Geometric Height Inequalities

Kefeng Liu

0. The Results. Let $f: X \to B$ be a fibration of a compact smooth algebraic surface over a compact Riemann surface B, denote by $g \geq 2$ the genus of a generic fiber of f and by q the genus of B. Let s be the number of singular fibers of f and $\omega_{X/B}$ be the relative dualizing sheaf. Let $C_1, \cdots, C_n$ be n mutually disjoint sections of f, and denote by D the divisor $\sum_{j=1}^{n} C_j$. Then the main result we are going to prove in this note is the following Theorem 0.1

Theorem 0.1. *If f is not isotrivial and semistable, then*

$$(\omega_{X/B} + D)^2 < (2g - 2 + n)(2q - 2 + s).$$

In fact our proof shows that, for any n,

$$(\omega_{X/B} + D)^2 = (2g - 2 + n)(2q - 2 + s).$$

if and only if f is isotrivial.

We can derive several corollaries from this theorem. To state the results, we first introduce some notations. Let k be the function field of B and $\bar{k}$ be its algebraic closure. For an algebraic point $P \in X(\bar{k})$, we let C_P be the corresponding horizontal curve on X. Let

$$h_K(P) = \frac{\omega_{X/B} \cdot C_P}{[k(P) : k]}, \quad d(P) = \frac{2g(\tilde{C}_P) - 2}{[k(P) : k]}$$

be respectively the geometric height and the geometric logarithmic discriminant of P. Here $\tilde{C}_P$ is the normalization of C_P and $[k(P) : k] = F \cdot C_P$, where

F is a generic fiber of f, is the degree of P. Let b_P be the number of ramification points on $\tilde{C}_P$ of the induced map $r: \tilde{C}_P \to B$. Write $d_P = [k(P) : k]$. Then we have

Theorem 0.2. *If f is semistable and not isotrivial, then*

$$h_K(P) < (1 + \frac{2g-2}{d_P})(d(P) + \frac{b_P}{d_P} + s) - \frac{\omega^2_{X/B}}{d_P}.$$

This theorem gives us a corollary about the geometric height inequality which is originally due to Vojta [V].

Corollary 0.3. *Given any $\varepsilon > 0$, there exists a constant $O_\varepsilon(1)$ depending on ε, s, g and q, such that*

$$h_K(P) \leq (2 + \varepsilon)d(P) + O_\varepsilon(1).$$

Vojta conjectured that the above inequality holds with $(2+\varepsilon)$ replaced by $(1+\varepsilon)$. Take $n = 1$ in Theorem 0.1, we get the following geometric height inequality, a weaker version of which was fisrt proved by Tan [Ta]. For simplicity we still assume f is semistable, the general case follows from the semistable reduction trick as used in [Ta].

Corollary 0.4. *Assume f is not isotrivial, then*

$$h_K(P) \leq (2g-1)(d(P) + s) - \omega^2_{X/B}.$$

The equality holds if and only if f is isotrivial.

We note that Theorem 0.2 is stronger than the Vojta's $(2+\varepsilon)$ inequality, but it is slightly weaker than the $(1+\varepsilon)$ conjectural height inequality. I hope that a modification of our method can be used to prove the $(1+\varepsilon)$ conjecture.

For the long history of height inequalities and the importance of the Vojta conjecture, we refer the reader to [La], Chapter VI, or [V2]. Especially such inequalities immediately imply the Mordell conjecture for functional field.

When $n = 0$, Theorem 0.1 has the following straightforward consequence [B], [Ta1],

Corollary 0.5. (Beauville Conjecture) *If $B = CP^1$ and f is semistable and not isotrivial, then $s \geq 5$.*

Recall that f is called a Kodaira fibration, if f is everywhere of maximal rank but not a complex analytic fiber bundle map. Let $c_1(X)$, $c_2(X)$ denote the first and second Chern classes of X. A very interesting consequence of Theorem 0.1 with $n = 0$ is the following Chern number inequality.

