ON MODULAR INVARIANCE AND RIGIDITY THEOREMS

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1. Introduction

Let M be a compact smooth manifold with group action and P be an elliptic operator on M which commutes with the action. Then the kernal and cokernal of P are representations of the action group. For an element g in the action group, the Lefschetz number of P at g is

$$F(g) = \mathrm{tr}_g \mathrm{Ker} P - \mathrm{tr}_g \mathrm{Coker} P.$$

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We say that P is rigid with respect to this group action, if $F(g)$ is independent of g. In the following, we will only consider S^1-action, in which case two well-known rigid elliptic operators are the signature and the Dirac operator. Obviously, if P is rigid with respect to S^1-action, then it is rigid with respect to any compact connected Lie group action.

Motivated by the work of Landweber-Stong, in [W] Witten derived a series of elliptic operators from LM, the loop space of M. The indices of these operators are the signature, $\hat{\mathfrak{A}}$-genus or the Euler characteristic of LM. He also derived some elliptic operators which do not have finite dimensional analogues. The cohomological aspects of these operators were discussed in detail by W. Lerche, B. Nilsson, A. Schellekens, N. Warner and many other physicists. Surprisingly the elliptic genus of Landweber-Stong turns out to be the index of one of these elliptic operators. Motivated by physics, Witten conjectured that these elliptic operators should be rigid with respect to S^1-action. These conjectures generalize the rigidity of the usual signature, Euler characteristic and Dirac operator to infinite dimensional manifolds.

After some partial work of Ochanine and Landweber-Stong, these remarkable conjectures were first proved by Taubes, then by Bott-Taubes. Hirzebruch and Krichever proved Witten's conjectures for almost complex manifold case. They used the very technical transfer argument. Many aspects of mathematics are involved in their proofs. Taubes used analysis of Fredholm operators; Krichever used cobordism; Bott-Taubes and Hirzebruch used the Atiyah-Bott-Segal-Singer Lefschetz fixed point formula.

In [Liu1] I observed that all of these operators have some kind of intrinsic symmetry under the action of the modular group $SL_2(\mathbf{Z})$, which actually implies their rigidity. This observation immediately gives a very simple and unified proof of the above conjectures of Witten. There the classical Jacobi theta-functions came into play in a very nice and crucial way. Strictly speaking, it is the theta-function expressions of the Lefschetz numbers of these elliptic operators that attracted me to the modularity argument.

This paper is the continuation of [Liu1] and is naturally divided into three parts. The main results which were circulated in my preprints [Liu2] and [Liu3] and in my thesis [Liu4], were anounced in [Liu5].

In the first part, by using the beautiful results of Kac-Peterson-Wakimoto about the modular invariance of the characters of affine Lie algebras, under a very natural assumption on the first equivariant Pontrjagin class, I prove the rigidity of the Dirac operator on loop space twisted by positive energy loop group representations of any level, while

the Witten rigidity theorems are the special cases of level 1. One can immediately construct many new rigid elliptic operators from this theorem. In this paper I have only considered the tensor products of level 1 representations and hope to discuss the general case in another paper. In the second part, I generalize the rigidity theorems in part I and [Liu1] to the so-called non-zero anomaly cases. As corollaries I obtain a series of interesting holomorphic Jacobi forms and many new vanishing theorems, especially an $\hat{\mathfrak{A}}$-vanishing theorem for loop spaces with spin structures. Using our result, G. Hoehn was able to characterize this loop space $\hat{\mathfrak{A}}$-vanishing theorem in terms of $MO < 8 >$-fibrations.

In the third part I discuss the relationships between these elliptic operators and the geometry of elliptic modular surfaces. We show that the Lefschetz numbers of these elliptic operators are holomorphic sections of certain holomorphic line bundles on some elliptic modular surfaces. In studying their degenerations to the singular fibers of the elliptic surfaces, we get some topological results for manifolds and bundles with group actions. This idea also gives a very natural algebro-geometric explanation of the transfer argument used in [BT], [H] and [Kri]. Finally in Appendix B, by a simple observation I prove a rigidity theorem for mod 2 elliptic genera which was also obtained by K. Ono independently.

While its rigidity property is basically clarified, many aspects of elliptic genus remain mysterious, notably the geometric construction of elliptic cohomology, its relationships with the monstrous moonshine, with vertex operator algebras, with mirror symmetry and with the Virasoro algebra. The study of these topics is under progress.

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I have presented the results in the 877 AMS meeting, in the topology and geometry seminars of MIT, Yale University, Columbia University,

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2. Loop Groups and Rigidity Theorems

In this part we prove the rigidity of the Dirac operator on loop space twisted by general positive energy loop group representations for both spin manifolds and almost complex manifolds.

After state Theorem 1, the main result in this part, we review some basic results in affine Lie algebra theory, especially the modular invariance of the characters of integrable highest weight modules. Then we give the construction of $\psi(E,V)$, which, used in Theorem 1, is a formal power series with coefficients in the K-group $K(M)$, from a positive energy representation E of $\tilde{L}\mathrm{Spin}(2l)$ of highest weight and a rank $2l$ spin vector bundle V on M. This construction is motivated by Brylinski's work [Br]. Some examples, including several new rigid elliptic operators are given in Section 2.4 as the corollaries of Theorem 1. From the point of view of loop group representation, our examples have exhausted all of the rigid elliptic genera. Theorem 1 is proved in Section 2.5. In Section 2.6, we discuss the rigidity theorems for almost complex manifolds.

2.1. **A General Rigidity Theorem.** Let $\tilde{L}\mathrm{Spin}(2l)$ denote the central extension of the loop group $L\mathrm{Spin}(2l)$ and E be a positive energy representation of it. See Section 2.2 for the definition of positive energy. Given a rank $2l$ spin vector bundle V on a spin manifold M, we can construct an element $\psi(E,V)$ in $K(M)[[q]]$ associated to E and V. Here $q=e^{2\pi i\tau}$ with τ in the upper half plane H is a parameter. See Section 2.3 for the construction. In this paper, by a vector bundle we always mean a real vector bundle, except otherwise specified. Let D denote the Dirac operator on M. Assume that there exists an S^1-action on M which lifts to V. For an equivariant vector bundle F, let $p(F)_{S^1}$ denote its first equivariant Pontrjagin class. See Appendix A for a geometric discussion about equivariant characteristic classes. Then we will prove the following:

Theorem 1: *For every positive energy representation E of $\tilde{L}Spin(2l)$ of highest weight of level m, if $p_1(M)_{S^1}=mp_1(V)_{S^1}$, then*

$$D\otimes\otimes_{n=1}^{\infty}S_{q^n}(TM)\otimes\psi(E,V)$$

is rigid.

Here recall that, for a vector bundle F,

$$S_t(F) = 1 + tF + t^2 S^2 F + \cdots$$

is the symmetric operation in $K(M)[[t]]$. Theorem 1 actually holds for any semisimple and simply connected Lie group, instead of $\mathrm{Spin}(2l)$. It actually holds in much more general situations. See Section 2.4 for the details.

If $m = 1$, we know that $\tilde{L}\mathrm{Spin}(2l)$ has four irreducible highest weight representations of positive energy which exactly give those elliptic operators considered by Witten [W], Bott-Taubes [BT] and [Liu1]. See the examples in Section 2.4. Therefore Theorem 1 includes all of the Witten rigidity theorems for spin manifolds.

In our proof, the actual use of the spin condition on M is the existence of the Dirac operator which we need to show that the modular transformations of the Lefschetz number of the above elliptic operator are still the Lefschetz numbers of some twisted Dirac operators. This shows that the modular invariance of the characters of the representations of affine Lie algebras discussed in Section 2.2 implies the rigidity of the elliptic operator in Theorem 1. This is surprising. We would like to know whether there is a finite dimensional analogue of this modular property which may explain the famous $\hat{\mathfrak{A}}$-vanishing theorem of Atiyah-Hirzebruch. We are also interested in giving an explanation of our results by using the geometry of loop space and physics.

2.2. **Affine Lie Algebras.** In Theorem 1 we need highest weight positive energy representations of $\tilde{L}\mathrm{Spin}(2l)$. This kind of representations can always be obtained by lifting the integrable highest weight representations of the affine Lie algebra $\hat{L}so(2l)$ associated to $so(2l)$, the Lie algebra of $\mathrm{Spin}(2l)$. In this section we review some basic facts about affine Lie algebras, especially the modular invariance of their characters.

Given a simple, simply connected compact Lie group G of rank l, let $\mathfrak{g}$ denote its Lie algebra. Let $\mathfrak{h}$ be the Cartan subalgebra, W be the Weyl group. Denote by $Q = \sum_{i=1}^{l} \mathbf{Z}\alpha_i$, where $\{\alpha_i\}$ is the root basis, the root lattice of $\mathfrak{g}$. The affine Lie algebra associated to $\mathfrak{g}$ is

$$\hat{L}\mathfrak{g} = \mathfrak{g} \otimes_{\mathfrak{R}} \mathbf{C}[\mathfrak{t}, \mathfrak{t}^{-1}] \oplus \mathbf{C}\mathfrak{K} \oplus \mathbf{C}\mathfrak{d}$$

where K, d are two operators on $\mathfrak{g}$. Explicitly K (resp. d) is the infinitesimal generator of the central element (resp. the rotation of S^1) of $\tilde{L}G$.

$\hat{L}\mathfrak{g}$ has the triangle decomposition

$$\hat{L}\mathfrak{g} = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_+$$

where $\hat{\mathfrak{n}}_\pm$ are the nilpotent subalgebras and

$$\hat{\mathfrak{h}} = \mathfrak{h} \otimes_R C \oplus \mathbf{C}K \oplus \mathbf{C}d$$

is the Cartan subalgebra. Let $\hat{\mathfrak{h}}^*$ be the dual of $\hat{\mathfrak{h}}$ with respect to the normalized symmetric invariant bilinear form $(\cdot,\cdot)$ on $\hat{L}\mathfrak{g}$ which extends the standard symmetric bilinear form on $\mathfrak{g}$, such that

$$(\mathbf{C}K + \mathbf{C}d,\ \mathfrak{g} \otimes_R C[t,t^{-1}]) = 0;\ (K,K) = 0,$$

$$(d,d) = 0, \quad (K,d) = 1.$$

Let $< \cdot,\cdot >$ denote the pairing between $\hat{\mathfrak{h}}$ and $\hat{\mathfrak{h}}^*$, then the level of $\lambda \in \hat{\mathfrak{h}}^*$ is defined to be $< \lambda, K >$.

$\hat{L}\mathfrak{g}$ falls into class $X_N^{(1)}$ in the classification of Kac-Moody algebras (see [Ka]). An $\hat{L}\mathfrak{g}$-module U is called a highest weight module with highest weight $\Lambda \in \hat{\mathfrak{h}}^*$ if there exists a nonzero vector $v_\Lambda \in U$ such that

$$\hat{\mathfrak{n}}_+(v_\Lambda) = 0;\ h(v_\Lambda) = \Lambda(h)v_\Lambda \text{ for } h \in \hat{\mathfrak{h}};\ \text{and}$$

$$U(\hat{L}\mathfrak{g})(v_\Lambda) = U$$

where $U(\hat{L}\mathfrak{g})$ is the universal enveloping algebra of $\hat{L}\mathfrak{g}$. If an irreducible representation $L(\Lambda)$ of $\hat{L}\mathfrak{g}$ is of highest weight Λ and the level of $\Lambda = k$ we say that $L(\Lambda)$ is of level k. It is called integrable if $\Lambda \in P_+$ where

$$P_+ = \{\lambda \in \hat{\mathfrak{h}}^* | (\lambda, \alpha_i) \in \mathbf{Z} \text{ and } \geq 0 \text{ for all } i\}$$

is the set of dominant integral weights.

An integrable highest weight representation $L(\Lambda)$ of $\hat{L}\mathfrak{g}$ can always be lifted to a representation of $\tilde{L}G$ which turns out to be irreducible and of positive energy. This lifted representation has the same level as $L(\Lambda)$. See [PS]. Recall that for each level there are only finitely many integrable highest weight representations induced from the irreducible representations of G.

An $\hat{L}\mathfrak{g}$-module V can be splitted into the form $\oplus_{\lambda\in\hat{\mathfrak{h}}^*} V_\lambda$, when restricted to the Cartan subalgebra $\hat{\mathfrak{h}}$. The formal Kac-Weyl character of V is defined to be

$$\mathrm{ch}_V = \sum_{\lambda\in\hat{\mathfrak{h}}^*} \mathrm{dim} V_\lambda e^\lambda.$$

The normalized character of $L(\Lambda)$ is

$$\chi_\Lambda = q^{m_\Lambda} \mathrm{ch}_{L(\Lambda)}$$

where

$$m_\Lambda = \frac{(\Lambda + 2\rho, \Lambda)}{2(m + h^\vee)} - \frac{m \dim \mathfrak{g}}{24(m + h^\vee)}$$

with $h^\vee =< \rho, K >$ the dual Coxeter number of $\mathfrak{g}$ and ρ half the sum of the positive roots. We call q^{m_Λ} the anomaly factor.

Let $M = \mathbf{Z}(W \cdot \theta)$, where θ is the long root in Q and W is the Weyl group of $\mathfrak{g}$, be a lattice in $\mathfrak{h}^*$. For any integer m, let

$$P_+^m = \{\lambda \in P_+ | < \lambda, K >= m\}$$

be the level m subset of the dominant integral weights. Let $\Lambda_0, \delta \in \hat{\mathfrak{h}}^*$ be the elements such that

$$\delta|_{\mathfrak{h} \oplus \mathbf{C}\mathfrak{K}} = 0, \ < \delta, d >= 1;$$

$$\Lambda_0|_{\mathfrak{h} \oplus \mathbf{C}\mathfrak{d}} = 0, \ < \Lambda_0, K >= 1.$$

Then χ_Λ can be expressed as

$$\chi_{L(\Lambda)} = \frac{A_{\Lambda+\rho}}{A_\rho} \qquad (+)$$

where

$$A_\lambda = \sum_{w \in W} \varepsilon(w) \Theta_{w(\lambda)}$$

with Θ_λ the classical theta-functions associated to the lattice M. If we choose an orthonormal basis of $\mathfrak{h}^* \otimes_{\mathfrak{R}} \mathbf{C}$, $\{v_i\}_{i=1}^l$, such that for $v \in \hat{\mathfrak{h}}^*$ one has

$$v = 2\pi i \sum_{s=1}^{l} z_s v_s - \tau \Lambda_0 + u\delta$$

where $z = \sum_{s=1}^l z_s v_s \in \mathfrak{h}^* \otimes_R \mathbf{C}$, then

$$\Theta_\lambda(z, \tau) = e^{2\pi i m u} \sum_{\gamma \in M + m^{-1}\bar{\lambda}} e^{\pi i m \tau(\gamma,\gamma) + 2\pi i m(\gamma, z)}.$$

Here $\bar{\lambda}$ means the orthogonal projection of λ from $\hat{\mathfrak{h}}^*$ to $\mathfrak{h}^* \otimes_{\mathfrak{R}} \mathfrak{C}$ with respect to the bilinear form $(\cdot, \cdot)$ and $\gamma = \sum_{i=1}^l \gamma_i v_i$ with

$$(\gamma, z) = \sum_{i=1}^{l} \gamma_i z_i.$$

Obviously $\chi_\Lambda(z, \tau)$ is well-defined for τ in the upper half plane. Another expression of χ_Λ is a finite sum

$$\chi_\Lambda(z, \tau) = \sum_{\lambda \in P^m \mathrm{mod}(mM + \mathbf{C}\delta)} c_\lambda^\Lambda(\tau) \Theta_\lambda(z, \tau)$$

where P^m is the level m element in the integral weight lattice and $\{c_\lambda^\Lambda(\tau)\}$ are some weight $-\frac{1}{2}l$ modular forms. They are called string functions in [Ka].

Recall that the modular transformation of

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$$

on $(t, \tau) \in \mathbf{C} \times \mathbf{H}$ is given by

$$g(t, \tau) = (\frac{t}{c\tau + d}, \frac{a\tau + b}{c\tau + d})$$

which defines a group action. Obviously two generators of $SL_2(\mathbf{Z})$,

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

act by

$$S(t, \tau) = (\frac{t}{\tau}, -\frac{1}{\tau}), \ T(t, \tau) = (t, \tau + 1).$$

The following theorem, which is due to Kac and Peterson, is one of the most beautiful results in affine Lie algebra theory. It is the easy consequence of the theta-function expression (+) of the character.

Theorem: *Let $\Lambda \in P_+^m$, then*

a)

$$\chi_\Lambda(\frac{z}{\tau}, -\frac{1}{\tau}) = e^{\pi i m \frac{(z,z)}{\tau}} \sum_{\Lambda' \in P_+^m mod \mathbf{C}\delta} S_{\Lambda,\Lambda'} \chi_{\Lambda'}(z, \tau)$$

for some complex numbers $S_{\Lambda,\Lambda'}$;

b)

$$\chi_\Lambda(z, \tau + 1) = e^{2\pi i m_\Lambda} \chi_\Lambda(z, \tau). \quad \square$$

In general, this theorem implies that, up to the factor $e^{c\pi i m \frac{(z,z)}{c\tau + d}}$, the complex vector space spanned by the characters of the highest weight modules of a given level is stable under the modular transformations. Note that we have slightly revised the statements in [Ka] to fit our purpose. Especially we have omitted considering the variable u. Instead we get the exponential factors. For $\alpha \in M$ we also have

$$\begin{aligned} \Theta_\lambda(z + \alpha, \tau) &= \Theta_\lambda(z, \tau); \\ \Theta_\lambda(z + \alpha\tau, \tau) &= e^{2\pi i m(z,\alpha) + \pi i m(\alpha,\alpha)} \Theta_\lambda(z, \tau). \end{aligned}$$

This, together with its transformation formulas under $SL_2(\mathbf{Z})$, means that χ_Λ is an l-variable Jacobi form of index $\frac{m}{2}$ and weight 0. See Section 3.2 of the next part for the definition of Jacobi forms. We refer

to Theorem 13.8 in [Ka] for the details of the above theorem. In the proof of Theorem 1 we will take

$$(z_1, \cdots, z_l) = (n_1 t, \cdots, n_l t)$$

for some integers $\{n_i\}$, this makes χ_Λ into a one variable Jacobi form.

