String Duality and Localization

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String Theory should be the final theory of the world, and should be unique.

But now there are *Five* different looking string theories.

Physicists: these theories should be equivalent, in a way dual to each other: their partition functions should be "equivalent".

Question: How to compute partition functions?

Localizations, modern version of residue theorem, on infinite dimensional spaces:

Path integrals; Reduce to integrals of Chern classes.

The identifications of partition functions of different theories have produced many surprisingly beautiful mathematics like the famous mirror formula.

The mathematical proofs of such formulas depend on **Localization Techniques** on various finite dimensional moduli spaces. More precisely integrals of Chern classes on moduli spaces:

Combined with various mathematics: Chern-Simons knot invariants, combinatorics of symmetric groups, Kac-Moody algebras' representations, Calabi-Yau, geometry and topology of moduli space of stable maps....

Localization has been very successful in proving many conjectures from physics, see my ICM2002 lecture for more examples.

Functorial Localization transfers computations on complicated spaces to simple spaces:

Connects computations of mathematicians and physicists.

Papers Containing the Results:

(1). *A Proof of a conjecture of Mariño-Vafa on Hodge Integrals*, JDG 2003.

(2). *A Formula of Two Partition Hodge Integrals*, math.AG/0310273.

C.-C. Liu, K. Liu and J. Zhou.

(3). *A Mathematical Theory of Topological Vertex*, math.AG/0408426.

(4). *Topological String Partition Functions as Equivariant Indices*, preprint.

J. Li, C.-C. Liu, K. Liu and J. Zhou.

Spirit of the Results:

(1). *Duality:* Gauge theory, Chern-Simons $\Longleftrightarrow$ Calabi-Yau in String theory.

(2). *Convolution formulas* encoded in the moduli spaces and in the combinatorics of symmetric groups. $\Rightarrow$ Differential equation: *cut-and-join* equation from both representation theory and geometry.

(3). Mathematical theory of *Topological Vertex*: Vafa group's works on duality for the past several years.

(4). *Integrality* in GW invariants $\Leftrightarrow$ *Indices* of elliptic operators in Gauge theory. (Gopakumar-Vafa conjecture).

I will first talk about the Marino-Vafa conjecture, and then several other results.

The Mariño-Vafa Conjecture:

To compute mirror formula for higher genus, we need to compute Hodge integrals (i.e. intersection numbers of λ classes and ψ classes) on the Deligne-Mumford moduli space of stable curves $\overline{\mathcal{M}}_{g,h}$, the most famous orbifold. It has been studied since Riemann, and by many Fields Medalists.

A point in $\overline{\mathcal{M}}_{g,h}$ consists of $(C, x_1, \ldots, x_h)$, a (nodal) curve and n smooth points on C.

The Hodge bundle $\mathbb{E}$ is a rank g vector bundle over $\overline{\mathcal{M}}_{g,h}$ whose fiber over $[(C, x_1, \ldots, x_h)]$ is $H^0(C, \omega_C)$. The λ classes are Chern Classes:

$$\lambda_i = c_i(\mathbb{E}) \in H^{2i}(\overline{\mathcal{M}}_{g,h}; \mathbb{Q}).$$

The cotangent line $T^*_{x_i}C$ of C at the i-th marked point x_i gives a line bundle $\mathbb{L}_i$ over $\overline{\mathcal{M}}_{g,h}$. The ψ classes are also Chern classes:

$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,h}; \mathbb{Q}).$$

Define

$$\Lambda_g^\vee(u) = u^g - \lambda_1 u^{g-1} + \cdots + (-1)^g \lambda_g.$$

Mariño-Vafa formula: Generating series of triple Hodge integrals

$$\int_{\overline{\mathcal{M}}_{g,h}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(\tau)\Lambda_g^\vee(-\tau-1)}{\prod_{i=1}^h (1-\mu_i\psi_i)},$$

can be expressed by close formulas of **finite** expression in terms of representations of symmetric groups, or Chern-Simons knot invariants. Here τ is a parameter.

Conjectured from large N duality between Chern-Simons and string theory.

Remark: Moduli space has been the sources of many interests from Math to Physics.

Mumford computed some low genus numbers. Witten conjecture is about the integrals of the ψ classes.