Corollary 0.6. *If f is a Kodaira fibration, then*

$$c_1(X)^2 < 3c_2(X).$$

For some special Kodaira fibrations, it is proved in [BPV], pp168, that $c_1(X)^2 < \frac{7}{3}c_2(X)$. This particularly implies that a Kodaira surface can not be uniformized by a ball. I was told by Tan and Tsai that this has been unknown for a long time.

The method to prove Theorem 0.1 was first used in [Liu] to prove the case of $n = 0$. In [Ta1], Corollary 0.5 was proved by establishing a weaker version of the $n = 0$ case of Theorem 0.1. Also in [Ta], a weaker version of Corollary 0.4 is proved. His method is algebro-geometric and is completely different from that of [Liu]. The key technique in [Liu] is the Schwarz-Yau lemma [Y] and the curvature computations of Wolpert and Jost [W], [J] for the Weil-Peterson metric on the moduli spaces of curves. Theorem 0.2 follows from Theorem 0.1 and the stablization theorem in [Kn].

This work is partially supported by an NSF grant. I would like to thank S. Lu, S. Tan, I. Tsai and S. T Yau for helpful discussions.

1. Moduli Spaces and the Schwarz-Yau Lemma. We will use the conventions in [W], [W1] for the geometry of moduli space of semistable curves. Let $\mathcal{M}_{g,n}$ be the moduli space of Riemann surfaces of genus g with n punctures. As in [W] or [TZ], we can consider $\mathcal{M}_{g,n}$ as the moduli space of complex structures on a fixed n-punctured Riemann surface R.

Let $\pi : \mathcal{T}_{g,n} \to \mathcal{M}_{g,n}$ be the universal curve and $\bar{\pi} : \bar{\mathcal{T}}_{g,n} \to \bar{\mathcal{M}}_{g,n}$ their Deligne-Mumford compactifications by adding nodal curves. The Poincare metric on each fiber of $\mathcal{T}_{g,n}$, which is a complete metric on the corresponding

n-punctured Riemann surface with constant curvature -1, patches together to give a smooth metric on the relative cotangent bundle $\Omega_{\mathcal{T}_{g,n}/\mathcal{M}_{g,n}}$. The push-down of $\Omega^{\otimes 2}_{\mathcal{T}_{g,n}/\mathcal{M}_{g,n}}$ by π which we denote by $\pi_!\Omega^{\otimes 2}_{\mathcal{T}_{g,n}/\mathcal{M}_{g,n}}$, is the cotangent bundle of $\mathcal{M}_{g,n}$. Recall that for any point $z \in \mathcal{M}_{g,n}$,

$$\pi_!\Omega^{\otimes 2}_{\mathcal{T}_{g,n}/\mathcal{M}_{g,n}}|_z = H^0(R_z, \Omega_z^{\otimes 2})$$

where $R_z = \pi^{-1}(z)$ and Ω_z denotes the cotangent bundle of R_z.

The Poincare metric on each fiber induces a natural inner product on $\pi_!\Omega^{\otimes 2}_{\mathcal{T}_{g,n}/\mathcal{M}_{g,n}}$, therefore on the tangent bundle of $\mathcal{M}_{g,n}$ which is just the well-known Weil-Peterson metric on $\mathcal{M}_{g,n}$ [W], [TZ].

Lemma 1.1. *The holomorphic sectional curvature of the Weil-Peterson metric on $\mathcal{M}_{g,n}$ is negative and* strictly *bounded from above by* $-\frac{1}{\pi(2g-2+n)}$.

Proof. This is basically due to Wolpert [W]. Let μ_α be a tangent vector on $\mathcal{M}_{g,n}$. We know that μ_α is represented by a harmonic Beltrami differential [TZ], [W]. Let dA denote the volume element of the Poincare metric on the n punctured Riemann surface. Assume μ_α is a unit vector, then we have $< \mu_\alpha, \mu_\alpha >= \int_R |\mu_\alpha|^2 dA = 1$ with respect to the induced metric. The computations in [W] and [J] actually tells us that the holomorphic sectional curvature of the Weil-Peterson metric is given by

$$-R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = -2 < \triangle|\mu_\alpha|^2, |\mu_\alpha|^2 >$$