Example: The irreducible highest weight $\hat{L}su(2)$-modules of level l are parametrized by an integer j. We denote them by $V_{j,l}$. The corresponding characters are given by

$$\chi_{j,l}(z,\tau) = \frac{\theta_{j+1,l+2}(z,\tau)}{\theta_{1,2}(z,\tau)}$$

where

$$\theta_{k,m}(z,\tau) = e^{2\pi i m u} \sum_{k \in \mathbf{Z} + \frac{k}{2m}} e^{2\pi i m (k^2 \tau + kz)} \text{ for } k \in \mathbf{Z} (\text{mod } 2m)$$

is the theta-function of degree (m,k).

One has

$$\chi_{j,l}(\frac{z}{\tau}, -\frac{1}{\tau}) = e^{\frac{\pi i l z^2}{\tau}} \sum_{k \in \mathbf{Z} (\text{mod} 2m\mathbf{Z})} A^l_{j,k} \chi_{k,l}(z,\tau)$$

where

$$A^l_{j,k} = \frac{\sqrt{2}}{\sqrt{l+2}} \sin \frac{\pi (j+1)(k+1)}{l+2}$$

as appeared in the now famous Verlinde formulas.

2.3. **The Construction of $\psi(E,V)$.** For a simply connected simple Lie group G, the positive energy representation E of the loop group $\tilde{L}G$ is characterized by the following properties:

(a) E is a direct sum of irreducible representations.

(b) Let R_θ be the rotation action of the loop by the angle θ, then R_θ acts on E as $\exp(-iA\theta)$ with A an operator of positive spectrum and the subspace $E_n = \{v \in E : \ R_\theta(v) = e^{in\theta} v\}$ is a finite dimensional representation of G.

(c) The action of $\tilde{L}G \rtimes S^1$ on E naturally extends to a smooth action of $\tilde{L}G \rtimes \text{Diff}^+(S^1)$, where $\text{Diff}^+(S^1)$ is the group of orientation preserving diffeomorphisms of S^1.

Assume that the infinitesimal generator K of the central element of $\tilde{L}G$ acts on E by $K \cdot v = mv$, for any $v \in E$ and a positive integer m, then m is called the level of E. As discussed in last section, positive energy representations of level m can always be lifted from the integrable representations of level m of the corresponding affine Lie algebra. See [PS] for the details of positive energy repersentations.

Consider $G = \mathrm{Spin}(2l)$. Since the representation E in Theorem 1 is of positive energy, one then has the decomposition $E = \oplus_{n\geq 0} E_n$ under the action of R_θ. Here each E_n is a finite dimensional representation of $\mathrm{Spin}(2l)$. Let Q_V be the frame bundle of V, then Q_V is a principal $\mathrm{Spin}(2l)$-bundle. For each E_n we can get an element $\tilde{E}_n \in K(M)$ associated to E_n and Q_V. Let us write formally

$$\psi(E, V) = \sum_{n\geq 0} \tilde{E}_n q^n \in K(M)[[q]].$$

This $\psi(E, V)$ is the desired element in Theorem 1.

Now let us discuss the characteristic classes of $\psi(E, V)$. For $G =$Spin$(2l)$, let $\{v_j\}_{j=1}^l$ be an orthonormal basis of $\hat{\mathfrak{h}}^*$, the dual of the Cartan subalgebra. The root basis $\{\alpha_j\}_{j=1}^l$ is given by

$$\{\alpha_1 = v_1 - v_2,\ \cdots\ \alpha_{l-1} = v_{l-1} - v_l,\ \alpha_l = v_{l-1} + v_l\}.$$

The root lattice is

$$Q = \{\sum_i k_i v_i | k_i \in \mathbf{Z}, \sum_i k_i \in 2\mathbf{Z}\}.$$

In this case the long root $\theta = v_1 + v_2$, the Weyl group W consists of all permutations and even number of sign changes of the v_j's. The lattice $M = Z(W \cdot \theta) = Q$. The dual Coxeter number $h^\vee = 2(l-1)$. The affine Lie albegra $\hat{L}so(2l)$ is of class $D_l^{(1)}$ in the notations of [Ka].

Let $R(\mathrm{Spin}(2l))$ denote the ring of $\mathrm{Spin}(2l)$ representations and by $H^*_{\mathrm{Spin}(2l)}(Q)$ the ring of characteristic polynomials. We have the characteristic map

$$\mathrm{ch}:\ R(\mathrm{Spin}(2l)) \to H^*_{\mathrm{Spin}(2l)}(Q)$$

which sends a representation to its character. Let $\{v_j\}_{j=1}^l$ also denote the standard character of $\mathfrak{h}$, then

$$H^*_{\mathrm{Spin}(2l)}(Q) = Q[[v_1, \cdots, v_l]]^W,$$

the W-invariant polynomials. By our choice of the coordinate in $\hat{\mathfrak{h}}^*$ in last section, we can view $\chi_\Lambda(z, \tau)$ as $\chi_\Lambda(v, \tau)$, where

$$v = (v_1, \cdots, v_l),$$

evaluated at

$$\oplus_{j=1}^l \begin{pmatrix} 0, & 2\pi i z_j \\ -2\pi i z_j, & 0 \end{pmatrix} \in \mathfrak{h} \otimes_R C.$$

Therefore for $E = L(\Lambda)$ we, consider $\psi(E, V)$ as an element in $R(\mathrm{Spin}(2l))[[q]]$, can write the character as

$$\mathrm{ch}(\psi(E, V)) = \mathrm{ch}_E(v, \tau) = q^{-m_\Lambda} \chi_\Lambda(v, \tau).$$

Under transgression, $\{\pm v_j\}_{j=1}^l$ lift to $\{\pm 2\pi i x_j\}_{j=1}^l$ where $\{\pm 2\pi i x_j\}_{j=1}^l$ are the formal Chern roots of V and the character map lifts to the Chern character. Therefore we only need to replace z_j by x_j in the character $\mathrm{ch}_E(z,\tau)$ when considering the Chern character of $\psi(E,V)$.

2.4. **Corollaries and Examples.** In this section we give several examples as the corollaries of Theorem 1. In the following, real bundles will be automatically complexified.

Example a) Let $\theta_3(v,\tau)$, $\theta_2(v,\tau)$, $\theta_1(v,\tau)$ and $\theta(v,\tau)$ be the classical Jacobi theta-functions. Recall that we have

$$\begin{aligned}
\theta_3(v,\tau) &= c\cdot\prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}}e^{2\pi iv})\prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}}e^{-2\pi iv})\\
\theta_2(v,\tau) &= c\cdot\prod_{n=1}^{\infty}(1-q^{n-\frac{1}{2}}e^{2\pi iv})\prod_{n=1}^{\infty}(1-q^{n-\frac{1}{2}}e^{-2\pi iv})\\
\theta_1(v,\tau) &= c\cdot q^{\frac{1}{8}}2\cos\pi v\prod_{n=1}^{\infty}(1+q^{n}e^{2\pi iv})\prod_{n=1}^{\infty}(1+q^{n}e^{-2\pi iv})\\
\theta(v,\tau) &= c\cdot q^{\frac{1}{8}}2\sin\pi v\prod_{n=1}^{\infty}(1-q^{n}e^{2\pi iv})\prod_{n=1}^{\infty}(1-q^{n}e^{-2\pi iv})
\end{aligned}$$

where $c=\prod_{n=1}^{\infty}(1-q^n)$. Let M and V be as in Theorem 1. Consider the case of level $m=1$ and the spin representation $S=S^+ + S^-$ of $\tilde{L}\mathrm{Spin}(2l)$. Here $S^{\pm}$ are the half spin representations. For a vector bundle F, let

$$\Lambda_t(F) = 1 + tF + t^2\Lambda^2 F + \cdots$$

be the wedge operation in $K(M)[[t]]$. Then one gets

$$\psi(S,V) = \triangle(V)\otimes_{n=1}^{\infty}\Lambda_{q^n}(V).$$

where $\triangle(V)=\triangle^+(V)\oplus\triangle^-(V)$ is the spinor bundle of V. In terms of the coordinate of $\mathfrak{h}^*\otimes_R\mathbf{C}$ introduced in Section 2.2, the normalized Kac-Weyl character is

$$\chi_S(z,\tau) = \frac{1}{\eta(\tau)^l}\prod_{i=1}^{l}\theta_1(z_i,\tau)$$

where $\eta(\tau)=q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind eta-function. The elliptic operator in Theorem 1 is

$$D\otimes\triangle(V)\otimes_{n=1}^{\infty}\Lambda_{q^n}(V)\otimes_{m=1}^{\infty}S_{q^m}(TM).$$

Example b) Similarly for $T=S^+-S^-$, one gets

$$\psi(T,V) = (\triangle^+(V)-\triangle^-(V))\otimes_{n=1}^{\infty}\Lambda_{-q^n}(V)$$

and the character

$$\chi_T(z,\tau) = \frac{1}{\eta(\tau)^l}\prod_{i=1}^{l}\theta(z_i,\tau).$$

This gives another elliptic operator

$$D\otimes(\triangle^+(V)-\triangle^-(V))\otimes_{n=1}^{\infty}\Lambda_{-q^n}(V)\otimes_{m=1}^{\infty}S_{q^m}(TM).$$

Note that the anomaly factor for Examples a) and b) is $q^{\frac{l}{12}}$.

Let Q_V be the frame bundle of V. From Q_V we naturally get a principal $L\mathrm{Spin}(2l)$ bundle LQ_V on LM. Actually LQ_V is the loop space of Q_V. If $p_1(V)=0$, we can further get a principal $\tilde{L}\mathrm{Spin}(2l)$ bundle $\tilde{Q}_V$ by central extension. One can associate S and T to $\tilde{Q}_V$ to get two vector bundles on LM which are the infinite dimensional analogues of $\triangle^+(V)\oplus\triangle^-(V)$ and $\triangle^+(V)-\triangle^-(V)$ respectively. See [Br] for the details. Since $D\otimes\otimes_{m=1}^{\infty}S_{q^m}(TM)$ corresponds to the Dirac operator on LM, the constructions in Examples a) and b) give, respectively, the signature and the Euler characteristic operator for the loop bundle LV which is the loop space of V, on LM.

Example c) $\tilde{L}\mathrm{Spin}(2l)$ has exactly four irreducible highest weight representations of level $m=1$. The remaining two are denoted by S'_+ and S'_+. Let $S'=S'_++S'_-$ and $T'=S'_+-S'_-$. Then we have

$$\chi_{S'}(z,\tau) = \frac{1}{\eta(\tau)^l}\prod_{i=1}^{l}\theta_2(z_i,\tau),$$

$$\chi_{T'}(z,\tau) = \frac{1}{\eta(\tau)^l}\prod_{i=1}^{l}\theta_3(z_i,\tau);$$

and respectively

$$\psi(S',V)=\otimes_{n=1}^{\infty}\Lambda_{-q^{n-\frac{1}{2}}}(V),$$
$$\psi(T',V)=\otimes_{n=1}^{\infty}\Lambda_{q^{n-\frac{1}{2}}}(V).$$

Their corresponding elliptic operators are

$$D\otimes\otimes_{n=1}^{\infty}\Lambda_{-q^{n-\frac{1}{2}}}(V)\otimes_{m=1}^{\infty}S_{q^m}(TM)$$

and

$$D\otimes\otimes_{n=1}^{\infty}\Lambda_{q^{n-\frac{1}{2}}}(V)\otimes_{m=1}^{\infty}S_{q^m}(TM)$$

respectively. The anomaly factor for both operators is $q^{-\frac{l}{24}}$.

The above examples are exactly those elliptic operators considered in [T], [BT], [W], [W1]. See also [Br]. By Theorem 1 all of these elliptic operators are rigid if $p_1(M)_{S^1}=p_1(V)_{S^1}$. Compare with the discussions in [Liu1].

Example d) Take $V = TM$ in the above examples, we get the elliptic operators discussed in [W]:

$$\begin{aligned} D &\otimes \triangle(M) \otimes_{n=1}^{\infty} \Lambda_{q^n}(TM) \otimes_{m=1}^{\infty} S_{q^m}(TM), \\ D &\otimes \otimes_{n=1}^{\infty} \Lambda_{-q^{n-\frac{1}{2}}}(TM) \otimes_{m=1}^{\infty} S_{q^m}(TM), \\ D &\otimes \otimes_{n=1}^{\infty} \Lambda_{q^{n-\frac{1}{2}}}(TM) \otimes_{m=1}^{\infty} S_{q^m}(TM), \end{aligned}$$

where $\triangle(M)$ is the spinor bundle of TM. Of course in this case, only the level 1 representations can satisfy the assumption of Theorem 1, except the trivial case $p_1(M)_{S^1} = 0$. Therefore we can say that for $V = TM$ in Theorem 1 the only possible rigid elliptic operators are given by the level 1 representations of $\tilde{L}\mathrm{Spin}(2l)$.

Example e) The virtual version of Example d), i.e. one replaces TM by $TM - \mathrm{dim}M$ to get

$$\begin{aligned} D &\otimes \triangle(M) \otimes_{n=1}^{\infty} \Lambda_{q^n}(TM - \mathrm{dim}M) \otimes_{m=1}^{\infty} S_{q^m}(TM - \mathrm{dim}M), \\ D &\otimes \otimes_{n=1}^{\infty} \Lambda_{-q^{n-\frac{1}{2}}}(TM - \mathrm{dim}M) \otimes_{m=1}^{\infty} S_{q^m}(TM - \mathrm{dim}M), \\ D &\otimes \otimes_{n=1}^{\infty} \Lambda_{q^{n-\frac{1}{2}}}(TM - \mathrm{dim}M) \otimes_{m=1}^{\infty} S_{q^m}(TM - \mathrm{dim}M). \end{aligned}$$

The indices of these operators are called universal elliptic genera.

We go further to consider the representation

$$P = S \oplus 2^l S' \oplus 2^l T'$$

of $\tilde{L}\mathrm{Spin}(2l)$ and take $V = TM$. The corresponding character is given by

$$\begin{aligned} \chi_P(z,\tau) &= \chi_S(z,\tau) + 2^k \chi_{S'}(z,\tau) + 2^k \chi_{T'}(z,\tau) \\ &= \frac{1}{\eta(\tau)^k}\Big(\prod_{i=1}^{k} \theta_1(z_i,\tau) + 2^k \prod_{i=1}^{k} \theta_2(z_i,\tau) + 2^k \prod_{i=1}^{k} \theta_3(z_i,\tau)\Big) \end{aligned}$$

where $k = \frac{1}{2}\mathrm{dim}M$. We would like to consider the virtual version of this example. The index of

$$D \otimes \otimes_{m=1}^{\infty} S_{q^m}(TM - \mathrm{dim}M) \otimes \psi(P, TM)_v$$

is an elliptic genus which gives modular forms of level 1. Here

$$\begin{aligned} \psi(P,TM)_v &= \triangle(M) \otimes_{n=1}^{\infty} \Lambda_{q^n}(TM - \mathrm{dim}M) \\ &+ 2^k \otimes_{n=1}^{\infty} \Lambda_{-q^{n-\frac{1}{2}}}(TM - \mathrm{dim}M) \\ &+ 2^k \otimes_{n=1}^{\infty} \Lambda_{q^{n-\frac{1}{2}}}(TM - \mathrm{dim}M) \end{aligned}$$

is the virtual version of $\psi(P,TM)$. Its modular property under $SL_2(\mathbf{Z})$ is easy to verify by using the transformation formulas of theta-functions. By Theorem 1 this genus is rigid. Without confusing the level of modular forms with the level of loop group representations, we say that this

elliptic genus is of level 1. This example solves a problem of Landweber in [La1] about the construction of level 1 elliptic genus. From the point of view of loop group representations, this genus seems to be the only possible rigid elliptic genus of level 1 for spin manifolds.

One can get more general rigid elliptic genera by considering

$$P_{a,b,c} = a(\tau)S \oplus b(\tau)S' \oplus c(\tau)T'$$

where $a(\tau), b(\tau), c(\tau)$ are modular forms over a modular subgroup $\Gamma(2K)$ for some positive integer $K \geq 1$. Then

$$D \otimes \otimes_{m=1}^{\infty} S_{q^m}(TM - \dim M) \otimes \psi(P_{a,b,c}, TM)_v$$

is an elliptic genus of level $2K$. Here

$$\begin{aligned} \psi(P_{a,b,c}, TM)_v &= a(\tau)\triangle(M) \otimes_{n=1}^{\infty} \Lambda_{q^n}(TM - \dim M) \\ &+ b(\tau) \otimes_{n=1}^{\infty} \Lambda_{-q^{n-\frac{1}{2}}}(TM - \dim M) \\ &+ c(\tau) \otimes_{n=1}^{\infty} \Lambda_{q^{n-\frac{1}{2}}}(TM - \dim M) \end{aligned}$$

is the virtual version of $\psi(P_{a,b,c}, TM)$ and lies in $K(M)[[q^{\frac{1}{2K}}]] \otimes \mathbf{C}$.

The proof of Theorem 1 works for more general loop group representations. Especially it works for the tensor product of two positive energy representations of highest weight and different level.

Example f) Consider the tensor product

$$Q = S \otimes S' \otimes T'$$

which is a level 3 representation. For an S^1-equivariant rank $2l$ spin vector bundle V with $3p_1(V)_{S^1} = p_1(M)_{S^1}$, by Theorem 1 we know that

$$D \otimes \otimes_{m=1}^{\infty} S_{q^m}(TM) \otimes \psi(Q, V)$$

is rigid. Here

$$\psi(Q, V) = \triangle(V) \otimes_{n=1}^{\infty} \Lambda_{q^n}(V) \otimes_{n=1}^{\infty} \Lambda_{-q^{n-\frac{1}{2}}}(V) \otimes_{n=1}^{\infty} \Lambda_{q^{n-\frac{1}{2}}}(V).$$

One can also consider level 2 representations $X = S \otimes S'$, $Y = S \otimes T'$ and $Z = S' \otimes T'$. As an easy corollary we know that, if the bundle V satisfies $p_1(M)_{S^1} = 2p_1(V)_{S^1}$, then

$$D \otimes \otimes_{m=1}^{\infty} S_{q^m}(TM) \otimes \psi(U, V),$$

for $U = X, Y$ or Z, is rigid. By taking the first two terms of the q-expansions we get the rigidity of the following elliptic operators

$$D \otimes \Lambda^2 V, \quad D \otimes S^2 V,$$

$$D \otimes \triangle(V) \otimes (TM + V + \Lambda^2 V),$$

$$D \otimes (TM + S^2(TM) + 2TM \otimes \Lambda^2 V + \Lambda^2 V \otimes \Lambda^2 V - 2V \otimes \Lambda^3 V),$$

$$D \otimes \triangle(V) \otimes (TM + V + S^2(TM) + V \otimes V + V \otimes TM + \Lambda^2 V + V \otimes \Lambda^2 V + TM \otimes \Lambda^2 V).$$

One can get more examples by taking tensor product of the basic representations, S, T, S' and T'.