Conifold transition: Resolve singularity in two ways:

Conifold X

$$\left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathbf{C}^4 : xw - yz = 0 \right\}$$

(1). Deformed conifold T^*S^3

$$\left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathbf{C}^4 : xw - yz = \epsilon \right\}$$

(ϵ real positive number)

(2). Resolved conifold $\tilde{X} = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{P}^1$

$$\left\{ ([Z_0, Z_1], \begin{pmatrix} x & y \\ z & w \end{pmatrix}) \in \mathbf{P}^1 \times \mathbf{C}^4 : \begin{array}{l} (x, y) \in [Z_0, Z_1] \\ (z, w) \in [Z_0, Z_1] \end{array} \right\}$$

$$\begin{array}{ccc} \tilde{X} & \subset & \mathbf{P}^1 \times \mathbf{C}^4 \\ \downarrow & & \downarrow \\ X & \subset & \mathbf{C}^4 \end{array}$$

Witten 92: The open topological string theory on the deformed conifold T^*S^3 is equivalent to Chern-Simons gauge thoery on S^3.

Gopakumar-Vafa 98, Ooguri-Vafa 00: The open topological string theory on the deformed conifold T^*S^3 is equivalent to the closed topological string theory on the resolved conifold $\tilde{X}$.

Vafa and his collaborators 98-Now: For the past years, Vafa et al developed these duality ideas into the most effective tool to compute GW invariants on toric Calabi-Yau manifolds:

High point: *Topological Vertex.*

We will give a rather complete mathematical theory. Start with Marinõ-Vafa formula.

Mathematical Consequence of the Duality:
Chern-Simons Partition function:

$\langle Z(U,V)\rangle = \text{exp}(-F(\lambda,t,V))$

U: holonomy of the $U(N)$ Chern-Simons gauge field around the knot $K \subset S^3$; V: $U(M)$ matrix

$\langle Z(U,V)\rangle$: Chern-Simons knot invariants of K.

$F(\lambda,t,V)$: Generating series of the open Gromov-Witten invariants of $(\tilde{X}, L_K)$, where L_K is a Lagrangian submanifold of the resolved conifold $\tilde{X}$ "canonically associated to" the knot K.

t'Hooft large N expansion, and canonical identifications of parameters similar to mirror formula: at level k:

$$\lambda = \frac{2\pi}{k+N}, \quad t = \frac{2\pi i N}{k+N}$$

Special case: When K is the unknot, $\langle Z(U,V)\rangle$ was computed in the zero framing by Ooguri-Vafa and in any framing $\tau \in \mathbb{Z}$ by Mariño-Vafa. Comparing with Katz-Liu's computations of $F(\lambda, t, V)$, Mariño-Vafa conjectured a striking formula about triple Hodge integrals in terms of Chern-Simons: representations and combinatorics of symmetric groups.

The framing in Mariño-Vafa's computations corresponds to choice of the circle action on the pair $(\tilde{X}, L_{\text{unknot}})$ in Katz-Liu's localization computations. Both choices are parametrized by an integer τ.

Question on General Duality: General knots in General three manifolds $\Leftrightarrow$ General Calabi-Yau?

Mariño-Vafa Conjecture:

Geometric side:

For every partition $\mu = (\mu_1 \geq \cdots \mu_{l(\mu)} \geq 0)$, define triple Hodge integral:

$$G_{g,\mu}(\tau) = A(\tau) \cdot \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(-\tau-1)\Lambda_g^\vee(\tau)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)},$$

with

$$A(\tau) = -\frac{\sqrt{-1}^{|\mu|+l(\mu)}}{|\operatorname{Aut}(\mu)|}[\tau(\tau+1)]^{l(\mu)-1} \prod_{i=1}^{l(\mu)} \frac{\prod_{a=1}^{\mu_i-1}(\mu_i\tau+a)}{(\mu_i-1)!}.$$

Introduce generating series

$$G_\mu(\lambda;\tau) = \sum_{g\geq 0} \lambda^{2g-2+l(\mu)} G_{g,\mu}(\tau).$$

Special case when $g = 0$:

$$\int_{\overline{\mathcal{M}}_{0,l(\mu)}} \frac{\Lambda_0^\vee(1)\Lambda_0^\vee(-\tau-1)\Lambda_0^\vee(\tau)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)} = \int_{\overline{\mathcal{M}}_{0,l(\mu)}} \frac{1}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)}$$

which is equal to $|\mu|^{l(\mu)-3}$ for $l(\mu) \geq 3$, and we use this expression to extend the definition to the case $l(\mu) < 3$.

Introduce formal variables $p = (p_1, p_2, \ldots, p_n, \ldots)$, and define

$$p_\mu = p_{\mu_1} \cdots p_{\mu_{l(\mu)}}$$

for any partition μ. ($\Leftrightarrow \mathrm{Tr}\, V^{\mu_j}$)

Generating series for all genera and all possible marked points:

$G(\lambda; \tau; p) = \sum_{|\mu| \geq 1} G_\mu(\lambda; \tau) p_\mu.$

Representation side:

χ_μ: the character of the irreducible representation of symmetric group $S_{|\mu|}$ indexed by μ with $|\mu| = \sum_j \mu_j$,

$C(\mu)$: the conjugacy class of $S_{|\mu|}$ indexed by μ.