where $\triangle = -2(D_0 - 2)^{-1}$ with D_0 the Laplacian of the Poincare metric on the n-punctured Riemann surface. Consider the orthogonal decomposition

$$|\mu_\alpha|^2 = \sum_j \psi_j + E$$

in terms of the eigenfunctions of D_0 on R. Here E is an Eisenstein series which belongs to the continuous spectrum of D_0, and ψ_j is the eigenfunctions of D_0 with eigenvalue λ_j. Note that $\lambda_j < 0$ and the continuous spectrum is contained in $(-\infty, -\frac{1}{4}]$. So E only has negative contribution to the sectional curvature. Let ψ_0 be the constant function term in the above decomposition, we then have

$$-R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} \leq 4\sum_j \frac{<\psi_j, \psi_j>}{\lambda_j - 2} < -2 < \psi_0, \psi_0 >= -2\psi_0^2 \int_R dA.$$

Since

$$\int_R \psi_0 dA = \int_R |\mu_\alpha|^2 dA = 1 \quad \text{and} \quad \int_R dA = 2\pi(2g-2+n)$$

we get $\psi_0 = \frac{1}{2\pi(2g-2+n)}$ which is the required result. As in the $n = 0$ case in [W], this upper bound can not be achieved. $\Box$

Note that the relative cotangent bundle $\Omega_{\mathcal{T}_{g,n}/\mathcal{M}_{g,n}}$ is a line bundle on $\mathcal{T}_{g,n}$. Its extension to $\bar{\mathcal{T}}_{g,n}$ is the universal dualizing sheaf. Let $c_1(\Omega)$ denote its first Chern form with respect to the Poincare metric for n-punctures Riemann surface. Let π_* denote the push-down of cohomology class by π, i.e. the integral along a generic fiber of π and ω_{WP} the Kahler class of the Weil-Peterson metric. The following lemma is also implicitly proved in [W], [W1] or [TZ].

Lemma 1.2. *On $\mathcal{M}_{g,n}$ we have*

$$\pi_*(c_1(\Omega)^2) = \frac{1}{2\pi^2}\omega_{WP}. \qquad (1)$$

Furthermore as currents $\pi_*(c_1(\Omega)^2)$ and ω_{WP} can be extended to $\bar{\mathcal{M}}_{g,n}$, and (1) *still holds as an equality of currents on $\bar{\mathcal{M}}_{g,n}$.*

Proof. Equality (1) for punctured surface is proved in [TZ], formula (5.3), following the argument of [W], Corollary 5.11.

It is easy to see that the metric on $\Omega_{\mathcal{T}_{g,n}/\mathcal{M}_{g,n}}$ is *good* in the sense of Mumford, from [W1], pp420, we know that (1) can be extended to $\bar{\mathcal{M}}_{g,n}$, and as currents $c_1(\Omega)$ is continuous on $\bar{\mathcal{T}}_{g,n}$ and represents the first Chern class of the universal dualizing sheaf of $\bar{\pi} : \bar{\mathcal{T}}_{g,n} \to \bar{\mathcal{M}}_{g,n}$. Also ω_{WP} has very mild sigularity near the compactification divisor $\bar{\mathcal{M}}_{g,n} - \mathcal{M}_{g,n}$, and as current on $\bar{\mathcal{M}}_{g,n}$ it is the limit of smooth positive Kahler forms in its cohomology class in $H^2(\bar{\mathcal{M}}_{g,n}, Q)$. $\Box$

We will denote by $\bar{\omega}_{WP}$ and respectively $\bar{\pi}_*(c_1(\Omega)^2)$ the extensions of ω_{WP} and $\pi_*(c_1(\Omega)^2)$ to $\bar{\mathcal{M}}_{g,n}$. So Lemma 1.2 tells us that, as currents on $\bar{\mathcal{M}}_{g,n}$, we have

$$\bar{\pi}_*(c_1(\Omega)^2) = \frac{1}{2\pi^2}\bar{\omega}_{WP} \quad (2).$$

Note that $\mathcal{M}_{g,n}$ is a V-manifold [W1]. The following lemma is a slight refinement of a special case of the general Schwarz-Yau lemma in [Y], and the proof is implicit in [Y], pp.201.