Example g) Take three non-negative integers a, b, c and consider the representation

$$Q_{a,b,c,} = S^{\otimes a} \otimes S'^{\otimes b} \otimes T'^{\otimes c}.$$

Consider the corresponding elliptic operator

$$D \otimes \otimes_{m=1}^{\infty} S_{q^m}(TM) \otimes \psi(Q_{a,b,c}, V)$$

where

$$\psi(Q_{a,b,c}, V) = (\triangle(V) \otimes_{n=1}^{\infty} \Lambda_{q^n}(V))^{\otimes a} \otimes (\otimes_{n=1}^{\infty} \Lambda_{-q^{n-\frac{1}{2}}}(V))^{\otimes b} \otimes (\otimes_{n=1}^{\infty} \Lambda_{q^{n-\frac{1}{2}}}(V))^{\otimes c}.$$

If $p_1(M)_{S^1} = (a+b+c)p_1(V)_{S^1}$, then this opreator is rigid. Actually it is easy to see that $\{S, S', T, T'\}$ generate a graded ring by tensor product, each homogeneous term of degree m gives a rigid elliptic operator, if the corresponding vector bundle V satisfies $p_1(M)_{S^1} = mp_1(V)_{S^1}$.

Example h) If we have another rank $2n$ spin vector bundle W such that

$$ap_1(V)_{S^1} + bp_1(W)_{S^1} = p_1(M)_{S^1}$$

for some non-negative integers a, b, then as a corollary of Theorem 1 we have that, for two highest weight positive energy representations E and F of level a and b of $\tilde{L}\mathrm{Spin}(2l)$ and $\tilde{L}\mathrm{Spin}(2n)$ respectively, the operator

$$D \otimes \otimes_{m=1}^{\infty} S_{q^m}(TM) \otimes \psi(E, V) \otimes \psi(F, W)$$

is rigid. One can also consider the tensor product of several bundles and several representations. This example may be used to study the equivariant splitting of TM. More interesting examples may be obtained by studying the explicit constructions of the higher level irreducible representations of $\tilde{L}\mathrm{Spin}(2l)$. It is also interesting to get examples from Lie groups other than $\mathrm{Spin}(2l)$.

2.5. **The Proof of Theorem 1.** To display our idea clearly, we first restrict our attention to the isolated fixed point case. By the discussions in Section 2.2 and Section 2.3 we can assume that E is an integrable highest weight module $L(\Lambda)$ of $\hat{L}so(2l)$ of level m.

Let $g = e^{2\pi i t} \in S^1$ be a generator of the action group and $\{p\} \subset M$ be the set of fixed points. Let

$$TM|_p = E_1 \oplus \cdots \oplus E_k, \quad k = \frac{1}{2}\mathrm{dim}M$$

be the decomposition of the tangent bundle into sum of the S^1-invariant 2-planes when restricted to the fixed points. Assume that g acts on E_j by $e^{2\pi i m_j t}$. Recall that $\{m_j\} \subset \mathbf{Z}$ is called the exponents of TM at

the fixed point p. See [BT] and [Liu1]. Choose the orientations of the E_j's compatibly with the orientation of M. Similarly let $\{n_\nu\}$ be the exponent of V at the fixed point p, i.e. one has the corresponding equivariant decomposition

$$V|_p = L_1 \oplus \cdots \oplus L_l$$

and g acts on L_ν by $e^{2\pi i n_\nu t}$.

Consider the following functions

$$H(t,\tau) = (2\pi i)^{-k} \prod_{j=1}^{k} \frac{\theta'(0,\tau)}{\theta(m_j t,\tau)},$$

$$c_E(t,\tau) = \chi_E(T,\tau)$$

where $T = (n_1 t, \cdots, n_l t)$ and $\chi_E(z,\tau) = q^{m_\Lambda} \mathrm{ch}_E(z,\tau)$ is the normalized Kac-Weyl character of the representation $E = L(\Lambda)$ of $\tilde{L}\mathrm{Spin}(2l)$.

Then it is not difficult to see that

$$F_E(t,\tau) = \sum_p H(t,\tau) c_E(t,\tau)$$

is the Lefschetz number of

$$q^{m_\Lambda} \cdot D \otimes \otimes_{n=1}^{\infty} S_{q^n}(TM - \mathrm{dim} M) \otimes \psi(E,V).$$

See Appendix A for the derivation of $F_E(t,\tau)$ in general. Obviously we can extend $F_E(t,\tau)$ to a (meromorphic) function on $\mathbf{C} \times \mathbf{H}$. The rigidity theorem is therefore equivalent to the proof that $F_E(t,\tau)$ is independent of t.

Lemma 5.1: *If* $p_1(M)_{S^1} = m p_1(V)_{S^1}$, *then* $F_E(t,\tau) = \sum_p H(t,\tau) c_E(t,\tau)$ *is invariant under the action*

$$t \to t + a\tau + b$$

for $a, b \in 2\mathbf{Z}$.

Proof: For $(a,b) \in (2\mathbf{Z})^2$, we have

$$H(t + a\tau + b, \tau) = e^{\pi i \sum_j m_j^2 (a^2\tau + 2at)} H(t,\tau);$$

$$c_E(t + a\tau + b, \tau) = e^{-m\pi i \sum_\nu n_\nu^2 (a^2\tau + 2at)} c_E(t,\tau)$$

which can be seen by the transformation formulas of theta-functions.

¿From Appendix A we know that, if $p_1(M)_{S^1} = m p_1(V)_{S^1}$, then $\sum_j m_j^2 = m \sum_\nu n_\nu^2$ for each fixed point and the exponential factors cancel each other. □

So the rigidity theorem is equivalent to that $F_E(t,\tau)$ is holomorphic in t. We will prove that $F_E(t,\tau)$ is actually holomorphic in two variables t and τ on $\mathbf{C} \times \mathbf{H}$.

Now we study the modular transformation of $SL_2(\mathbf{Z})$ on $F_E(t,\tau)$. Recall that

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$$

acts on $(t,\tau) \in \mathbf{C} \times \mathbf{H}$ by

$$g(t,\tau) = (\frac{t}{c\tau + d}, \frac{a\tau + b}{c\tau + d}).$$

One has the following

Lemma 5.2: *For any* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$, *we have*

$$F_E(g(t,\tau)) = (c\tau + d)^k F_{gE}(t,\tau)$$

where $gE = \sum_\mu a_\mu E_\mu$ *is a finite complex linear combination of positive energy representations of* $\tilde{L}Spin(2l)$ *of highest weight of level* m.

Actually the function

$$F_{gE}(t,\tau) = \sum_\mu \sum_p H(t,\tau) c_{E_\mu}(t,\tau)$$

is the complex linear combination of the corresponding Lefschetz numbers.

Proof: We use the theorem of Kac-Peterson in Section 2.2, which tells us the actions of the two generators

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

of $SL_2(\mathbf{Z})$ on the characters. In general this gives

$$\chi_E(g(z,\tau)) = e^{mc\pi i \sum_j z_j^2/(c\tau+d)} \chi_{gE}(z,\tau)$$

where $gE = \sum_\mu a_\mu E_\mu$ is a finite linear combination of positive energy representations of $\tilde{L}\mathrm{Spin}(2l)$ of highest weight of level m. Here $\{a_\mu\}$ are some complex numbers and E_μ is a representation of $\tilde{L}\mathrm{Spin}(2l)$ of highest weight Λ_μ and level m, i.e. $E_\mu = L(\Lambda_\mu)$. Here we define

$$\chi_{gE}(z,\tau) = \sum_\mu a_\mu \chi_{E_\mu}(z,\tau)$$

by complex linear extension and we also extend the elliptic operator associated to gE and its Lefschetz number linearly to $K(M) \otimes_Z \mathbf{C}$. We have the corresponding elliptic operator

$$\sum_\mu a_\mu q^{m_{\Lambda_\mu}} D \otimes \otimes_{n=1}^\infty S_{q^n}(TM - \mathrm{dim}M) \otimes \psi(E_\mu, V)$$

whose Lefschetz number is

$$F_{gE}(t,\tau)=\sum_p\sum_\mu a_\mu H(t,\tau)c_{E_\mu}(t,\tau).$$

It is easy to see that

$$c_E(g(t,\tau))=e^{cm\pi i\sum_\nu n_\nu^2t^2/(c\tau+d)}\sum_\mu a_\mu c_{E_\mu}(t,\tau).$$

Since we also have

$$H(g(t,\tau))=(c\tau+d)^k e^{-c\pi i\sum_j m_j^2t^2/(c\tau+d)}H(t,\tau)$$

for any $g\in SL_2(\mathbf{Z})$. By the condition on equivariant Pontrjagin classes, the exponential factors cancel each other, which gives the lemma. □

One actually only needs to check Lemma 5.2 for the two generators of $SL_2(Z)$, S and T. The following lemma is a generalization of Proposition 6.1 in [BT] or Lemma 1.3 in [Liu1]. The proof is essentially the same. For completeness, we give the detail.

Lemma 5.3: *For any $g\in SL_2(\mathbf{Z})$, the function $F_{gE}(t,\tau)$ is holomorphic in (t,τ) for $t\in\mathbf{R}$ and $\tau\in\mathbf{H}$.*

Proof: Let $z=e^{2\pi it}$ and $N=\max\{|m_j|\}$ where m_j runs through the exponents of all fixed points. The expressions

$$c_E(t,\tau)=\sum_{\lambda\in P^k(\mathrm{mod}kM+\mathbf{C}\delta)}c_\lambda^\Lambda(\tau)\Theta_\lambda(T,\tau)$$

and

$$H(t,\tau)=(2\pi i)^{-k}\prod_{j=1}^k\frac{\theta'(0,\tau)}{\theta(m_jt,\tau)}$$

tell us that $F_{gE}(t,\tau)$ has a convergent Laurent series expansion of the form

$$\sum_\mu a_\mu q^{m_{\Lambda_\mu}}\sum_{j=0}^\infty b_{j\mu}^g(z)q^j$$

in the domain $|q|^{\frac{1}{N}}<|z|<|q|^{-\frac{1}{N}}$. Here $\{b_{j\mu}^g(z)\}$ are rational functions of z with possible poles on the unit circle.

But considered as a formal power series of q,

$$\otimes_{n=1}^\infty S_{q^n}(TM)\otimes(\sum_\mu a_\mu q^{m_{\Lambda_\mu}}\psi(E_\mu,V))=\sum_\mu a_\mu q^{m_{\Lambda_\mu}}\sum_{j=0}^\infty V_{j\mu}^g q^j$$

with $V_{j\mu}^g\in K(M)\otimes_Z\mathbf{C}$. Note that the terms in the above two μ-sums correspond to each other. Apply Lefschetz fixed point formula to each

$V^g_{j\mu}$, we get that, for $|z| = 1$, each $b^g_{j\mu}(z)$ is the Lefschetz number of an elliptic operator. This implies that

$$b^g_{j\mu}(z) = \sum_{m=-N(j)}^{N(j)} a^{g\mu}_{mj} z^m,$$

for $N(j)$ some positive integer depending on j and $a^{g\mu}_{mj}$ complex number. Since both sides are analytic functions of z, this equality holds for any $z \in \mathbf{C}$.

On the other hand, multiply $F_{gE}(t,\tau)$ by

$$f(z) = \prod_p \prod_{j=1}^{k} (1 - z^{m_j})$$

where the product runs over all of the fixed points $\{p\}$, we get a holomorphic function which then has a convergent power series expansion of the form

$$\sum_\mu a_\mu q^{m_{\Lambda_\mu}} \sum_{j=0}^{\infty} c^g_{j\mu}(z) q^j$$

with $\{c^g_{j\mu}(z)\}$ polynomial functions, in the domain $|q|^{\frac{1}{N}} < |z| < |q^{-\frac{1}{N}}$. Compare the above two expansions, one gets that for each j the equality

$$c^g_{j\mu}(z) = f(z) \cdot b^g_{j\mu}(z)$$

holds. So by the Hilbert Nullstellensatz, we know that

$$\sum_\mu a_\mu q^{m_{\Lambda_\mu}} \sum_{j=0}^{\infty} b^g_{j\mu}(z) q^j = \sum_\mu a_\mu q^{m_{\Lambda_\mu}} \sum_{j=0}^{\infty} (\frac{c^g_{j\mu}(z)}{f(z)}) q^j$$

is holomorphic in the domain $|q|^{\frac{1}{N}} < |z| < |q^{-\frac{1}{N}}$. Obviously $R \times H$ lies inside this domain. □

Now let us **prove Theorem 1**. At this point the proof is almost identical to our new proof of the Witten rigidity theorems in [Liu1].

By Lemma 5.1, we know that $F_E(t,\tau)$ is a doubly periodic meromorphic function in t, therefore to get the rigidity theorem, we only need to prove that $F_E(t,\tau)$ is holomorphic on $\mathbf{C} \times \mathbf{H}$.

First note that, as a meromorphic function on $\mathbf{C} \times \mathbf{H}$, all of the possible polar divisors of $F_E(t,\tau)$ can be expressed in the form $t = \frac{n(c\tau+d)}{A}$ with A, n, c, d integers, $A \neq 0$ and c, d prime to each other.

Lemma 5.3 tells us that the divisor $t = \frac{n}{A}$ is not the polar divisor of $F_{gE}(t,\tau)$ for any g and any integers A, n.

For any polar divisor $t = \frac{n(c\tau+d)}{A}$ of $F_E(t,\tau)$ with $(c,d) = 1$, we can find integers a, b such that $ad - bc = 1$ and consider the matrix

$g = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in SL_2(\mathbf{Z})$. Since

$$F_{gE}(t,\tau) = (-c\tau + a)^{-k} F_E(\frac{t}{-c\tau + a}, \frac{d\tau - b}{-c\tau + a}),$$

it is easy to see that, if $t = \frac{n(c\tau+d)}{A}$ is the polar divisor of $F_E(t,\tau)$, then a polar divisor of $F_{gE}(t,\tau)$ is given by

$$\frac{t}{-c\tau + a} = \frac{n(c\frac{d\tau+b}{-c\tau+a} + d)}{A}$$

which exactly gives $t = \frac{n}{A}$. This is a contradiction to Lemma 5.3. So $F_E(t,\tau)$ is holomorphic on $\mathbf{C} \times \mathbf{H}$. This proves Theorem 1 for the isolated fixed point case.

Now we discuss the general fixed point case. Obviously we only need to verify the transformation formulas used above.

Let $\{M_\alpha\}$ be the fixed submanifolds of the circle action and

$$TM|_{M_\alpha} = E_1 \oplus \cdots \oplus E_h \oplus TM_\alpha$$

be the equivariant decomposition of TM with respect to the S^1-action. We denote the Chern root of E_γ by $2\pi i x_\gamma$ and the Chern roots of $TM_\alpha \otimes \mathbf{C}$ by $\{\pm 2\pi i y_j\}$. Assume that g acts on E_γ by $e^{2\pi i m_\gamma t}$.

Similarly let

$$V|_{M_\alpha} = L_1 \oplus \cdots \oplus L_l$$

be the equivariant decomposition of V restricted to M_α. Assume that g acts on L_ν by $e^{2\pi i n_\nu t}$. Here some n_ν may be zero. We denote the Chern root of L_ν by $2\pi i u_\nu$.

Let $2k_\alpha$ denote the dimension of M_α. Then the Lefschetz number of

$$q^{m_\Lambda} \cdot D \otimes \otimes_{n=1}^{\infty} S_{q^n}(TM - \mathrm{dim}M) \otimes \psi(E,V)$$

is given by

$$F_E(t,\tau) = \sum_{M_\alpha} (\prod_{j=1}^{k_\alpha} (2\pi i y_j F(y_j,\tau))(\prod_{\gamma=1}^{h} F(x_\gamma + m_\gamma t, \tau)) c_E(u+t,\tau))[M_\alpha]$$

where

$$F(x,\tau) = (2\pi i)^{-1} \frac{\theta'(0,\tau)}{\theta(x,\tau)}, \ c_E(u+t,\tau) = \chi_E(U+T,\tau)$$

with $U+T = (u_1+n_1 t, \cdots, u_l+n_l t)$. See Appnedix A for the derivation of $F_E(t,\tau)$.

Since $gE = \sum_\mu a_\mu E_\mu$, the corresponding elliptic operator is

$$\sum_\mu a_\mu q^{m_{\Lambda_\mu}} D \otimes (\otimes_{n=1}^{\infty} S_{q^n}(TM - \mathrm{dim}M) \otimes \psi(E_\mu, V))$$

whose Lefschetz number is

$$F_{gE}(t,\tau)=\sum_{M_\alpha}(\prod_{j=1}^{k_\alpha}(2\pi i y_j F(y_j,\tau))(\prod_{\gamma=1}^{h}F(x_\gamma+m_\gamma t,\tau))c_{gE}(u+t,\tau))[M_\alpha]$$

with

$$c_{gE}(u+t,\tau)=\sum_\mu a_\mu c_{E_\mu}(u+t,\tau)$$

as in Lemma 5.2.

Let us first check the modular transformation of $g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL_2(\mathbf{Z})$. We have

$$F_E(\frac{t}{c\tau+d},\frac{a\tau+b}{c\tau+d})$$

$$=\sum_{M_\alpha}(\prod_{j=1}^{k_\alpha}(2\pi i y_j F(y_j,\frac{a\tau+b}{c\tau+d}))(\prod_{\gamma=1}^{h}(F(x_\gamma+\frac{m_\gamma t}{c\tau+d},\frac{a\tau+b}{c\tau+d}))c_E(u+\frac{t}{c\tau+d},\frac{a\tau+b}{c\tau+d}))[M_\alpha]$$

$$=(c\tau+d)^k\sum_{M_\alpha}(\prod_{j=1}^{k_\alpha}2\pi i y_j F((c\tau+d)y_j,\tau))\prod_{\gamma=1}^{h}(F((c\tau+d)x_\gamma+m_\gamma t,\tau))c_{gE}((c\tau+d)u+t,\tau))[M_\alpha].$$

Here to cancel the exponential factors one needs

$$\sum_j y_j^2+\sum_\gamma(x_\gamma+m_\gamma t)^2=m\sum_\nu(u_\nu+n_\nu t)^2$$

which is exactly the localization of the equality $p_1(M)_{S^1}=mp_1(V)_{S^1}$ to M_α. See Appendix A.