Introduce:

$$\mathcal{W}_\mu(\lambda) = \prod_{1\le a<b\le l(\mu)} \frac{\sin[(\mu_a-\mu_b+b-a)\lambda/2]}{\sin[(b-a)\lambda/2]}$$

$$\cdot\frac{1}{\prod_{i=1}^{l(\nu)}\prod_{v=1}^{\mu_i} 2\sin[(v-i+l(\mu))\lambda/2]}.$$

This has an interpretation in terms of *quantum dimension* in Chern-Simons knot theory.

Define:

$$R(\lambda;\tau;p) = \sum_{n\ge 1}\frac{(-1)^{n-1}}{n}\sum_\mu[(\sum_{\cup_{i=1}^n \mu^i=\mu}$$

$$\prod_{i=1}^n \sum_{|\nu^i|=|\mu^i|}\frac{\chi_{\nu^i}(C(\mu^i))}{z_{\mu^i}} e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_{\nu^i}\lambda/2}\mathcal{W}_{\nu^i}(\lambda)]p_\mu$$

where μ^i are sub-partitions of μ, $z_\mu = \prod_j \mu_j! j^{\mu_j}$ and $\kappa_\mu = |\mu| + \sum_i(\mu_i^2 - 2i\mu_i)$ for a partition μ: standard for representations.

Mariño-Vafa Conjecture:

$G(\lambda;\tau;p) = R(\lambda;\tau;p)$.

Remark: (1). This is a formula:

G: Geometry = R: Representations

Representations of symmetric groups are essentially combinatorics.

(2). Each $G_\mu(\lambda,\tau)$ is given by a *finite and closed* expression in terms of representations of symmetric groups:

$G_\mu(\lambda,\tau) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} (\sum_{\cup_{i=1}^n \mu^i=\mu} \prod_{i=1}^n$

$\sum_{|\nu^i|=|\mu^i|} \frac{\chi_{\nu^i}(C(\mu^i))}{z_{\mu^i}} e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_{\nu^i}\lambda/2} \mathcal{W}_{\nu^i}(\lambda)$

$G_\mu(\lambda,\tau)$ gives triple Hodge integrals for moduli spaces of curves of all genera with $l(\mu)$ marked points.

(3). Equivalent expression:

$G(\lambda;\tau;p)^\bullet = \exp\left[G(\lambda;\tau;p)\right] =$

$\sum_{|\mu|\geq 0}\sum_{|\nu|=|\mu|}\frac{\chi_\nu(C(\mu))}{z_\mu}e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_\nu\lambda/2}\mathcal{W}_\nu(\lambda)$

(4). Mariño-Vafa: this formula gives values for all Hodge integrals up to three Hodge classes.

Taking Taylor expansion in τ on both sides, various Hodge integral identities have been derived by C.-C. Liu, K. Liu and Zhou.

For example, as easy consequences of the MV formula and the cut-and-join equation, we have unified simple proofs of the λ_g conjecture,

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \lambda_g = \binom{2g+n-3}{k_1, \ldots, k_n} \frac{2^{2g-1}-1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!},$$

for $k_1 + \cdots + k_n = 2g - 3 + n$, and the following identities for Hodge integrals:

$\int_{\overline{\mathcal{M}}_g} \lambda_{g-1}^3 = \int_{\overline{\mathcal{M}}_g} \lambda_{g-2}\lambda_{g-1}\lambda_g = \frac{1}{2(2g-2)!} \frac{|B_{2g-2}|}{2g-2} \frac{|B_{2g}|}{2g},$

B_{2g} are Bernoulli numbers. And

$\int_{\overline{\mathcal{M}}_{g,1}} \frac{\lambda_{g-1}}{1-\psi_1} = b_g \sum_{i=1}^{2g-1} \frac{1}{i} -$

$\frac{1}{2} \sum_{\substack{g_1+g_2=g \\ g_1,g_2>0}} \frac{(2g_1-1)!(2g_2-1)!}{(2g-1)!} b_{g_1} b_{g_2},$

where

$$b_g = \begin{cases} 1, & g = 0, \\ \frac{2^{2g-1}-1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!}, & g > 0. \end{cases}$$

Idea of Proof: (with Chiu-Chu Liu, Jian Zhou)

The first proof is based on the *Cut-and-Join* equation: a beautiful match of Combinatorics and Geometry.