Lemma 1.3. *Let M be a complete Riemann surface with curvature bounded from below by a constant K_1. Let N be an Hermitian V-manifold with holomorphic sectional curvature* strictly *bounded from above by a negative constant K_2. Then for any* non-constant *holomorphic map from M to N, one has*

$$f^*\omega_N < \frac{K_1}{K_2}\omega_M.$$

Where ω_M, ω_N denote respectively the Kahler forms of M and N.

Proof. All of the computations in [Y] is done on the domain manifold M, so the formula (21) in [Y] still holds even when N is a V-manifold. Next by taking lower limit of $\varepsilon \to 0$ on both sides of the formula (21) in [Y] and then substitute in the bounds for curvatures. □

In fact we only need the following integral version of the above inequality

$$\int_M f^*\omega_N < \frac{K_1}{K_2}\int_M \omega_M.$$

Now we come back to the semistable family introduced at the beginning of the paper. By deleting the n sections, we can view $f: X \to B$ as a family of n-punctured Riemann surfaces. Let $c_1(\Omega_{X/B})$ be the first Chern form of the relative cotangent bundle with respect to the complete Poincare metric on the n-punctured Riemann surface. Of course $c_1(\Omega_{X/B})$ is first computed on the smooth part $X - f^{-1}(S)$, then since the Poincare metric is good in Mumford sense, as in Lemma 2, it can be extended to X as a current and represents the first Chern class of the relative dualizing sheaf of the n-punctured family. Let $\omega_{X/B}$ be the relative dualizing sheaf of f considered as a family of compact Riemann surfaces, and D be the divisor of sections. We will also use $\omega_{X/B}$ and D to denote their corresponding first Chern classes. Let Q be the rational number field.

Lemma 1.4. *As cohomology classes in $H^2(X, Q)$, we have*

$$c_1(\Omega_{X/B}) = \omega_{X/B} + D.$$

Proof. Let R be a Riemann surface with n punctures, and $\bar{R}$ be its compactification. Let $c_1(\Omega)_R$ be the first Chern form of the cotangent bundle of R with respect to the complete Poincare metric on R. Let $c_1(\Omega)_{\bar{R}}$ the first Chern class of the cotangent bundle of $\bar{R}$, then as currents on $\bar{R}$ we obviously have

$$c_1(\Omega)_R = c_1(\Omega)_{\bar{R}} + \delta$$

where δ is the delta-function of the punctures.

Lemma 1.4 is just a family version of this formula. □

2. The Proof of Theorem 0.1. Now we can prove Theorem 0.1. Let $f : X \to B$ be the family of semistable curves with n mutually disjoint sections $\{C_j\}$. Then f induces a holomorphic map

$$h : \ B \to \bar{\mathcal{M}}_{g,n}.$$

Let S denote the set of points over which the fibers are singular. Then s is just the number of points in S. Since f is not isotrivial, from Lemma 1.3 we know that $s \geq 3$ when $q = 0$ and $s \geq 1$ when $q = 1$ [Liu], otherwise we must have $q > 1$. Here recall that q is the genus of B. The restriction of h to $B - S$ induces a holomorphic map

$$h_0 : \ B - S \to \mathcal{M}_{g,n}.$$

Equip $B - S$ with the complete Poincare metric whose Kahler form we denote by ω_P. On $\mathcal{M}_{g,n}$ we have the Weil-Peterson metric ω_{WP}. Recall that the Poincare metric has constant curvature -1. By applying Lemmas 1.1, 1.3 to h_0, one has

$$h_0^* \omega_{WP} < \pi(2g - 2 + n)\omega_P.$$

Integrate over $B - S$, we get

$$\int_{B-S} h_0^*\omega_{WP} < \pi(2g-2+n)\cdot 2\pi(2q-2+s).$$

Now Lemma 1.2 tells us

$$\int_{B-S} h_0^*\omega_{WP} = \int_B h^*\bar{\omega}_{WP} = 2\pi^2\int_B h^*(\bar{\pi}_* c_1(\Omega)^2) = 2\pi^2\int_B f_*(c_1(\Omega_{X/B})^2)$$

where $c_1(\Omega_{X/B})$ is the first Chern class of the relative dualizing sheaf of $f:\ X\to B$ with respect to the Poincare metric for n-punctured Riemann surface.