But since we only need the k_α-th homogeneous terms of the polynomials in x's, y's and u's, one gets

$$(\prod_{j=1}^{k_\alpha}y_j F((c\tau+d)y_j,\tau))\prod_{\gamma=1}^{h}(F((c\tau+d)x_\gamma+m_\gamma t,\tau))c_{gE}((c\tau+d)u+t,\tau))[M_\alpha]$$

$$=(\prod_{j=1}^{k_\alpha}y_j F(y_j,\tau))\prod_{\gamma=1}^{h}(F(x_\gamma+m_\gamma t,\tau))c_{gE}(u+t,\tau))[M_\alpha].$$

Therefore

$$F_E(g(t,\tau))=(c\tau+d)^k F_{gE}(t,\tau)$$

as in the isolated fixed point case.

We leave to the reader to check the action of $t\to t+a\tau+b$ for $a,b\in 2\mathbf{Z}$. Note that, for this, one only needs the conditions

$$\sum_j m_j^2=m\sum_\nu n_\nu^2,\ \sum_\gamma m_\gamma x_\gamma=m\sum_\nu n_\nu u_\nu$$

which are easy consequences of the localization of the equality of the first equivariant Pontrjagin classes.

This finishes the proof of Theorem 1.

2.6. **Almost Complex Manifolds I.** Now let X be a compact almost complex manifold of complex dimension k and W be a complex vector bundle of rank l on X. Here by complex bundle we mean a real bundle with a complex structure. One has the decompositions

$$TX \otimes \mathbf{C} = T'X \oplus T''X,\ W \otimes \mathbf{C} = W' \oplus W''.$$

Assume that there exists an S^1-action on X which lifts to W and preserves the complex structures of X and W.

Following Witten, consider the fiberwise multiplication action by a complex number $y = e^{2\pi i\alpha}$ on W' and by y^{-1} on W'' respectively. In this way we get a real G_y-equivariant bundle V^α such that $V^\alpha \otimes \mathbf{C} = W' \oplus W''$. Here G_y denotes the multiplicative group generated by y. One notes that V^α is actually isomorphic to W viewed as a real bundle.

If $w_2(W) = 0$ which is equivalent to $c_1(W) \equiv 0(\mathrm{mod}2)$, then V^α is a $\mathrm{Spin}(2l)$-vector bundle and the method in Section 2.3 can be used to get an element $\psi(E, V^\alpha)$, associated to V^α and a positive energy representation E of $\tilde{L}\mathrm{Spin}(2l)$ of highest weight.

Let $\bar{\partial}$ denote the anti-holomorphic differential on X. Assume furthermore that $w_2(X) = 0$ and denote the Dirac operator on X by D. Recall that $D = \bar{\partial} \otimes K^{-\frac{1}{2}}$ with $K = \det TX'$. Consider the equivariant elliptic operator

$$D \otimes \otimes_{n=1}^{\infty} S_{q^n}(TX \otimes C) \otimes \psi(E, V^\alpha)$$

under the action of $G_y \times S^1$. Take $\alpha = \frac{1}{N}$ for some positive integer N. For a complex vector bundle F, let $p_1(F)_{S^1} = c_1^2(F)_{S^1} - 2c_2(F)_{S^1}$ be the first equivariant Pontrjagin class of the underlying real bundle. We have

Theorem 2: *For any positive energy representation E of $\tilde{L}Spin(2l)$ of highest weight of level m, if $w_2(X) = w_2(W) = 0$, $c_1(W) \equiv 0(modN)$ and $p_1(X)_{S^1} = mp_1(W)_{S^1}$, then the $G_y \times S^1$-equivariant elliptic operator*

$$D \otimes \otimes_{n=1}^{\infty} S_{q^n}(TX \otimes C) \otimes \psi(E, V^\alpha)$$

is rigid with respect to the S^1-action.

Proof: We only give a sketch for isolated fixed point case and leave the general fixed point case to the reader.

Let $\{p\}$ be the fixed points of the S^1-action. Choose the orientation compatible decompositions of TX' and W' at each fixed point. One then has

$$TX'|_p = \oplus_{j=1}^k E_j, \quad W'|_p = \oplus_{\nu=1}^l L_\nu.$$

Assume that $g = e^{2\pi it}$ acts on E_j and L_ν by $e^{2\pi i m_j t}$ and $e^{2\pi i n_\nu t}$ respectively. Write

$$\chi_E^\alpha(z,\tau) = \chi_E(z+\alpha,\tau)$$

where $z+\alpha = (z_1+\alpha,\cdots,z_l+\alpha)$. First by the same method as in Section 2.5, we get that

$$F_E^\alpha(t,\tau) = \sum_p H(t,\tau) c_E^\alpha(t,\tau),$$

where

$$c_E^\alpha(t,\tau) = \chi_E^\alpha(T,\tau),$$

is the Lefschetz number of

$$q^{m_\Lambda} \cdot D \otimes \otimes_{n=1}^\infty S_{q^n}(TX \otimes C - \mathrm{dim}X) \otimes \psi(E, V^\alpha)$$

at $y \times e^{2\pi it} \in G_y \times S^1$. Here $H(t,\tau)$, T and m_Λ have the same expressions as in Section 2.5. As in the proof of Theorem 1, one first verifies that $F_E^\alpha(t,\tau)$ is doubly periodic with respect to the action

$$T \to t + a\tau + b$$

for $a, b \in N\mathbf{Z}$. Then one can check that, for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$,

$$F_E^\alpha(g(t,\tau)) = (c\tau+d)^k F_{gE}^{\alpha(c\tau+d)}(t,\tau)$$

where $F_{gE}^{\alpha(c\tau+d)}(t,\tau)$ is the Lefschetz number of

$$e^{lm\pi i c\alpha^2(c\tau+d)} \sum_\mu a_\mu q^{m_{\Lambda_\mu}} D \otimes L^{mc\alpha} \otimes \otimes_{n=1}^\infty S_{q^n}(TX \otimes C - \mathrm{dim}X) \otimes \psi(E_\mu, V^{\alpha(c\tau+d)}).$$

Here $L = \mathrm{det}W'$ and $\{a_\mu\}$ are some complex numbers. Also $V^{\alpha(c\tau+d)}$ is the corresponding equivariant bundle with respect to the fiber multiplication by $y = e^{2\pi i\alpha(c\tau+d)}$ and $\psi(E_\mu, V^{\alpha(c\tau+d)})$ is the element in $K(X)[[q]]$ associated to $V^{\alpha(c\tau+d)}$ and the positive energy representation $E_\mu = L(\Lambda_\mu)$ of $\tilde{L}\mathrm{Spin}(2l)$ of highest weight Λ_μ of level m. Recall $gE = \sum_\mu a_\mu E_\mu$. In terms of local data, we conveniently write

$$F_{gE}^{\alpha(c\tau+d)}(t,\tau) = \sum_p H(t,\tau) c_{gE}^{\alpha(c\tau+d)}(t,\tau)$$

with

$$c_{gE}^{\alpha(c\tau+d)}(t,\tau) = e^{lm\pi i c\alpha^2(c\tau+d)} \sum_\mu a_\mu e^{2\pi i m c\alpha \sum_\nu n_\nu t} \cdot \chi_{E_\mu}^{\alpha(c\tau+d)}(t,\tau).$$

By our discussions in Section 2.5, these two properties, together with Lemma 5.3 of that section, are enough for Theorem 2. □

As the applications of Theorem 2, we give some examples. We will use the same notations S, T, S' and T' as in Section 2.4 to denote the four highest weight representations of $\tilde{L}\mathrm{Spin}(2l)$ of level 1.

Example A) Take $m = 1$ and $E = T$, One easily sees that

$$\psi(V^\alpha, T) = L^{\frac{1}{2}} \otimes \otimes_{n=0}^{\infty} \Lambda_{-y^{-1}q^n} W'' \otimes_{n=1}^{\infty} \Lambda_{-yq^n} W'.$$

We get the rigidity of $\bar{\partial} \otimes (K^{-1} \otimes L)^{\frac{1}{2}} \otimes \Theta_q^\alpha(TX|W)$. Here $L = \det W'$, $K = \det T'X$ and

$$\Theta_q^\alpha(TX|W) = \otimes_{n=0}^{\infty} \Lambda_{-y^{-1}q^n} W'' \otimes_{n=1}^{\infty} \Lambda_{-yq^n} W' \otimes_{n=1}^{\infty} S_{q^n} T'X \otimes_{n=1}^{\infty} S_{q^n} T''X.$$

Actually, in [Liu1] we have proved this result by assuming the slightly weaker condition $w_2(W) = w_2(X)$, $c_1(W) \equiv 0(\mathrm{mod} N)$ and $p_1(W)_{S^1} = p_1(X)_{S^1}$. Take $W = TX$, we get the rigidity theorem of Hirzebruch [H], i.e. the rigidity of $\bar{\partial} \otimes \Theta_q^\alpha(TX)$. Here

$$\Theta_q^\alpha(TX) = \otimes_{n=0}^{\infty} \Lambda_{-y^{-1}q^n} T''X \otimes_{n=1}^{\infty} \Lambda_{-yq^n} T'X \otimes_{n=1}^{\infty} S_{q^n} T'X \otimes_{n=1}^{\infty} S_{q^n} T''X.$$

These examples were also discussed by Witten in [W].

Example B) Take $m = 1$ and $E = S,\ S'$ or T', one gets the rigidity of $\bar{\partial} \otimes (K^{-1} \otimes L)^{\frac{1}{2}} \otimes P_q^\alpha(TX|W)$, $D \otimes Q_q^\alpha(TX|W)$ and $D \otimes R_q^\alpha(TX|W)$. Take $W = TX$, one gets the rigidity of $\bar{\partial} \otimes P_q^\alpha(TX)$, $D \otimes Q_q^\alpha(TX)$ and $D \otimes R_q^\alpha(TX)$. The rigidity of these operators were proved in [Liu1], Proposition 2.1. Here recall that

$$\begin{aligned}
P_q^\alpha(TX|W) &= \otimes_{n=0}^{\infty} \Lambda_{y^{-1}q^n} W'' \otimes_{n=1}^{\infty} \Lambda_{yq^n} W' \otimes_{n=1}^{\infty} S_{q^n} T'X \otimes_{n=1}^{\infty} S_{q^n} T''X, \\
Q_q^\alpha(TX|W) &= \otimes_{n=1}^{\infty} \Lambda_{-y^{-1}q^{n-\frac{1}{2}}} W'' \otimes_{n=1}^{\infty} \Lambda_{-yq^{n-\frac{1}{2}}} W' \otimes_{n=1}^{\infty} S_{q^n} T'X \otimes_{n=1}^{\infty} S_{q^n} T''X, \\
R_q^\alpha(TX|W) &= \otimes_{n=1}^{\infty} \Lambda_{y^{-1}q^{n-\frac{1}{2}}} W'' \otimes_{n=1}^{\infty} \Lambda_{yq^{n-\frac{1}{2}}} W' \otimes_{n=1}^{\infty} S_{q^n} T'X \otimes_{n=1}^{\infty} S_{q^n} T''X;
\end{aligned}$$

and take $W = TX$, one gets

$$\begin{aligned}
P_q^\alpha(TX) &= \otimes_{n=0}^{\infty} \Lambda_{y^{-1}q^n} T''X \otimes_{n=1}^{\infty} \Lambda_{yq^n} T'X \otimes_{n=1}^{\infty} S_{q^n} T'X \otimes_{n=1}^{\infty} S_{q^n} T''X, \\
Q_q^\alpha(TX) &= \otimes_{n=1}^{\infty} \Lambda_{-y^{-1}q^{n-\frac{1}{2}}} T''X \otimes_{n=1}^{\infty} \Lambda_{-yq^{n-\frac{1}{2}}} T'X \otimes_{n=1}^{\infty} S_{q^n} T'X \otimes_{n=1}^{\infty} S_{q^n} T''X, \\
R_q^\alpha(TX) &= \otimes_{n=1}^{\infty} \Lambda_{y^{-1}q^{n-\frac{1}{2}}} T''X \otimes_{n=1}^{\infty} \Lambda_{yq^{n-\frac{1}{2}}} T'X \otimes_{n=1}^{\infty} S_{q^n} T'X \otimes_{n=1}^{\infty} S_{q^n} T''X.
\end{aligned}$$

Example C) Consider the tensor products of S, T, S' and T', one can get some higher level rigidity theorems, especially the examples in Section 4 of [Liu1]. We omit the details here. In fact $\{S, T, S', T'\}$ form a ring by tensor product, each homogeneous term of degree m gives a rigid elliptic operator, if the corresponding bundle W satisfies $c_1(W) \equiv 0(\mathrm{mod} N)$, $w_2(X) = w_2(W) = 0$ and $p_1(X)_{S^1} = m \cdot p_1(W)_{S^1}$.

Take $m = 1$, $W = TX$ and consider the following elliptic operator

$$\sum_g \bar{\partial} \otimes \otimes_{n=1}^{\infty} S_{q^n}(T'X - \mathrm{dim}X) \otimes_{n=1}^{\infty} S_{q^n}(T''X - \mathrm{dim}X) \otimes \psi(V^{\alpha(c\tau+d)}, T)_v$$

where the sum is over

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})/\Gamma_1(N).$$

The index of this operator gives a rigid elliptic genus of level 1 for compact almost complex manifolds with $c_1 \equiv 0(\mathrm{mod}N)$. Explicitly

$$\psi(V^{\alpha(c\tau+d)}, T)_v = L^{c\alpha} \otimes \otimes_{n=0}^{\infty} \Lambda_{-y^{-1}q^n}(W'' - \mathrm{dim}W) \otimes_{n=1}^{\infty} \Lambda_{-yq^n}(W' - \mathrm{dim}W)$$

with $y = e^{2\pi i\alpha(c\tau+d)}$. Also recall

$$\Gamma_1(N) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | c \equiv 0(\mathrm{mod}N),\ a \equiv d \equiv 1(\mathrm{mod}N)\}$$

It is not difficult to see that when $c_1(W) = 0$ Theorem 2 actually holds for any complex number α, this generalizes the result of Krichever to higher level loop group representations. See the corresponding discussions in [Liu1].

3. Jacobi Forms and Rigidity Theorems

In this part we generalize the the rigidity theorems in last part and in [Liu1] to the non-zero anomaly case from which we derive a family of holomorphic Jacobi forms. As corollaries we get many vanishing theorems, especially an $\hat{\mathfrak{A}}$-vanishing theorem for loop space.

3.1. **Non-zero Anomaly.** Let M be a compact smooth spin manifold of dimension $2k$ with an S^1-action and V be a rank $2l$ equivariant spin vector bundle on it. We consider the equivariant cohomology group of M, $H^*_{S^1}(M, \mathbf{Z})$. Obviously $H^*_{S^1}(M, \mathbf{Z})$ is a module over $H^*(BS^1, \mathbf{Z})$ induced by the projection

$$\pi : \quad M \times_{S^1} ES^1 \to BS^1.$$

Recall that

$$H^*_{S^1}(M, \mathbf{Z}) = H^*(M \times_{S^1} ES^1, \mathbf{Z}).$$

Let $p_1(V)_{S^1},\ p_1(M)_{S^1} \in H^*_{S^1}(M, \mathbf{Z})$ be the equivariant first Pontrjagin classes of V and TM respectively. See Appendix A for a geometric discussion of equivariant characteristic classes. From our previous discussions, one knows that the condition $p_1(V)_{S^1} = p_1(M)_{S^1}$ puts very strong restriction on the characteristic numbers of M and V. Actually this condition governs the modular invariance of the elliptic operators

discussed in [Liu1] and last part which is one of the essential reasons for their rigidity.

In this part we consider the situation when $p_1(V)_{S^1} - p_1(M)_{S^1} \in H^*_{S^1}(M, \mathbf{Z})$ is equal to the pull-back of an element in $H^*(BS^1, \mathbf{Z})$. Since

$$H^*(BS^1, \mathbf{Z}) = \mathbf{Z}[[u]]$$

with u a generator of degree 2, we know that this is equivalent to

$$p_1(V)_{S^1} - p_1(M)_{S^1} = n \cdot \pi^* u^2$$

with n an integer. We call n the anomaly of rigidity. The reason for this will be clear in the following. Follow [Liu1], we introduce the following elements in $K(M)[[q^{\frac{1}{2}}]]$:

$$\begin{aligned}
\Theta'_q(TM|V)_v &= \otimes_{n=1}^{\infty} \Lambda_{q^n}(V - \mathrm{dim}V) \otimes_{m=1}^{\infty} S_{q^m}(TM - \mathrm{dim}M), \\
\Theta_q(TM|V)_v &= \otimes_{n=1}^{\infty} \Lambda_{-q^{n-\frac{1}{2}}}(V - \mathrm{dim}V) \otimes_{m=1}^{\infty} S_{q^m}(TM - \mathrm{dim}M), \\
\Theta_{-q}(TM|V)_v &= \otimes_{n=1}^{\infty} \Lambda_{q^{n-\frac{1}{2}}}(V - \mathrm{dim}V) \otimes_{m=1}^{\infty} S_{q^m}(TM - \mathrm{dim}M), \\
\Theta^*_q(TM|V)_v &= \otimes_{n=1}^{\infty} \Lambda_{-q^n}(V - \mathrm{dim}V) \otimes_{m=1}^{\infty} S_{q^m}(TM - \mathrm{dim}M).
\end{aligned}$$

One of our main results in this part is the following theorem which generalizes the rigidity theorems to the non-zero anomaly case:

Theorem 3: *Let M and V be as above. Assume*

$$p_1(V)_{S^1} - p_1(M)_{S^1} = n \cdot \pi^* u^2,$$

then the Lefschetz numbers of $D \otimes \triangle(V) \otimes \Theta'_q(TM|V)_v$, $D \otimes \Theta_q(TM|V)_v$, $D \otimes \Theta_{-q}(TM|V)_v$ are holomorphic Jacobi forms of index $\frac{n}{2}$ and weight k over $(2\mathbf{Z})^2 \rtimes \Gamma$ with Γ equal to $\Gamma_0(2)$, $\Gamma^0(2)$, Γ_θ respectively. The Lefschetz number of $D \otimes (\triangle^+(V) - \triangle^-(V)) \otimes \Theta^*_q(TM|V)_v$ is a holomorphic Jacobi form of index $\frac{n}{2}$ and weight $k - l$ over $\mathbf{Z}^2 \rtimes SL_2(Z)$

Here by Lefschetz number we actually mean its extension from unit circle to complex plane. See the discussions in Section 3.2 for definitions of the modular subgroups appeared in Theorem 3.

