e)-1}}{1-\lambda_{ab}^{a(e)}\mu_{ab}^{-l(e)-1}}.$$

After taking the limit as $t_1\to 0$ then taking the limit as $t_2\to 0$, one obtains

$$\begin{aligned}
&\sum_{n\geq 0} p^n\chi_{-y}((\mathbb{CP}_1\times\mathbb{CP}_1)^{[n]})\\
=\;&\left(\sum_{n\geq 0}p^n\sum_{P\in\mathcal{P}(n)} y^n\right)\cdot\left(\sum_{n\geq 0}p^n\sum_{P\in\mathcal{P}(n)} y^{n+r(P)}\right)^2\cdot\left(\sum_{n\geq 0}p^n\sum_{P\in\mathcal{P}(n)} y^{n-r(P)}\right)\\
=\;&\prod_{m\geq 0}\frac{1}{(1-y^mp^{m+1})(1-y^{m+1}p^{m+1})(1-y^{m+2}p^{m+1})}\\
=\;&\exp\left(\sum_{n>0}\frac{p^n}{n}\frac{1+2y^n+y^{2n}}{1-(yp)^n}\right) = \exp\left(\sum_{n>0}\frac{p^n}{n}\frac{\chi_{-y^n}(\mathbb{CP}_1\times\mathbb{CP}_1)}{1-(yp)^n}\right)
\end{aligned}$$

Since χ_y is a complex genus, by §1.3, we have completed the proof of (3).

References

[1] M.F. Atiyah, R. Bott, The moment map and equivariant cohomology, **Topology 23** (1984), 1-28.

[2] J.B. Carrell, D.I. Lieberman, *Holomorphic vector fields and Kaehler manifolds*, **Invent. Math. 21** (1973), 303–309.

[3] J. Cheah, *On the cohomology of Hilbert schemes of points*, **J. Algebraic Geom. 5** (1996), no. 3, 479–511.

[4] R. Dijkgraaf, G. Moore, E. Verlinde, H. Verlinde, *Elliptic genera of symmetric products and second quantized strings*, **Comm. Math. Phys. 185** (1997), no. 1, 197–209.

[5] G. Ellingsrud, L. Göttsche, M. Lehn, *On the Cobordism Class of the Hilbert Scheme of a Surface*, preprint, math.AG/9904095.

[6] G. Ellingsrud, A. Strømme, *On the cohomology of the Hilbert scheme of points in the plane* **Invent. Math. 87** (1987), 343-352.

[7] J. Fogarty, *Algebraic families on algebraic surfaces*, **Amer. J. Math. 10** (1968), 511-522.

[8] H. Göttsche, W. Soergel, *Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces*, **Math. Ann. 296** (1993), no. 2, 235–245.

[9] M. Haiman, *t,q-Catalan numbers and the Hilbert scheme.* Selected papers in honor of Adriano Garsia (Taormina, 1994). **Discrete Math. 193** (1998), no. 1-3, 201–224.

[10] F. Hirzebruch, **Topological ethods in algebraic geometry**, Grundl. der math. Wiss. 131, Springer Verlag, Berlin, 1978.

[11] J. Milnor, *O the cobordism ring Ω^* and a complex analogue*, **Amer. J. Math. 82** (1960), 505-521.

[12] J. Zhou, *Delocalized equivariant cohomology of symmetric products*, preprint, math.DG/9910028.

Department of Mathematics, Stanford University, Stanford, CA 94305
E-mail address: kefeng@math.stanford.edu

Department of Mathematics, Texas A&M University, College Station, TX 77843
E-mail address: cyan@math.tamu.edu

Department of Mathematics, Texas A&M University, College Station, TX 77843
E-mail address: zhou@math.tamu.edu