¿From Lemma 1.4 we get

$$\int_B f_*(c_1(\Omega_{X/B})^2) = \int_X c_1(\Omega_{X/B})^2 = (\omega_{X/B}+D)^2.$$

This finishes the proof of Theorem 0.1.

3. The Proof of Theorem 0.2

Let $d=d_P=[k(P):k]$. We consider the pull-back family by $r:\ \tilde{C}_P\to B$ which gives us a semistable family $f_P:\ X_P\to\tilde{C}_P$. The pull-back of the horizontal curve C_P splits into d sections of f_P. Let $\tilde{S}$ be the union of the set of points in $\tilde{C}_P$ where r ramifies and those points where the fibers of $f:\ X\to B$ are singular. Then these d sections, which we denote by $\{C_j\}$, intersect only over the fibers above $\tilde{S}$. Let D denote the divisor $\sum_j C_j$. It is easy to see that the number of points in $\tilde{S}$ is at most $b+ds$. Here recall that b is the number of points where r ramifies and s is the number of singular fibers of f.

By using the stablization theorem in [Kn], Theorem 2.4, we get a family of semistable d-punctured curves $\tilde{f}:\ Y_P\to\tilde{C}_P$. The singular fibers of $\tilde{f}$ are those above S. There is a fiberwise contraction map $c:\ Y_P\to X_P$ such that the d mutually disjoint sections of $\tilde{f}$, $\{\tilde{C}_j\}$, are mapped to $\{C_j\}$ correspondingly. Let us write $\tilde{D}=\sum_j\tilde{C}_j$.

By applying Theorem 0.1 to $\tilde{f}$, we get

$$(\omega_{Y_P/\tilde{C}_P}+\tilde{D})^2 < (2g-2+d)(2\tilde{q}-2+b+ds)$$

where $\tilde{q}$ denotes the genus of $\tilde{C}_P$.

Lemma 1.6 in [Kn] tells us that

$$(\omega_{Y_P/\tilde{C}_P} + \tilde{D})^2 = (\omega_{X_P/C_P} + D)^2.$$

We deduce from this the following inequality

$$\omega_{X_P/C_P}^2 + \sum_{j=1}^{d} \omega_{Y/C_P} \cdot C_j < (2g-2+d)(2\tilde{q}-2+b+ds).$$

Now $\omega_{X_P/C_P}^2 = d\omega_{X/B}^2$, and $\omega_{X_P/C_P} \cdot C_j = \omega_{X/B} \cdot C_P$. Therefore we have

$$d\omega_{X/B}^2 + d\omega_{X/B} \cdot C_P < (2g-2+d)(2\tilde{q}-2+b+ds).$$

Divide both sides of the inequality by d^2, we get

$$h_K(P) < (1 + \frac{2g-2}{d})(d(P) + \frac{b}{d} + s) - \frac{\omega_{X/B}^2}{d}$$

which is exactly Theorem 0.2. $\square$

4. The Proofs of the Corollaries

To prove Corollary 0.3, we take a large integer N such that, when $d \geq N$, $\frac{4g-4}{d} < \varepsilon$. The Hurwitz formula tells us that $b < 2\tilde{q} - 2$. So we get from Theorem 0.2, for $d \geq N$

$$h_K(P) < (2+\varepsilon)d(P) + A$$

where $A = s(1+\frac{\varepsilon}{2}) - \frac{\omega_{X/B}^2}{d}$.

Next we take another large number L such that for any algebraic point P of degree less than N, one has $h_K(P) < L$. Obviously L depends on N, therefore ε, s and q. Then the inequality in Corollary 0.3 holds with $O_\varepsilon(1) = A + L$.