As a corollary of Theorem 3 we have the following vanishing theorems for loop space.

Corollary 4.1: *Let M, V and n be as in Theorem 3. If $n = 0$, the Lefschetz numbers of the elliptic operators in Theorem 3 are independent of the generators of S^1. If $n < 0$, then these Lefschetz numbers are identically zero, especially the indices of these elliptic operators are zero.*

This explains the reason that we call n the anomaly. There are some other corollaries by applying several simple facts about Jacobi forms in [EZ] to our situation. We believe that the applications of certain

deeper results in Jacobi form theory may bring new light to elliptic genus theory.

It is very interesting to discuss the operator

$$D \otimes \otimes_{m=1}^{\infty} S_{q^m}(TM - \mathrm{dim}M),$$

which corresponds to the Dirac operator on LM. One notes that this operator is the same as

$$\prod_{n=1}^{\infty}(1-q^n)^{2k} \cdot D \otimes \otimes_{m=1}^{\infty} S_{q^m}(TM).$$

We will prove the following $\hat{\mathfrak{A}}$-vanishing theorem for loop space.

Theorem 4: *If $p_1(M)_{S^1} = n \cdot \pi^* u^2$ for some integer n, then the Lefschetz number, especially the index, of*

$$D \otimes \otimes_{m=1}^{\infty} S_{q^m}(TM - dimM)$$

is zero.

We note that $p_1(M)_{S^1} = n \cdot \pi^* u^2$ is the equivariant spin condition on LM. If M is 2-connected or the S^1-action is induced from an S^3-action, then this condition is equivalent to the condition $p_1(M) = 0$ which is the spin condition on LM. See the discussion in Section 3.4.

We remark that Witten has predicted Theorem 3 by considerations from physics. See the discussions in [W1]. We may view the results here as part of his famous rigidity theorems.

It is also interesting to generalize Theorem 3 to higher level cases. In last part, we considered the Dirac operator on loop space twisted by some element $\psi(E,V) \in K(M)[[q]]$ associated to a spin vector bundle V of rank $2l$ and a positive energy representation E of $\tilde{L}\mathrm{Spin}(2l)$ of highest weight of level m. Our theorem there says that, if $p_1(M)_{S^1} = mp_1(V)_{S^1}$, then

$$D \otimes \otimes_{n=1}^{\infty} S_{q^n}(TM) \otimes \psi(E,V)$$

is rigid. Using a refinement of the modular invariance of the characters of the highest weight modules of affine Lie algebras given by Kac-Wakimoto, we will show the following.

Theorem 5: *Let M, V and E be as above. If*

$$mp_1(V)_{S^1} - p_1(M)_{S^1} = n \cdot \pi^* u^2,$$

then

$$q^{m_\Lambda} D \otimes \otimes_{n=1}^{\infty} S_{q^n}(TM - dimM) \otimes \psi(E,V)$$

is a holomorphic Jacobi form of index $\frac{n}{2}$ and weight k over $(2\mathbf{Z})^2 \rtimes \Gamma(N(m))$. $\square$

Here $N(m)$ is an integer depending on the level m and given in [Ka], and m_Λ is as given in Section 2.2 of last part. As a corollary one has that, if $n < 0$, the Lefschetz number of the above elliptic operator must be zero, so is its index. If $n = 0$, Theorem 5 gives the rigidity theorem. There are similar theorems for almost complex manifolds which will be discussed in Section 3.3.

We organize this part in the following way. In Section 3.2 we prove Theorem 3 and Theorem 5. In Section 3.3, we discuss the the corresponding theorems for almost complex manifolds. In Section 3.4 we prove Theorem 4 and give some vanishing theorems by combining several simple facts in the theory of Jacobi forms with the theorems in Sections 3.2 and 3.3.

3.2. **Proofs of Theorems 3 and 5.** Recall that a (meromorphic) Jacobi form of index m and weight l over $L \rtimes \Gamma$, where L is an integral lattice in the complex plane $\mathbf{C}$ preserved by the modular subgroup $\Gamma \subset SL_2(\mathbf{Z})$, is a (meromorphic) function $F(t,\tau)$ on $\mathbf{C} \times \mathbf{H}$ such that

$$\begin{aligned} 1)\quad F(\frac{t}{c\tau+d}, \frac{a\tau+b}{c\tau+d}) &= (c\tau+d)^l e^{2\pi i m \frac{ct^2}{c\tau+d}} F(t,\tau), \\ 2)\quad F(t+\lambda\tau+\mu,\tau) &= e^{-2\pi i m(\lambda^2\tau+2\lambda t)} F(t,\tau), \end{aligned}$$

where $(\lambda,\mu) \in L$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. If F is holomorphic on $\mathbf{C} \times \mathbf{H}$, we say that F is a holomorphic Jacobi form. It is important for us to emphasize this point, since the key point of our theorems is to prove that those Lefschetz numbers are holomorphic Jacobi forms.

Jacobi forms can be viewed as sections of holomorphic line bundles on the elliptic modular surface

$$X_\Gamma^L = \mathbf{C} \times \mathbf{H}/L \rtimes \Gamma.$$

See [EZ] and [Kr]. Also see Section 4.1 for more detail about elliptic modular surface. Obviously $F(t,\tau)$ is holomorphic iff it is a holomorphic section.

Now let us start to prove Theorem 3. We first prove that the Lefschetz numbers of the elliptic operators in Theorem 3 are, possibly meromorphic, Jacobi forms over the corresponding modular subgroups.

Let us first consider the isolated fixed point case. Let $g = e^{2\pi i t} \in S^1$ be a generator of the action group and $\{p\} \subset M$ be the set of fixed points. Let $\{m_j\}, \{n_\nu\} \subset \mathbf{Z}$ be the exponents of TM and V respectively, at the fixed point p. See Section 2.5 for the geometric meaning of these local data.

Denote the Lefschetz numbers of $2^{-l}\cdot D\otimes\triangle(V)\otimes\Theta'_q(TM|V)_v, D\otimes\Theta_q(TM|V)_v,\ D\otimes\Theta_{-q}(TM|V)_v$ and $2^{-l}\cdot D\otimes(\triangle^+(V)-\triangle^-(V))\otimes\Theta^*_q(TX|V)_v$ by $F^V_{d_s}(t,\tau)$, $F^V_D(t,\tau)$, $F^V_{-D}(t,\tau)$ and $F^V_{D^*}(t,\tau)$ respectively. Apply the Atiyah-Bott-Segal-Singer Lefschetz fixed point formula, one has

$$\begin{aligned}
F^V_{d_s}(t,\tau) &= (2\pi i)^{-k}\sum_p \frac{\theta'(0,\tau)^k}{\theta_1(0,\tau)^l}\frac{\prod_{\nu=1}^l\theta_1(n_\nu t,\tau)}{\prod_{j=1}^k\theta(m_j t,\tau)},\\
F^V_D(t,\tau) &= (2\pi i)^{-k}\sum_p \frac{\theta'(0,\tau)^k}{\theta_2(0,\tau)^l}\frac{\prod_{\nu=1}^l\theta_2(n_\nu t,\tau)}{\prod_{j=1}^k\theta(m_j t,\tau)},\\
F^V_{-D}(t,\tau) &= (2\pi i)^{-k}\sum_p \frac{\theta'(0,\tau)^k}{\theta_3(0,\tau)^l}\frac{\prod_{\nu=1}^l\theta_3(n_\nu t,\tau)}{\prod_{j=1}^k\theta(m_j t,\tau)},\\
F^V_{D^*}(t,\tau) &= (2\pi i)^{l-k}\sum_p \theta'(0,\tau)^{k-l}\frac{\prod_{\nu=1}^l\theta(n_\nu t,\tau)}{\prod_{j=1}^k\theta(m_j t,\tau)}.
\end{aligned}$$

Here $\theta(t,\tau),\theta_\mu(t,\tau)$ for $\mu=1,2,3$ are the four classical Jacobi theta-functions and

$$\theta'(0,\tau)=\frac{\partial}{\partial t}\theta(t,\tau)|_{t=0},\ \theta_\mu(0,\tau)=\theta_\mu(t,\tau)|_{t=0}.$$

Similarly let us denote the Lefschetz number of

$$D\otimes\otimes_{m=1}^\infty S_{q^m}(TM-\mathrm{dim}M)$$

by $H(t,\tau)$, then

$$H(t,\tau)=(2\pi i)^{-k}\sum_p\prod_{j=1}^k\frac{\theta'(0,\tau)}{\theta(m_j t,\tau)}.$$

As Lefschetz numbers, the F^V's and H are only defined for $t\in\mathbf{R}$ i.e. for $z=e^{2\pi i t}\in S^1$. But we can obviously extend them to well-defined meromorphic functions for $(t,\tau)\in\mathbf{C}\times\mathbf{H}$. This follows easily from the infinite product expressions of the theta-functions. In the following, when mention F^V's and H, we actually mean their extensions.

Recall the three modular subgroups

$$\begin{aligned}
\Gamma_0(2) &= \{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbf{Z})|c\equiv 0(\mathrm{mod}\ 2)\},\\
\Gamma^0(2) &= \{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbf{Z})|b\equiv 0(\mathrm{mod}\ 2)\},\ \text{and}\\
\Gamma_\theta &= \{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbf{Z})|\begin{pmatrix} a & b\\ c & d\end{pmatrix}\equiv\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}\ \text{or}\ \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}(\mathrm{mod}\ 2)\}.
\end{aligned}$$

First one has the following

Lemma 2.1: *If $p_1(V)_{S^1} - p_1(M)_{S^1} = n \cdot \pi^* u^2$, then*
$F^V_{d_s}(t,\tau)$ is a Jacobi form over $(2\mathbf{Z})^2 \rtimes \Gamma_0(2)$;
$F^V_D(t,\tau)$ is a Jacobi form over $(2\mathbf{Z})^2 \rtimes \Gamma^0(2)$;
$F^V_{-D}(t,\tau)$ is a Jacobi form over $(2\mathbf{Z})^2 \rtimes \Gamma_\theta$;
If $p_1(M)_{S^1} = -n \cdot u^2$, then $H(t,\tau)$ is a Jacobi form over $(2\mathbf{Z})^2 \rtimes SL_2(\mathbf{Z})$.
All of them are of index $\frac{n}{2}$ and weight k;
$F^V_{D^*}(t,\tau)$ is a Jacobi form of index $\frac{n}{2}$ and weight $k-l$ over $\mathbf{Z}^2 \rtimes SL_2(\mathbf{Z})$.

Proof: The condition $p_1(V)_{S^1} - p_1(M)_{S^1} = n \cdot \pi^* u^2$ implies that

$$\sum_\nu n_\nu^2 - \sum_j m_j^2 = n$$

for each fixed point. See Appendix A. First under the actions of $a, b \in 2\mathbf{Z}$, one has

$$\theta_\mu(m_j(t + a\tau + b), \tau) = e^{-\pi i m_j^2 (a^2\tau + 2at)} \theta_\mu(m_j t, \tau)$$

and

$$\theta_\mu(n_\nu(t + a\tau + b), \tau) = e^{-\pi i n_\nu^2 (a^2\tau + 2at)} \theta_\mu(n_\nu t, \tau)$$

for $\theta_\mu = \theta, \theta_1, \theta_2$ or θ_3. Therefore

$$\frac{\prod_{\nu=1}^l \theta_\mu(n_\nu(t + a\tau + b), \tau)}{\prod_{j=1}^k \theta_\mu(m_j(t + a\tau + b), \tau)} = e^{-\pi i n (a^2\tau + 2at)} \frac{\prod_{\nu=1}^l \theta_\nu(n_\nu t, \tau)}{\prod_{j=1}^k \theta(m_j t, \tau)}.$$

So we get that these F^V's satisfy the condition 2) in the definition of Jacobi forms. Similarly if $p_1(M)_{S^1} = -n \cdot \pi^* u^2$, then $\sum_j m_j^2 = -n$ for each fixed point, one can get

$$H(t + a\tau + b) = e^{-\pi i n (a^2\tau + 2at)} H(t, \tau).$$

On the other hand, we have the well-known modular transformation formulas for theta-functions under the action of the generators $S, T \in SL_2(\mathbf{Z})$. See [Ch] or [Liu1]. For S, we have

$$\begin{aligned}
\frac{\theta'(0, -\frac{1}{\tau})^k}{\theta_1(0, -\frac{1}{\tau})^l} \frac{\prod_{\nu=1}^l \theta_1(\frac{n_\nu t}{\tau}, -\frac{1}{\tau})}{\prod_{j=1}^k \theta(\frac{m_j t}{\tau}, -\frac{1}{\tau})} &= \tau^k e^{\frac{\pi i n t^2}{\tau}} \frac{\theta'(0,\tau)^k}{\theta_2(0,\tau)^l} \frac{\prod_{\nu=1}^l \theta_2(n_\nu t, \tau)}{\prod_{j=1}^k \theta(m_j t, \tau)}, \\
\frac{\theta'(0, -\frac{1}{\tau})^k}{\theta_3(0, -\frac{1}{\tau})^l} \frac{\prod_{\nu=1}^l \theta_3(\frac{n_\nu t}{\tau}, -\frac{1}{\tau})}{\prod_{j=1}^k \theta(\frac{m_j t}{\tau}, -\frac{1}{\tau})} &= \tau^k e^{\frac{\pi i n t^2}{\tau}} \frac{\theta'(0,\tau)^k}{\theta_3(0,\tau)^l} \frac{\prod_{\nu=1}^l \theta_3(n_\nu t, \tau)}{\prod_{j=1}^k \theta(m_j t, \tau)}; \\
\frac{\theta'(0, -\frac{1}{\tau})^k}{\theta(0, -\frac{1}{\tau})^l} \frac{\prod_{\nu=1}^l \theta(\frac{n_\nu t}{\tau}, -\frac{1}{\tau})}{\prod_{j=1}^k \theta(\frac{m_j t}{\tau}, -\frac{1}{\tau})} &= \tau^k e^{\frac{\pi i n t^2}{\tau}} \frac{\theta'(0,\tau)^k}{\theta(0,\tau)^l} \frac{\prod_{\nu=1}^l \theta(n_\nu t, \tau)}{\prod_{j=1}^k \theta(m_j t, \tau)}.
\end{aligned}$$

Therefore

$$F^V_{d_s}(\frac{t}{\tau},-\frac{1}{\tau})=\tau^k e^{\frac{\pi i n t^2}{\tau}}F^V_D(t,\tau),\quad F^V_{-D}(\frac{t}{\tau},-\frac{1}{\tau})=\tau^k e^{\frac{\pi i n t^2}{\tau}}F^V_{-D}(t,\tau)$$

and

$$F^V_{D^*}(\frac{t}{\tau},-\frac{1}{\tau})=\tau^{k-l} e^{\frac{\pi i n t^2}{\tau}}F^V_{D^*}(t,\tau).$$

For $H(t,\tau)$, we have

$$H(\frac{t}{\tau},-\frac{1}{\tau})=\tau^k e^{\frac{\pi i n t^2}{\tau}}H(t,\tau).$$

Similarly under the action of T, we have

$$F^V_{d_s}(t,\tau+1)=F^V_{d_s}(t,\tau),\ F^V_D(t,\tau+1)=F^V_{-D}(t,\tau)$$

and

$$F^V_{D^*}(t,\tau+1)=F^V_{D^*}(t,\tau).$$

For $H(t,\tau)$, one has

$$H(t,\tau+1)=H(t,\tau).$$

One knows that T and ST^2ST generate $\Gamma_0(2)$. Also $\Gamma^0(2)$ and Γ_θ are conjugate to $\Gamma_0(2)$ by S and TS respectively. So the assertions for $F^V_{d_s}(t,\tau)$, $F^V_D(t,\tau)$ and $F^V_{-D}(t,\tau)$ follow easily from the above formulas. Also easily follow the cases for $F^V_{D^*}(t,\tau)$ and $H(t,\tau)$. □

The above proof gives some transformation formulas of the F^V's and H which are crucial for the proof of Theorem 3. We single them out as a lemma.

Lemma 2.2: *If $p_1(V)_{S^1}-p_1(M)_{S^1}=n\cdot\pi^*u^2$, we have*

$$F^V_{d_s}(\tfrac{t}{\tau},-\tfrac{1}{\tau})=\tau^k e^{\frac{\pi i n t^2}{\tau}}F^V_D(t,\tau),\ F^V_{d_s}(t,\tau+1)=F^V_{d_s}(t,\tau);$$
$$F^V_{-D}(\tfrac{t}{\tau},-\tfrac{1}{\tau})=\tau^k e^{\frac{\pi i n t^2}{\tau}}F^V_{-D}(t,\tau),\ F^V_D(t,\tau+1)=F^V_{-D}(t,\tau);$$
$$F^V_{D^*}(\tfrac{t}{\tau},-\tfrac{1}{\tau})=\tau^{k-l} e^{\frac{\pi i n t^2}{\tau}}F^V_{D^*}(t,\tau),\ F^V_{D^*}(\tau,\tau+1)=F^V_{D^*}(t,\tau).$$

If $p_1(M)_{S^1}=-n\cdot\pi^*u^2$,

$$H(\frac{t}{\tau},-\frac{1}{\tau})=\tau^k e^{\frac{\pi i n t^2}{\tau}}H(t,\tau),\ H(t,\tau+1)=H(t,\tau).\quad\square$$

For $g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbf{Z})$, let us use the notation

$$F(g(t,\tau))|_{m,k}=(c\tau+d)^{-k}e^{-2\pi i m c t^2/(c\tau+d)}F(\frac{t}{c\tau+d},\frac{a\tau+b}{c\tau+d})$$

to denote the action of g on a Jacobi form F of index m and weight k.

Lemma 2.2 tells us that, for $F \in \{F^{V'}s, H\}$, its modular transformation $F(g(t,\tau))|_{\frac{n}{2},k}$ is still one of the F^V's or H. Similar to Lemma 5.3 of Section 2.5, we have the following

Lemma 2.3: *For any $g \in SL_2(\mathbf{Z})$, let $F(t,\tau)$ be one of the F^V's or H, then $F(g(t,\tau))|_{\frac{n}{2},k}$ is holomorphic in (t,τ) for $t \in \mathbf{R}$ and $\tau \in \mathbf{H}$.* □

For this lemma, it is crucial that the F^V's and H are the Lefschetz numbers of elliptic operators. This is also the place where the spin conditions on M and V come in. The following lemma can be viewed as a summary of our key techniques.

Lemma 2.4: *For a (meromorphic) Jacobi form $F(t,\tau)$ of index m and weight k over $L \rtimes \Gamma$, assume that F may only have polar divisors of the form $t = \frac{c\tau+d}{l}$ in $C \times H$ for some integers c, d and $l \neq 0$. If $F(g(t,\tau))|_{m,k}$ is holomorphic for $t \in \mathbf{R}, \tau \in \mathbf{H}$ for every $g \in SL_2(\mathbf{Z})$, then $F(t,\tau)$ is holomorphic for any $t \in \mathbf{C}$ and $\tau \in \mathbf{H}$.*

Proof: Since the possible polar divisors of $F(t,\tau)$ can be written in the form $t = \frac{n(c\tau+d)}{l}$ with $(c,d) = 1$. We can always find integers a, b such that $ad - bc = 1$. Take $g = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in SL_2(\mathbf{Z})$. Since

$$F(g(t,\tau)) = F(\frac{t}{-c\tau + a}, \frac{d\tau + b}{-c\tau + a}),$$

it is easy to see that, if $t = \frac{n(c\tau+d)}{l}$ is the polar divisor of $F(t,\tau)$, a polar divisor of $F(g(t,\tau))$ is given by solving the equation

$$\frac{t}{-c\tau + a} = \frac{n(c\frac{d\tau-b}{-c\tau+a} + d)}{l}$$

which exactly gives $t = \frac{n}{l}$. This is a contradiction to the assumption. □

Now it is easy to **prove Theorem 3**. By Lemmas 2.1, 2.2 and 2.3 we know that the F^V's and H satisfy the assumptions of Lemma 2.4. In fact all of their possible polar divisors are of the form $t = \frac{c\tau+d}{m}$ where c, d are integers and m is one of the exponents $\{m_j\}$. This easily follows from their theta-function expressions. So Lemma 2.4 gives Theorem 3 for the isolated fixed point case.

For the general fixed point case, one only needs to verify the transformation formulas of the F^V's and H under the action of $(2\mathbf{Z})^2 \rtimes SL_2(\mathbf{Z})$. We only check the operator $D \otimes \triangle(V) \otimes \Theta'_q(TM|V)_v$ and leave the other cases to the reader. Let us keep the notations as in Section 2.5 of last part. In terms of those data the Lefschetz number $F^V_{d_s}(t,\tau)$ is then

given by

$$F^V_{d_s}(t,\tau)=\sum_{M_\alpha}(\prod_{j=1}^{k_\alpha}(2\pi i y_j F(y_j,\tau))(\prod_{\gamma=1}^{h}F(x_\gamma+m_\gamma t,\tau))(\prod_{\nu=1}^{l}F_1(u_\nu+n_\nu t,\tau))[M_\alpha]$$

where

$$F(x,\tau)=(2\pi i)^{-1}\frac{\theta'(0,\tau)}{\theta(x,\tau)},\quad F_1(x,\tau)=\frac{\theta_1(x,\tau)}{\theta_1(0,\tau)}$$

and $2k_\alpha$ is the dimension of M_α.

First recall that the condition on the first equivariant Pontrjagin classes implies the equality

$$\sum_\nu(u_\nu+n_\nu t)^2-(\sum_j y_j^2+\sum_\gamma(x_\gamma+m_\gamma t)^2)=n\cdot t^2$$

for each fixed point. See Appendix A. This means

$$\sum_\nu n_\nu^2-\sum_\gamma m_\gamma^2=n,\ \sum_\nu n_\nu u_\nu=\sum_\gamma m_\gamma x_\gamma,\ \text{and}\ \sum_\nu u_\nu^2=\sum_j y_j^2+\sum_\gamma x_\gamma^2.$$

Apply the transformation formulas of the theta-functions, we easily get

(1) Under the action $t\to t+a\tau+b$ with $a,b\in 2\mathbf{Z}$:

$$\theta_\mu(x_\gamma+m_\gamma(t+a\tau+b),\tau)=e^{-\pi i(m_\gamma^2(a^2\tau+2at)+2am_\gamma x_\gamma)}\theta_\mu(x_\gamma+m_\gamma t,\tau)$$

and

$$\theta_\mu(u_\nu+n_\nu(t+a\tau+b),\tau)=e^{-\pi i(n_\nu^2(a^2\tau+2at)+2an_\nu u_\nu)}\theta_\mu(u_\nu+n_\nu t,\tau)$$

for θ_μ one of the four Jacobi theta-functions. Combine these with the equalities derived from the condition $p_1(V)_{S^1}-p_1(M)_{S^1}=n\cdot\pi^*u^2$, we get

$$F^V_{d_s}(t+a\tau+b,\tau)=e^{-\pi in(a^2\tau+2at)}F^V_{d_s}(t,\tau).$$

(2) Under the action of $SL_2(\mathbf{Z})$. We only check the action of $S=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ and leave to the reader to check the action of T. We have

$$\theta(x_\gamma+\frac{m_\gamma t}{\tau},-\frac{1}{\tau})=\frac{1}{i}\sqrt{\frac{\tau}{i}}e^{\pi i\frac{(\tau x_\gamma+m_\gamma t)^2}{\tau}}\theta(\tau x_\gamma+m_\gamma t,\tau),$$

$$\theta(y_j,-\frac{1}{\tau})=\frac{1}{i}\sqrt{\frac{\tau}{i}}e^{\pi i\frac{(\tau y_j)^2}{\tau}}\theta(\tau y_j,\tau),\ \theta'(0,-\frac{1}{\tau})=\frac{\tau}{i}\sqrt{\frac{\tau}{i}}\theta'(0,\tau),$$

$$\theta_1(u_\nu+\frac{n_\nu t}{\tau},-\frac{1}{\tau})=\sqrt{\frac{\tau}{i}}e^{\pi i\frac{(\tau u_\nu+n_\nu t)^2}{\tau}}\theta_2(\tau u_\nu+n_\nu t,\tau).$$

Write

$$F_2(x,\tau)=(2\pi i)^{-1}\frac{\theta_2(x,\tau)}{\theta_2(0,\tau)}$$

and put the above equalities together, we get

$$F^V_{d_s}(\frac{t}{\tau},-\frac{1}{\tau})$$

$$=\sum_{M_\alpha}(\prod_{j=1}^{k_\alpha}(2\pi i y_j F(y_j,-\frac{1}{\tau}))(\prod_{\gamma=1}^{h}F(x_\gamma+\frac{m_\gamma t}{\tau},-\frac{1}{\tau}))(\prod_{\nu=1}^{l}F_1(u_\nu+\frac{n_\nu t}{\tau},-\frac{1}{\tau})[M_\alpha]$$

$$=\tau^k e^{\pi i\frac{nt^2}{\tau}}\sum_{M_\alpha}(\prod_{j=1}^{k_\alpha}(2\pi i y_j F(\tau y_j,\tau))(\prod_{\gamma=1}^{h}F(\tau x_\gamma+m_\gamma t,\tau))(\prod_{\nu=1}^{l}F_2(\tau u_\nu+n_\nu t,\tau))[M_\alpha]$$

$$=\tau^k e^{\pi i\frac{nt^2}{\tau}}\sum_{M_\alpha}(\prod_{j=1}^{k_\alpha}(2\pi i y_j F(y_j,\tau))(\prod_{\gamma=1}^{h}F(x_\gamma+m_\gamma t,\tau))(\prod_{\nu=1}^{l}F_2(u_\nu+n_\nu t,\tau))[M_\alpha]$$

$$=\tau^k e^{\pi i\frac{nt^2}{\tau}}F^V_D(t,\tau).$$

For the second equality one needs

$$\sum_\nu(u_\nu+n_\nu t)^2-(\sum_j y_j^2+\sum_\gamma(x_\gamma+m_\gamma t)^2)=n\cdot t^2$$

which is exactly the localization of $p_1(V)_{S^1}-p_1(M)_{S^1}=n\cdot\pi^*u^2$; for the third equality one notes that we only need the k_α-th homogeneous term in the expansion in $\{y_j,x_\gamma,u_\nu\}$. This finishes the proof of Theorem 3.

Proof of Theorem 5. We only discuss isolated fixed point case. The general fixed point case is the same as above. Take $E=L(\Lambda)$ to be a level m highest weight representation of $\tilde{L}\mathrm{Spin}(2l)$. Recall that the Lefschetz number of

$$q^{m_\Lambda}D\otimes\otimes_{n=1}^{\infty}S_{q^n}(TM-\mathrm{dim}M)\otimes\psi(E,V)$$

is given by

$$F_E(t,\tau)=\sum_p H(t,\tau)c_E(t,\tau)$$

where the sum is over the fixed points of $g=e^{2\pi it}$ with

$$H(t,\tau)=(2\pi i)^{-k}\prod_{j=1}^{k}\frac{\theta'(0,\tau)}{\theta(m_j t,\tau)}$$

and

$$c_E(t,\tau)=\chi_E(T,\tau).$$

Here $T=(n_1t,\cdots,n_lt)$ and $\chi_E(z,\tau)=q^{m_\Lambda}\mathrm{ch}_E(z,\tau)$ is the normalized Kac-Weyl character of the representation $E=L(\Lambda)$ of $\tilde{L}\mathrm{Spin}(2l)$. See Sections 2.2 and 2.5 for the notations.

First we extend $F_E(t,\tau)$ to a (meromorphic) function on $\mathbf{C}\times\mathbf{H}$ and then verify the following properties:

(a) Under the action of $(a,b) \in (2\mathbf{Z})^2$, we have

$$H(t+a\tau+b,\tau) = e^{\pi i \sum_j m_j^2(a^2\tau+2at)} H(t,\tau);$$

$$c_E(t+a\tau+b,\tau) = e^{-m\pi i \sum_\nu n_\nu^2(a^2\tau+2at)} c_E(t,\tau).$$

These follow from the transformation formulas of theta-functions. Since

$$mp_1(V)_{S^1} - p_1(M)_{S^1} = n \cdot \pi^* u^2$$

implies $m\sum_\nu n_\nu^2 - \sum_j m_j^2 = n$ for each fixed point, we immediately get

$$F_E(t+a\tau+b,\tau) = e^{\pi i n(a^2\tau+2at)} F_E(t,\tau).$$

(b) Under the action of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$, we have

$$H(g(t,\tau)) = H(\frac{t}{c\tau+d}, \frac{a\tau+b}{c\tau+d})$$

$$= (c\tau+d)^k e^{-c\pi i \sum_j m_j^2 t^2/(c\tau+d)} H(t,\tau)$$

which can be shown by using the transformation formulas of $\theta(t,\tau)$ and $\theta'(0,\tau)$.

On the other hand, by a theorem of Kac-Peterson-Wakimoto ([Ka], Ch. 13), there exists an integer $N(m)$ such that, for any $g \in \Gamma(N(m))$ where

$$\Gamma(N(m)) = \{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) | g \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\mathrm{mod} N(m))\},$$

one has

$$c_E(g(t,\tau)) = e^{mc\pi i \sum_\nu n_\nu^2 t^2/(c\tau+d)} c_E(t,\tau).$$

Therefore, if $mp_1(V)_{S^1} - p_1(M)_{S^1} = n \cdot \pi^* u^2$, we have

$$F_E(g(t,\tau)) = (c\tau+d)^k e^{\pi i n c t^2/(c\tau+d)} F_E(t,\tau).$$

Obviously, (a) and (b) together with Lemma 2.4 imply Theorem 5. We leave the consideration of the general fixed point case to the reader.

3.3. **Almost Complex Manifolds II.** Now we consider the case of almost complex manifolds. For simplicity we restrict ourselves to the isolated fixed point case and leave the general fixed point case to the reader.

Let W be a complex vector bundle, i.e. a real vector bundle with a comlex structure, of rank l on a compact almost complex manifold M of dimension k. Assume that there exists an S^1-action on M with respect to which W is equivariant. Assume the action preserves the complex structures of M and W.

Recall

$$\Gamma_1(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) | c \equiv 0(\mathrm{mod} N),\ a \equiv d \equiv 1(\mathrm{mod} N)\}$$

and the decompositions

$$TM \otimes \mathbf{C} = TM' \oplus TM'', \quad W \otimes \mathbf{C} = W' \oplus W''.$$

Let $L = \det W'$, $K = \det TM'$ and

$$\begin{aligned} \Theta_q^\alpha(TM|W)_v = \ & \otimes_{n=0}^\infty \ \Lambda_{-y^{-1}q^n}(W'' - \dim W) \otimes_{n=1}^\infty \Lambda_{-yq^n}(W' - \dim W) \\ & \otimes_{n=1}^\infty \ S_{q^n}(T'M - \dim M) \otimes_{n=1}^\infty S_{q^n}(T''M - \dim M) \end{aligned}$$

with $y = e^{2\pi i\alpha}$ an N-th root of unity. Then we have

Proposition 3.1: *If $w_2(W) = w_2(M)$, $c_1(W) \equiv 0(modN)$ for some positive integer N and*

$$p_1(W)_{S^1} - p_1(M)_{S^1} = n \cdot \pi^* u^2,$$

then the Lefschetz number of $\bar\partial \otimes (K^{-1} \otimes L)^{\frac{1}{2}} \otimes \Theta_q^\alpha(TM|W)_v$ is a holomorphic Jacobi form of index $\frac{n}{2}$ and weight k over $(N\mathbf{Z})^2 \rtimes \Gamma_1(N)$. $\square$

Here recall $p_1(\cdot)_{S^1}$ means the equivariant first Pontrjagin class of the intrinsic real bundle and $\bar\partial$ is the anti-holomorphic derivative.

We actually have more results. In fact all of the virtual versions of the elliptic operators in Proposition 2.1 of [Liu1] give holomorphic Jacobi forms. We summarize this in the following

Proposition 3.2: *Let M and W be as above. If $w_2(M) = w_2(W) = 0$, $c_1(W) \equiv 0(modN)$ and*

$$p_1(W)_{S^1} - p_1(M)_{S^1} = n \cdot \pi^* u^2,$$

then $\bar\partial \otimes (K^{-1} \otimes L)^{\frac{1}{2}} \otimes P_q^\alpha(TM|W)_v$, $D \otimes Q_q^\alpha(TM|W)_v$ and $D \otimes R_q^\alpha(TM|W)_v$ are holomorphic Jacobi forms of index $\frac{n}{2}$ and weight k over $(2N\mathbf{Z})^2 \rtimes \Gamma_1(2N)$. $\square$

Here recall $D = \bar\partial \otimes K^{-\frac{1}{2}}$ is the Dirac operator on M which exists by the assumption and

$$\begin{aligned} P_q^\alpha(TM|W)_v \ &= \ \otimes_{n=0}^\infty \Lambda_{y^{-1}q^n}(W'' - \dim W) \otimes_{n=1}^\infty \Lambda_{yq^n}(W' - \dim W) \\ & \otimes_{n=1}^\infty \ S_{q^n}(T'M - \dim M) \otimes_{n=1}^\infty S_{q^n}(T''M - \dim M), \\ Q_q^\alpha(TM|W)_v \ &= \ \otimes_{n=1}^\infty \Lambda_{-y^{-1}q^{n-\frac{1}{2}}}(W'' - \dim W) \otimes_{n=1}^\infty \Lambda_{-yq^{n-\frac{1}{2}}}(W' - \dim W) \\ & \otimes_{n=1}^\infty \ S_{q^n}(T'M - \dim M) \otimes_{n=1}^\infty S_{q^n}(T''M - \dim M), \\ R_q^\alpha(TM|W)_v \ &= \ \otimes_{n=1}^\infty \Lambda_{y^{-1}q^{n-\frac{1}{2}}}(W'' - \dim W) \otimes_{n=1}^\infty \Lambda_{yq^{n-\frac{1}{2}}}(W' - \dim W) \\ & \otimes_{n=1}^\infty \ S_{q^n}(T'M - \dim M) \otimes_{n=1}^\infty S_{q^n}(T''M - \dim M). \end{aligned}$$

For the proofs of Propositions 3.1 and 3.2 we have to introduce the following elliptic operators:

$$\bar{\partial} \otimes (K^{-1} \otimes L)^{\frac{1}{2}} \otimes L^{c\alpha} \otimes \Theta_q^{\alpha(c\tau+d)}(TM|W)_v$$

for Proposition 3.1; and

$$\begin{aligned} \bar{\partial} \quad & \otimes (K^{-1} \otimes L)^{\frac{1}{2}} \otimes L^{c\alpha} \otimes P_q^{\alpha(c\tau+d)}(TM|W)_v, \\ D \quad & \otimes L^{c\alpha} \otimes Q_q^{\alpha(c\tau+d)}(TM|W)_v, \\ D \quad & \otimes L^{c\alpha} \otimes R_q^{\alpha(c\tau+d)}(TM|W)_v \end{aligned}$$

for Proposition 3.2. Here the bundles $\Theta_q^{\alpha(c\tau+d)}$, $P_q^{\alpha(c\tau+d)}$, $Q_q^{\alpha(c\tau+d)}$ and $R_q^{\alpha(c\tau+d)}$ are the same as the Θ_q^{α}, P_q^{α}, Q_q^{α} and R_q^{α} respectively, but replacing α by $\alpha(c\tau + d)$. We denote their Lefschetz numbers by $F^{\alpha(c\tau+d)}(t,\tau)$, $P^{\alpha(c\tau+d)}(t,\tau)$, $Q^{\alpha(c\tau+d)}(t,\tau)$ and $R^{\alpha(c\tau+d)}(t,\tau)$ respectively. Let $\{m_j\}, \{n_\nu\}$ be the exponents of $T'M, W'$ respectively. See Section 2.6. Then in terms of the theta-functions we have

$$\begin{aligned} F^{\alpha(c\tau+d)}(t,\tau) &= (2\pi i)^{-k} \sum_p e^{2\pi i c\alpha \sum n_\nu t} \frac{\theta'(0,\tau)^k}{\theta(\alpha(c\tau+d),\tau)^l} \frac{\prod_{\nu=1}^l \theta(n_\nu t + \alpha(c\tau+d),\tau)}{\prod_{j=1}^k \theta(m_j t,\tau)}; \\ P^{\alpha(c\tau+d)}(t,\tau) &= (2\pi i)^{-k} \sum_p e^{2\pi i c\alpha \sum n_\nu t} \frac{\theta'(0,\tau)^k}{\theta_1(\alpha(c\tau+d),\tau)^l} \frac{\prod_{\nu=1}^l \theta_1(n_\nu t + \alpha(c\tau+d),\tau)}{\prod_{j=1}^k \theta(m_j t,\tau)}, \\ Q^{\alpha(c\tau+d)}(t,\tau) &= (2\pi i)^{-k} \sum_p e^{2\pi i c\alpha \sum n_\nu t} \frac{\theta'(0,\tau)^k}{\theta_2(\alpha(c\tau+d),\tau)^l} \frac{\prod_{\nu=1}^l \theta_2(n_\nu t + \alpha(c\tau+d),\tau)}{\prod_{j=1}^k \theta(m_j t,\tau)}, \\ R^{\alpha(c\tau+d)}(t,\tau) &= (2\pi i)^{-k} \sum_p e^{2\pi i c\alpha \sum n_\nu t} \frac{\theta'(0,\tau)^k}{\theta_3(\alpha(c\tau+d),\tau)^l} \frac{\prod_{\nu=1}^l \theta_3(n_\nu t + \alpha(c\tau+d),\tau)}{\prod_{j=1}^k \theta(m_j t,\tau)}. \end{aligned}$$

Obviously when $c = 0, d = 1$ we recover the Lefschetz numbers of the operators in Propositions 3.1 and 3.2.