To get such L, we make a base change of degree $N!$, $\pi: C \to B$ and let $\tilde{f}: \tilde{X} \to C$ be the pull-back fibration. Then for any algebraic point P of degree $d < N$, $C_P \times_B C$ gives a section of $\tilde{f}$. By applying Theorem 0.1 to $\tilde{f}$ with $n = 1$, we get

$$h_K(P) < (2g-1)(\frac{\tilde{q}-2+\tilde{s}}{N!}) - \omega_{X/B}^2$$

where $\tilde{q}$ and $\tilde{s}$ are respectively the genus of C and the number of singular fibers of $\tilde{f}$. Take L to be the number on the left hand side of this inequality, we are done.

Next we recall the following well-known formulas for the Chern numbers of a semistably fibered algebraic surface $f: X \to B$:

$$c_1(X)^2 = \omega_{X/B}^2 + 8(g-1)(q-1), \; c_2(X) = 4(g-1)(q-1) + \delta$$

where δ is the number of double points on the fibers. With these two formulas we see that *Corollary 0.4* follows from Theorem 0.1 by taking $n = 1$ while *Corollary 0.6* follows by taking $n = 0$.

The proof of *Corollary 0.5* follows from a result of Beauville [B] and Theorem 0.1 with $n = 0$. In fact Beauville proved that $s \geq 4$ and $s = 4$ if and only if $\chi(O_X) = 1$. On the other hand we have the following inequality

$$\deg f_*\omega_{X/B} \leq (4 - \frac{4}{g})^{-1}\omega_{X/B}^2$$

proved by Xiao and Harris-Cornalba. Then recall that

$$\deg f_*\omega_{X/B} = \chi(O_X) - (g-1)(q-1).$$

By taking $n = 0$ in Theorem 0.1, we get

$$\chi(O_X) + g - 1 < \frac{g}{2}(-2+s)$$

which gives $s > 4$.

References

[BPV] Barth, W., Peters, P., Van de Ven, A., *Compact Complex Surfaces*, Springer-Verlag 1984.

[B] Beauville, A., Le nombre minimum de fibres singullieres D'une courbe stable sur P^1, Asterisque (1981), 86, 97-108.

[J] Jost, J., Harmonic maps and the geometry of Teichmuller spaces, Ann. Acad. Sci. Fenn. Ser. A, Math. 16, no. 1, (1991), 13-48,

[Kn] Knutson, F.: The projectivity of the moduli space of stable curves II, Math. Scand, 52 (1983), 161-199.

[La] Lang, S., *Number Theory* III, Encyclopaedia of Mathematical Sciences, vol.60, Spinger-Verlag, 1991.

[Liu] Liu, K., Remarks on the geometry of moduli spaces, Harvard preprint 1991, to appear in Proc. AMS.

[Ta] Tan, S., Height inequality of algebraic points on curves over functional fields, preprint 1994

[Ta1] Tan, S., On the invariants of base changes of pencils of curves, II, preprint 1994

[Ta2] Tan, S., The minimal number of singular fibers of a semistable curves over P^1, MPI preprint 1995

[TZ] Takhtajan, L., Zograf, P., A local index theorem for families of $\bar{\partial}$-operators on punctured Riemann surfaces and a new Kahler metric on their moduli spaces, Commun. Math. Phys. 137, 399-426 (1991).

[V] Vojta, P., Diophantine inequalities and Arakelov theory, in Lang, S., *Introduction to Arakelov Theory,* (1988), Springer-Verlag, 155-178.

[V1] Vojta, P., On algebraic points on curves, Compositio Mathematica 78 (1991), 29-36.

[V2] Vojta, P., Diophantine Approximations and value distribution theory, Lecture Notes in Math. 1329, Springer-Verlag, 1987.

[W] Wolpert, S., Chern forms and the Riemann tensor for the moduli space of curves, Invent. Math. 85 (1985), 119-145

[W1] Wolpert, S., The hyperbolic metric and the geometry of the universal curves, J. Diff. Geom. 31 (1990), 417-472.

[W2] Wolpert, S.,Homology theory of moduli space of stable curves, Ann. Math. 118 (1983), 491-523.

[Y] Yau, S.-T., A general Schwarz lemma for Kahler manifolds, Amer. J. Math. vol. 100, no. 1 (1978), 197-203.

Department of Mathematics
M. I. T.
Cambridge, MA 02139.
liu@math.mit.edu