Follow the same method as before, one can check the modularity of $F^\alpha(t,\tau)$, $P^\alpha(t,\tau)$, $Q^\alpha(t,\tau)$ and $R^\alpha(t,\tau)$ under the actions of the corresponding groups. Also for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$ one has the following transformation formulas:

$$F^\alpha(g(t,\tau))|_{\frac{n}{2},k} = F^{\alpha(c\tau+d)}(t,\tau),$$

and

$$P^\alpha(g(t,\tau))|_{\frac{n}{2},k} = U^{\alpha(c\tau+d)}(t,\tau)$$

where $U^{\alpha(c\tau+d)}(t,\tau)$ is one of the $P^{\alpha(c\tau+d)}(t,\tau)$, $Q^{\alpha(c\tau+d)}(t,\tau)$ and $R^{\alpha(c\tau+d)}(t,\tau)$. It is quite easy to show that they are preserved by the corresponding modular subgroups. Consult [Liu1].

Together with Lemma 1.4 in [Liu1], we can prove Propositions 3.1 and 3.2 in the same way as that of proving Theorem 3.

Some higher level elliptic operators for almost complex manifolds were discussed in last part. Under the assumption

$$mp_1(W)_{S^1} - p_1(M)_{S^1} = n \cdot \pi^* u^2,$$

one can get holomorphic Jacobi forms of index $\frac{n}{2}$ and weight k over $(2N\mathbf{Z})^2 \rtimes \Gamma(2N(m))$ for the elliptic operator in Theorem 2 in Section 2.6. We omit the details.

3.4. **Vanishing Theorems for Loop Space.** In this section we apply some simple facts in the theory of Jacobi forms to our situation and get certain topological results for manifolds with S^1-actions. It is conceivable that the applications of some deeper theory of Jacobi forms might give much deeper topological results. This should be an interesting topic for further studies.

The following lemma which is Theorem 1.2 in [EZ] can be easily proved by using the property 2) of Jacobi forms in Section 3.2 and by considering the integral of $\frac{\frac{\partial F_t}{\partial t}}{F}$ around the boundary of the fundamental domain of the lattice L.

Lemma 4.1: *Let F be a holomorphic Jacobi form of index m and weight k. Then for fixed τ, $F(t,\tau)$, if not identically zero, has exactly $2m$ zeroes in any fundamental domain for the action of the lattice on* $\mathbf{C}$. □

This tells us that there are no holomorphic Jacobi form of negative index, therefore, if $m < 0$, F must be identically zero. If $m = 0$, it is easy to see that F must be independent of t.

The following lemma is Theorem 2.2 in [EZ] and can be proved by using the property 2) of Jocobi forms as given in Section 3.2.

Lemma 4.2: *Let F be a holomorphic Jacobi form of index m and weight k. Assume that F has Fourier development $\sum_{l,r} c(l,r)q^l z^r$. Then $c(l,r)$ depends only on $4lm - r^2$ and on $r(mod\, 2m)$. If $m = 1$ or m is prime, then $c(l,r)$ depends only on $4lm - r^2$. If $m = 1$ and k is odd, then F is identically zero.*

Combining Lemmas 4.1 and 4.2 with Theorem 3, we have the following result:

Corollary 4.1: *Let M, V and n be as in Theorem 3. If $n = 0$, the Lefschetz numbers of the elliptic operators in Theorem 3 are independent of the generators of S^1. If $n < 0$ or $n = 2$ and $k{=}\frac{1}{2}dimM$ is odd, then these Lefschetz numbers are identically zero, especially the indices of these elliptic operators are zero.* □

We know that, when $k = \frac{1}{2}\dim M$ is odd, the indices of these elliptic operators should be zero by Atiyah-Singer index formula, since the degree of characteristic classes of a compact real manifold are of $4l$. But it is not so obvious that their Lefschetz numbers should be zero.

One can also get the following results from the above lemmas and Theorem 5,

Corollary 4.2: *Let M, V, E and n be as in Theorem 5. If $n = 0$, the Lefschetz number of*

$$q^{m_\Lambda} D \otimes \otimes_{n=1}^{\infty} S_{q^n}(TM - dimM) \otimes \psi(E, V)$$

is independent of the generator of S^1. If $n < 0$, this Lefschetz number is identically zero, especially its index is zero.

For almost complex manifolds, we have

Corollary 4.3: *a) Let M, W and n be as in Proposition 3.1. If $n = 0$, the Lefschetz number of $\bar{\partial} \otimes (K^{-1} \otimes L)^{\frac{1}{2}} \otimes \Theta_q^\alpha(TM|W)_v$ is independent of the generator of S^1. If $n < 0$ or $n = 2$ and k is odd, this Lefschetz number is identically zero, especially the index of this operator is zero.*

b) Under the assumptions of Proposition 3.2, the same conclusions hold for $\bar{\partial} \otimes (K^{-1} \otimes L)^{\frac{1}{2}} \otimes P_q^\alpha(TM|W)_v$, $D \otimes Q_q^\alpha(TM|W)_v$ and $D \otimes R_q^\alpha(TM|W)_v$. □

Another quite interesting consequence of the above discussions is the following

Theorem 4: *Let M be a compact spin manifold with an S^1-action, if $p_1(M)_{S^1} = -n \cdot \pi^* u^2$ for some integer n, then the Lefschetz number, especially the index, of*

$$D \otimes \otimes_{m=1}^{\infty} S_{q^m}(TM - dimM)$$

is zero. □

Proof: In fact, from the proof of Lemma 2.1 we find that the case $n > 0$ can never happen, since the condition on the first equivariant Pontrjagin class tells us that $\sum m_j^2 = -n$ for each fixed point. The case $n = 0$ implies that all the exponents $\{m_j\}$ are zero which means that the S^1-action can not have fixed point. This in turn implies the vanishing of any characteristic number. For $n < 0$, one can apply Lemmas 2.1, 2.4 and 4.1 to get the result. □

Theorem 4 may be viewed as a loop space analogue of the $\hat{\mathfrak{A}}$-vanishing theorem of Atiyah-Hirzebruch [AH] for compact spin manifolds with S^1-actions. It is easy to show that, if M is 2-connected, the condition $p_1(M)_{S^1} = -n \cdot \pi^* u^2$ is actually equivalent to $p_1(M) = 0$ which is the spin condition on LM. Because of the special role played by Dirac operator on spin manifold, Theorem 4 suggests that there should be much

more interesting theory lying behind this vanishing theorem. When the S^1-action is induced from an S^3-action, A. Dessai informed me that the condition $p_1(M)_{S^1} = -n \cdot \pi^* u^2$ is also equivalent to $p_1(M) = 0$.

One can draw more corollaries from our theorems. Here we only give several examples.

(1) If $p_1(M)_{S^1} - mp_1(V)_{S^1} = n \cdot \pi^* u^2$ with the integer $n \leq 0$, then $D \otimes V$ and $D \otimes \triangle(V) \otimes V$ are rigid.

(2) If $p_1(V)_{S^1} = n \cdot \pi^* u^2$ with $n \leq 0$, then $d_s \otimes V$ is rigid.

(3) If $n < 0$, then the indices of the above operators vanish.

These results can be derived from the tensor products of the four level 1 irreducible representations of $\tilde{L}\mathrm{Spin}(2l)$. See Section 2.4.

3.5. **Appendix A: The Derivation of $F_E(t, \tau)$.** In this section we describe a simple way to derive $F_E(t, \tau)$ which is the local expression in the Lefschetz fixed point formula for the elliptic operator in Theorem 1. We also discuss equivariant characteristic classes from the point of view of differential geometry.

Still let M be a $2k$-dimensional compact smooth spin manifold with S^1-action, and V be an equivariant spin vector bundle of rank $2l$ on M. All elliptic operators on M are twisted Dirac operators. Consider elliptic operator $D \otimes V$ and denote its Lefschetz number at $g = e^{2\pi i t} \in S^1$ with respect to the S^1-action by $L(t, V)$.

It is very interesting to understand the Lefschetz fixed point formula in the category of equivariant cohomology. First every equivariant vector bundle V on M has an equivariant extension which is the bundle

$$p: \ V \times_{S^1} ES^1 \to M \times_{S^1} ES^1.$$

The characteristic classes of this bundle are the equivariant characteristic classes of V. We denote the equivariant $\hat{\mathfrak{A}}$-class by $\hat{\mathfrak{A}}_{S^1}$ and the equivariant Chern character by ch_{S^1}.

Using differential geometry, we can give explicit formulas for these equivariant classes. Let D_V be a covariant derivative on V, ω_V be the S^1-invariant connection form and $\Omega_V = D_V \omega_V$ be the curvature matrix. Let $J_V = i_X \omega_V$ where X is the Killing vector field generated by the S^1-action and i_X is the contraction operator. Since ω_V is S^1-invariant, it is easy to see that

$$i_X \Omega_V = -D_V J_V.$$

Similarly we have the corresponding J_M and Ω_M for the tangent bundle TM of M. Replace the generator u of $H^*(BS^1, Z) = Z[[u]]$ by t, we

then can use

$$\begin{aligned}\hat{\mathfrak{A}}_{S^1}(M) &= \det{}^{\frac{1}{2}} \frac{(\Omega_M + tJ_M)/2}{\sinh(\Omega_M + tJ_M)/2}, \\ \mathrm{ch}_{S^1} V &= \mathrm{tr}\ e^{\Omega_V + tJ_V}\end{aligned}$$

in practical computations. Here, modulo torsion, we have used the identification of equivariant cohomology with the cohomology of the complex $(\Omega^*_{S^1}(M), d + ti_X)$, where $\Omega^*_{S^1}(M)$ is the S^1-invariant C^∞-differential forms on M. See [AB1] and [BV] for further details about this identification.

When restricted to the fixed point set, using the notations of Section 2.5, we can formally write

$$\begin{aligned}J_V &= \oplus_{\nu=1}^{l} \begin{pmatrix} 0 & 2\pi i n_\nu \\ -2\pi i n_\nu & 0 \end{pmatrix} \\ \Omega_V &= \oplus_{\nu=1}^{l} \begin{pmatrix} 0 & 2\pi i u_\nu \\ -2\pi i u_\nu & 0 \end{pmatrix}.\end{aligned}$$

One has similar expressions for J_M and Ω_M in terms of the $\{y_j, m_\gamma, x_\gamma\}$ in Section 2.5. Denote by

$$\pi_* : \ H^*_{S^1}(M, Z) \to H^*(BS^1, Z)$$

where

$$\pi : \ M \times_{S^1} ES^1 \to BS^1$$

is the canonical projection, the push-forward map. We then have the following identities

$$\begin{aligned}L(t, V) &= \pi_*(\hat{\mathfrak{A}}_{S^1}(M)\mathrm{ch}_{S^1} V) \\ &= \sum_{M_\alpha} \frac{i^*_\alpha(\hat{\mathfrak{A}}_E(M)\mathrm{ch}_E V)}{E(\nu_\alpha)}[M_\alpha]\end{aligned}$$

where $E(\nu_\alpha)$ is the equivariant Euler class of the normal bundle of M_α in M. The second equality is called Bott localization.

Let $i_\alpha : \ M_\alpha \to M$ be the inclusion and i^*_α denote the induced homomorphism in equivariant cohomology. In terms of the local data

on M_α, we have

$$\begin{aligned} i_\alpha^* \mathrm{ch}_{S^1} V &= \sum_{\nu=1}^{l} e^{2\pi i(u_\nu + n_\nu t)}, \\ E(\nu_\alpha) &= \prod_{\gamma=1}^{h} (x_\gamma + m_\gamma t), \quad \text{and} \\ i_\alpha^* \hat{\mathfrak{A}}(M) &= \hat{\mathfrak{A}}(M_\alpha) \prod_{\gamma=1}^{h} \frac{x_\gamma + m_\gamma t}{e^{\pi i(x_\gamma + m_\gamma t)} - e^{-\pi i(x_\gamma + m_\gamma t)}}. \end{aligned}$$

One then notes that $i_\alpha^* \mathrm{ch}_{S^1} \otimes_{n=1}^{\infty} S_{q^n}(TM)$ is the inverse of

$$\prod_{n=1}^{\infty} \prod_{j=1}^{k_\alpha} (1 - e^{2\pi i y_j} q^n)(1 - e^{-2\pi i y_j} q^n) \prod_{\gamma=1}^{h} (1 - e^{2\pi i(x_\gamma + m_\gamma t)} q^n)(1 - e^{-2\pi i(x_\gamma + m_\gamma t)} q^n),$$

and

$$q^{m_\Lambda} i_\alpha^* \mathrm{ch}_{S^1} \psi(E, V) = c_E(u + t, \tau) = \chi_E(U + T, \tau)$$

with $U + T = (u_1 + n_1 t, \cdots, u_l + n_l t)$. Also recall that the Jacobi theta-function

$$\theta(v, \tau) = q^{\frac{1}{8}} 2 \sin \pi v \prod_{n=1}^{\infty} (1 - q^n)(1 - e^{2\pi i v} q^n)(1 - e^{-2\pi i v} q^n).$$

Put these formulas together, we get the expression of $F_E(t, \tau)$.

One also has explicit expressions for the equivariant Pontrjagin classes

$$\begin{aligned} i_\alpha^* p(V)_{S^1} &= \prod_{\nu=1}^{l} (1 + (u_\nu + n_\nu t)^2), \\ i_\alpha^* p(M)_{S^1} &= \prod_{j=1}^{k_\alpha} (1 + y_j^2) \prod_{\gamma=1}^{h} (1 + (x_\gamma + m_\gamma t)^2). \end{aligned}$$

Therefore

$$\begin{aligned} i_\alpha^* p_1(V)_{S^1} &= \sum_{\nu=1}^{l} (u_\nu + n_\nu t)^2, \\ i_\alpha^* p_1(M)_{S^1} &= \sum_{j=1}^{k_\alpha} y_j^2 + \sum_{\gamma=1}^{h} (x_\gamma + m_\gamma t)^2. \end{aligned}$$

These are the formulas of the first equivariant Pontrjagin classes which we used in our proofs of the rigidity and vanishing theorems.

4. Elliptic Genera and Elliptic Surfaces

In this part we use the geometry of elliptic modular surface to study the topology of manifolds with S^1-actions. We also use this idea to explain the algebraic geometry behind the transfer argument in [BT], [H] and [Kri]. From now on we only consider level 1 case and assume that the anomalies vanish. So we are in the situation that the Witten rigidity theorems hold. For explicity we only consider the isolated fixed point case.

4.1. **Localization and Elliptic Surfaces.** For a positive integer K, consider the open elliptic surface

$$X_{\Gamma(K)} = \mathbf{C} \times \mathbf{H}/\mathbf{Z}^2 \rtimes \Gamma(K)$$

where the action of $(a, b) \in Z^2$ is given by

$$(t, \tau) \to (t + a\tau + b, \tau)$$

and $g \in \Gamma$ acts by modular transformation. Let $\bar{X}_{\Gamma(K)}$ be the toric compactification of $X_{\Gamma(K)}$ by adding singular fibers and consider the natural projection

$$\pi : \ \bar{X}_{\Gamma(K)} \to \bar{Y}_{\Gamma(K)}$$

where $\bar{Y}_{\Gamma(K)}$ is the the compactification of $\mathbf{H}/\Gamma(K)$ by adding cusps. The fiber over $\tau \in Y_{\Gamma(K)}$ is

$$\pi^{-1}(\tau) = C/Z + Z\tau.$$

For $K > 2$ the singular fibers of π, lying only above the cusps of $\bar{Y}_{\Gamma(K)}$, are equivalent to each other and are K-gons of rational curves. Explicitly a singular fiber is given by

$$\cup_{\nu=0}^{K-1} \Theta_\nu, \ \text{with } \Theta_\nu \cdot \Theta_{\nu+1} = 1$$

where $\Theta_\nu \simeq \mathbf{C}P^1$. We know that Θ_ν has self-intersection -2 and is covered by two affine charts W_ν^0 and W_ν^1, where the coordinates (u_ν, v_ν) of W_ν^0 can be chosen such that $\Theta_\nu|_{W^0}$ is given by $v_\nu = 0$, then the coordinates of W_ν^1 are $(u_\nu^{-1}, u_\nu^2 v_\nu)$. From this the following relations can be deduced easily

$$u_{\nu+1} = v_\nu^{-1}, \ v_{\nu+1} = u_\nu v_\nu^2;$$

$$u_\nu v_\nu = q_K, \ u_\nu^{\nu+1} v_\nu^\nu = z$$

where $q_K = e^{\frac{2\pi i \tau}{K}}, z = e^{2\pi i t}$.

We keep the conventions of last two parts, i.e. M is a dimension $2k$ spin manifold with an S^1-action and V is a rank $2l$ equivariant spin vector bundle on it. First let us consider the behaviors of $F_{d_s}^V(t, \tau)$ and $F_{D^*}^V(t, \tau)$ around Θ_ν by using the above local coordinates. Let

$F^V_{d_s}(u_\nu^{\nu+1}v_\nu^\nu,(u_\nu v_\nu)^K)$ denote $F^V_{d_s}(t,\tau)$, but replacing (z,q) by the local coordinates u_ν, v_ν on Θ_ν. We use the same notations for $F^V_{D^*}(t,\tau)$ as well as for the theta-functions in the fixed point formula expressions.

Let us first take $V = TM$ and simply write the corresponding $F^V_{d_s}(t,\tau)$ as $F_{d_s}(t,\tau)$. Denote by $d_s = D\otimes\triangle(M)$ the signature operator on M.

(1) On Θ_0, $z = u_0, q = (u_0v_0)^{2m}$. It is easy to see that, when $v_0 = 0$, one has $q = 0$ and $\Theta'_q(TM)_v = 1$, therefore

$$F_{d_s}(t,\tau) = \text{the Lefschetz number of } d_s \text{ on } M.$$

(2) On Θ_ν for $\nu > 0$. We assume that $m_j\nu = 2ml_j + k_j$ with $k_j \geq 0$. Here recall that the m_j's are the exponents of TM at the fixed points. See Section 2.5. When v_ν goes to zero, one has

$$\frac{\theta'(0,(u_\nu v_\nu)^{2m})}{\theta_1(0,(u_\nu v_\nu)^{2m})} \to 1$$

and

$$\frac{\theta_1((u_\nu v_\nu)^{m_j\nu}u_\nu^{m_j},(u_\nu v_\nu)^{2m})}{\theta((u_\nu v_\nu)^{m_j\nu}u_\nu^{m_j},(u_\nu v_\nu)^{2m})} = (-1)^{l_j}\frac{\theta_1((u_\nu v_\nu)^{k_j}u_\nu^{m_j},(u_\nu v_\nu)^{2m})}{\theta((u_\nu v_\nu)^{k_j}u_\nu^{m_j},(u_\nu v_\nu)^{2m})}$$

$$\to \begin{cases} (-1)^{l_j} & \text{if } k_j \neq 0; \\ (-1)^{l_j}\frac{1+u_\nu^{m_j}}{1-u_\nu^{m_j}} & \text{if } k_j = 0. \end{cases}$$

Since M is spin, $\{(-1)^{\sum_j l_j}\}$ have the same parity for different points in one connected component of M_{2m} which is the fixed submanifold of the cyclic group $\mathbf{Z}_{2m} \subset S^1$ (see Lemma 8.1 in [BT]). Then the limiting terms sum up to the Lefschetz number of the signature operator on $M_{2m} = \cup_i M^i_{2m}$ where $\{M^i_{2m}\}$ denote the connected components of M_{2m}.

By rigidity theorems we know that $F_{d_s}(t,\tau)$ is independent of t, so we have

Theorem 6:

$$\sum_{M^i_{2m}}(-1)^{\sum_j l_j} sign(M^i_{2m}) = sign(M),$$

where sign$(\cdot)$ *denotes the signature, i.e. the index of* d_s. □

The constancy of $F^V_{d_s}(t,\tau)$ and $F^V_{D^*}(t,\tau)$ can also give some interesting topological results which we would like to leave to the reader to verify. For example assume M and V are spin with $p_1(V)_{S^1} = p_1(M)_{S^1}$. Let $n_j\nu = 2mp_j + q_j$ with $q_j \geq 0$, where the n_j's are the exponents of V at the fixed points. we then have

$$\mathrm{Ind}(D\otimes\triangle(V)) = \sum_{M^i_{2m}}(-1)^{\sum_j l_j}\mathrm{Ind}(D^i_m\otimes\triangle(V^i_m)),$$

$$e(V) = \sum_{M^i_{2m}} (-1)^{\sum_j l_j - \sum_j p_j} e(V^i_m),$$

where D^i_m denotes the Dirac operator on M^i_{2m}, V^i_m is the $\mathbf{Z}_{2m}$ invariant part of V restricted to M^i_{2m} and $e(\cdot)$ denotes the Euler number.

We leave the discussions of the case of almost complex manifolds to the reader. For example let M and W be as in Section 3.3. Let $\{m_j\}$ be the exponents of TM' and

$$Td_y(M) = \text{the index of } \bar{\partial} \otimes \Lambda_{-y^{-1}} T''M,$$

then one can easily get

$$Td_y(M) = \sum_{M^i_{2m}} y^{-\sum_j l_j} Td_y(M^i_{2m})$$

where $\{l_j\}$ are integers such that $m_j \nu = 2ml_j + k_j$ with $k_j \geq 0$ as in the spin case and $\cup_i M^i_{2m}$ are the fixed point submanifolds of $\mathbf{Z}_{2m}$.

The following corollary also corresponds to the singular fibers of the elliptic surface $\bar{X}_{\Gamma(2m)}$. We use the same notations as above.

Corollary 1.1: *a) For spin case, if $w_2(V) = 0$ and $p_1(V)_{S^1} = p_1(M)_{S^1}$, then $D \otimes \triangle(V)$, $D \otimes (\triangle^+(V) - \triangle^-(V))$ and $D \otimes V$ are rigid.*

b) For almost complex case, if $p_1(W)_{S^1} = p_1(M)_{S^1}$, $w_2(M) = w_2(W)$ and $c_1(W) \equiv 0 (mod N)$, then $\bar{\partial} \otimes (K^{-1} \otimes L)^{\frac{1}{2}} \otimes L^{\frac{s}{N}}$ for $-N < s < 1$ and $\bar{\partial} \otimes (K^{-1} \otimes L)^{\frac{1}{2}} \otimes \Lambda_{-y^{-1}} W''$ are rigid for y an N-th root of unity.

Recall that $L = \det W'$ and $K = \det TM'$.

Proof: At $q = 0$,

$$\Theta'_q(TM|V)_v = 1; \ \Theta^*_q(TM|V)_v = 1; \text{ and } \Theta^\alpha_q(TM|W)_v = \Lambda_{-y^{-1}} W''.$$

Also V is the second term in the q-expansion of $\Theta_q(TM|V)_v$. □

As observed by Bott, we even do not know a direct proof of the rigidity of $D \otimes TM$ without using the Witten rigidity theorems. Therefore it will be interesting to find a simple direct proof of the above corollary. Our proof of the Witten rigidity theorems is, in some sense, representation theoretic, since the modular invariance is essentially related to the characters of the representations of affine Lie algebras. It will be interesting to find a representation theoretic proof of the $\hat{\mathfrak{A}}$-vanishing theorem of Atiyah-Hirzebruch, which may bring some new light to rigidity theorems. The geometric relationship between the rigidity of the elliptic operators on loop space and their modular invariance is still mysterious at present.

4.2. **Transfer and Elliptic Surfaces.** In this section we study in more detail the behavior of $F_{d_s}(t, \tau)$ around the singular fibers of the elliptic modular surface $\bar{X}_{\Gamma(2m)}$, for any positive integer $m > 1$. It may

be interesting to see that the expression of $F_{d_s}(t,\tau)$ on Θ_ν discussed in last section naturally invites us to the transfer argument which is the crucial technique in [BT], [H] and [Kri]. We leave the considerations of the other elliptic operators to the reader. For simplicity we consider the non-virtual version of $F_{d_s}(t,\tau)$, that is we consider elliptic operator

$$D \otimes \triangle(M) \otimes \otimes_{n=1}^{\infty} S_{q^n} TM \otimes_{m=1}^{\infty} \Lambda_{q^m} TM$$

and still denote its Lefschetz number by $F_{d_s}(t,\tau)$. In terms of the theta-functions, we have

$$F_{d_s}(t,\tau) = i^{-k} \sum_p \prod_{j=1}^{k} \frac{\theta_1(m_j t,\tau)}{\theta(m_j t,\tau)}.$$

See [Liu1].

Let M and $M_{2m} = \cup_i M_{2m}^i$ be as in last section. Let $p \in M_{2m}^i$ be a fixed point of the S^1-action and still let $\{E_j\}$ be the line bundles in the equivariant decomposition of TM restricted at p. Then according to the action of the $\mathbf{Z}_{2m} \subset S^1$ we have

$$TM|_{M_{2m}^i} = TM_{2m}^i \oplus E_1 \oplus \cdots \oplus E_h.$$

and the S^1 acts on E_j by $e^{2\pi i m_j t}$ with $k_j \neq 0$ where $m_j \nu = 2ml_j + k_j$ as in last section. Note that those E_j's which have $k_j = 0$ are absorbed into TM_{2m}^i. It is easy to see that

$$\frac{\theta_1((u_\nu v_\nu)^{m_j \nu} u_\nu^{m_j}, (u_\nu v_\nu)^{2m})}{\theta((u_\nu v_\nu)^{m_j \nu} u_\nu^{m_j}, (u_\nu v_\nu)^{2m})} = (-1)^{l_j} \mathrm{ch}_{S^1} \Theta_{w_\nu}(E_j)$$

where $w_\nu = u_\nu v_\nu = q^{\frac{1}{2m}}$ and

$$\Theta_{w_\nu}(E_j) = \frac{\Lambda_{w_\nu^{k_j}} E_j}{\Lambda_{-w_\nu^{k_j}} E_j} \otimes_{n=1}^{\infty} \left(\frac{\Lambda_{w_\nu^{2mn+k_j}} E_j}{\Lambda_{-w_\nu^{2mn+k_j}} E_j} \otimes \frac{\Lambda_{w_\nu^{2mn+k_j}} E_j^*}{\Lambda_{-w_\nu^{2mn+k_j}} E_j^*}\right).$$

Here ch_{S^1} is the equivariant Chern character restricted to M_{2m}^i and E_j^* denotes the complex dual of E_j .

For a vector bundle F, write

$$\Theta_q'(F) = \otimes_{m=1}^{\infty} S_{q^m} F \otimes_{n=1}^{\infty} \Lambda_{q^n} F.$$

Then one gets

$$\begin{aligned} F_{d_s}(u_\nu^{\nu+1} v_\nu^\nu, (u_\nu v_\nu)^{2m}) \quad &= \quad \sum_{M_{2m}^i} (-1)^{\sum l_j} (\text{the Lefschetz number of} \\ & d_s^i \otimes \quad \Theta_q(TM_{2m}^i) \otimes_{j=1}^{m} \Theta_{w_\nu}(E_j) \text{ on } M_{2m}^i) \end{aligned}$$

where d_s^i is the signature operator on M_{2m}^i. By Proposition 6.1 of [BT] or Lemma 2.3 of [Liu1], see also Lemma 5.3 in Section 2.5, one knows

that $\{1 - z^{m_j} q^{-2ml_j} = 0\}$, which is the same as $\{1 - u_\nu^{m_j} = 0\}$ in the (u_ν, v_ν) coordinate, is not the polar divisor of $F_{d_s}(t, \tau)$. Note that only the $k_j = 0$ terms in the fixed point formula may contribute polar divisors of the form $\{1 - u_\nu^{m_j} = 0\}$ in the neiborhood of Θ_ν, and they are eliminated by the above expression. In this way one can consider other components of the singular fibers and prove that all of the possible polar divisors can not happen. In fact all of the polar divisors of $F_{d_s}(t, \tau)$ can be transformed into the form $\{1 - u_\nu^{m_j} = 0\}$ around some singular component Θ_ν. Here we have omitted the consideration of the action of $-1 \in S^1$ and would like to refer the readers to [BT], [H] and [Kri] for the details of this transfer argument. Note that we come up to this argument from a different point of view from that of [BT], [H] and [Kri]. It is quite interesting to relate this technique to the geometry of elliptic modular surfaces. For example we find that the 'transfer' to the $\mathbf{Z}_{2m}$ fixed point submanifold in [BT] corresponds to the 'transfer' from Θ_0 to Θ_ν for $\nu \neq 0$ on the singular fibers of $\bar{X}_{\Gamma(2m)}$. Our proof of the Witten rigidity theorems is, in some sense, a global transfer, because we have used the whole elliptic surface. Modular group action interchanges different singular fibers and transforms the Θ_ν of one singular fiber to the Θ_0 of another singular fiber. The proofs of [BT], [H] and [Kri] are, in a sense, local transfer, because they worked around one singular fiber.

4.3. **Appendix B: A Mod 2 Rigidity Theorem.** Let M be an $8k + 1$ or $8k + 2$ dimensional compact smooth spin manifold. Let $\triangle(M) = \triangle^+(M) \oplus \triangle^-(M)$ be the $\mathbf{Z}_2$-graded spinor bundle on M, and let

$$D\colon \triangle^+(M) \to \triangle^-(M)$$

be the Dirac operator. Given a real vector bundle E on M we can form the twisted Dirac operator $D \otimes E$ and obtain a skew adjoint or skew Hermitian elliptic operator, which gives well-defined index

a) $\dim_{\mathbf{R}}$ Ker $D \otimes E$ mod 2, if $\dim M = 8k + 1$
b) $\dim_{\mathbf{C}}$ Ker $D \otimes E$ mod 2, if $\dim M = 8k + 2$

as topological invariant. We write it as $\mathrm{Ind}_2 D \otimes E$. This index can be naturally extended to a homomorphism from the real K-group $KO(M)$ to $\mathbf{Z}_2$.

Recall that a modular form $f(\tau)$ over a modular subgroup Γ is a holomorphic function on the upper half plane H, with the following transformation law

$$f(\frac{a\tau + b}{c\tau + d}) = \chi(g)(c\tau + d)^k f(\tau)$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $\chi : \Gamma \to C^*$ is a character of Γ. The integer k is called the weight of f. We also assume that f is holomorphic at $\tau = i\infty$.

The power series expansion of $f(\tau)$ in $q_N = e^{\frac{2\pi i\tau}{N}}$ for some positive integer N is called the Fourier expansion of f. We denote the ring of modular forms over a modular subgroup Γ with integral Fourier coefficients by $M^{\mathbf{Z}}(\Gamma)$. For $f(\tau) = \sum_{j=0}^{\infty} a_j q_N^j \in M^Z(\Gamma)$ and a prime number p, we consider the modulo p reduction of f which is given by $\bar{f}(\tau) = \sum_{j=0}^{\infty} \bar{a}_j q_N^j$ where $\bar{a}_j$ is $a_j \mathrm{mod} p$. We call $\bar{f}(\tau)$ a mod p modular form.

From number theory, we know that $M^{\mathbf{Z}}(\Gamma_0(2))$ has an integral basis consisting of two elements. We also know that $\Gamma^0(2)$ and Γ_θ are conjugate to $\Gamma_0(2)$. Let $\Theta_q(TM|V)_v$ and $\Theta_{-q}(TM|V)_v$ be as in Section 3.1. Take $V = TM$ and denote the corresponding elements by $\Theta_q(TM)_v$ and $\Theta_{-q}(TM)_v$ respectively. In [Liu] we proved the following

Theorem B1: *Let M be a dimension $8k+1$ or $8k+2$ compact smooth spin manifold . Then the following mod 2 indices are mod 2 modular forms over the corresponding modular groups*

1): $Ind_2(D \otimes \Theta_q(TM)_v$ over $\Gamma^0(2)$;

2): $Ind_2(D \otimes \Theta_{-q}(TM)_v$ over Γ_θ. $\square$

The proof is essentially an index formula interpretation of an idea of Ochanine. See [Liu] for the detail. The mod 2 modular forms in this theorem are called mod 2 elliptic genera.

Now we prove a kind of mod 2 rigidity theorem which was also obtained by K. Ono independently. First recall that an odd type involution on a spin manifold can not be lifted to an action on the spin structure. See [AB] II for a detailed discussion about odd involutions. The proof is very simple. Our purpose is to motivate the study of mod p rigidity in topology.

Theorem B2: *The existence of an odd type involution on an $8k+1$ dimensional compact smooth spin manifold implies the vanishing of the mod 2 elliptic genera.*

Proof: We only consider $D \otimes \Theta_q(M)_v$. Let T be the odd involution on M. Naturally T induces an action on TM which we still denote by T. Since T is of odd type, only its double cover can be lifted to the spin bundle. See [AH]. We denote the lifting by $\hat{T}$. Then $\hat{T}^2 = -1$ on the spinor bundle.

By choosing a T invariant metric on M, we can assume that both T and $\hat{T}$ commute with the action of the skew-adjoint operator $P =$

$D \otimes \Theta_q(TM)_v$. Consider the action

$$S = \hat{T} \otimes T$$

on

$$\triangle^+(M) \otimes \Theta_q(TM)_v;$$

then

$$S^2 = \hat{T}^2 \otimes T^2 = -1.$$

This is because T acts on the virtual bundle $TM - \dim M$ by involution and $\hat{T}^2$ acts on $\triangle^+(M)$ by -1.

S^2 induces identity action on M and $\Theta_q(TM)_v$, while S induces a non-trivial action on

$$\Gamma(\triangle^+(M) \otimes \Theta_q(TM)_v.$$

Since S commutes with P, naturally it induces an action on

$$\mathrm{Ker}(P \otimes \Theta_q(TM)_v$$

which satisfies $S^2 = -1$.

As a family of (virtual) real vector spaces, each term in the q-expansion of $\mathrm{Ker}(P \otimes \Theta_q(TM)_v$ has a complex structure S, so the dimension is even and

$$\dim_R \mathrm{Ker}(P \otimes \Theta_q(TM)_v \equiv 0 (\mathrm{mod}\ 2). \qquad \square$$

It is easy to see that this argument can be used to prove that $\mathrm{Ind}_2 D \otimes E = 0$ for any T-equivariant vector bundle E.

Especially the $\hat{\mathfrak{A}}$-invariant and the Brown-Kervaire invariant vanish. For this see [Liu0].

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