Cut-and-Join: The combinatorics and geometry:

Combinatorics: Denote by $[s_1, \cdots, s_k]$ a k-cycle in the permutation group:

Cut: a k-cycle is cut into an i-cycle and a j-cycle:

$[s, t] \cdot [s, s_2, \cdots, s_i, t, t_2, \cdots t_j]$

$$= [s, s_2, \cdots, s_i][t, t_2, \cdots t_j].$$

Join: an i-cycle and a j-cycle are joined to an $(i+j)$-cycle:

$$[s, t] \cdot [s, s_2, \cdots, s_i][t, t_2, \cdots t_j]$$

$$= [s, s_2, \cdots, s_i, t, t_2, \cdots t_j].$$

Such operations can be organized into differential equations: cut-and-join equation.

Geometry: In the moduli of stable maps:

Cut: One curve split into two lower degree or lower genus curves.

Join: Two curves joined together to give a higher genus or higher degree curve.

The combinatorics and geometry of cut-and-join are reflected in the following two differential equations, like heat equation. It is equivalent to a series of linear ODE systems:

Proved either by direct computations in combinatorics or by localizations on moduli spaces of relative stable maps in geometry:

Combinatorics: Computation:

Theorem 1:

$$\begin{aligned}\frac{\partial R}{\partial \tau} &= \frac{1}{2}\sqrt{-1}\lambda \sum_{i,j=1}^{\infty}\Bigg((i+j)p_i p_j \frac{\partial R}{\partial p_{i+j}} \\ &\quad + ijp_{i+j}\left(\frac{\partial R}{\partial p_i}\frac{\partial R}{\partial p_j} + \frac{\partial^2 R}{\partial p_i \partial p_j}\right)\Bigg)\end{aligned}$$

Geometry: Localization:

Theorem 2:

$$\begin{aligned}\frac{\partial G}{\partial \tau} &= \frac{1}{2}\sqrt{-1}\lambda \sum_{i,j=1}^{\infty}\Bigg((i+j)p_i p_j \frac{\partial G}{\partial p_{i+j}} \\ &\quad + ijp_{i+j}\left(\frac{\partial G}{\partial p_i}\frac{\partial G}{\partial p_j} + \frac{\partial^2 G}{\partial p_i \partial p_j}\right)\Bigg)\end{aligned}$$

Initial Value: $\tau = 0$, Ooguri-Vafa formula:

$$G(\lambda, 0, p) = \sum_{d=1}^{\infty} \frac{p_d}{2d \sin\left(\frac{\lambda d}{2}\right)} = R(\lambda, 0, p).$$

The solution is unique! Therefore

$G(\lambda;\tau;p) = R(\lambda;\tau;p)$.

Remark: (1). Cut-and-join equation is encoded in the geometry of the moduli spaces of stable maps: convolution formula of the form: (disconnected version: $G^\bullet = \exp G$)

$$G^\bullet_\mu(\lambda,\tau) = \sum_{|\nu|=|\mu|} \Phi^\bullet_{\mu,\nu}(-\sqrt{-1}\tau\lambda) z_\nu K^\bullet_\nu(\lambda)$$

where $\Phi^\bullet_{\mu,\nu}$ is series of double Hurwitz numbers, z_ν the combinatorial constants. Equivalently this gives the explicit solution of the cut-and-join equation, with initial value $K^\bullet(\lambda)$, the integrals of Euler classes on moduli of relative stable maps.

(2). Witten conjecture is about KdV equations. But the Marinõ-Vafa formula gives *closed formula!*

Taking limits in τ and μ_i's one can obtain the Witten conjecture (Okounkov-Pandhrapande).

Same argument gives a simple and geometric proof of the ELSV formula for Hurwitz numbers.

The proof of the combinatorial cut-and-join formula is based on Burnside formula and various simple results in symmetric functions.

The proof of the geometric cut-and-join formula used *Functorial Localization Formula*:

$f: X \to Y$ equivariant map. $F \subset Y$ a fixed component, $E \subset f^{-1}(F)$ fixed components in $f^{-1}(F)$. Let $f_0 = f|_E$, then

For $\omega \in H_T^*(X)$ an equivariant cohomology class, we have identity on F:

$$f_{0*}[\frac{i_E^*\omega}{e_T(E/X)}] = \frac{i_F^*(f_*\omega)}{e_T(F/Y)}.$$

This formula, which is a generalization of Atiyah-Bott localization to relative setting, has been applied to various settings to prove the conjectures from physics.

It is used to push computations on complicated moduli space to simpler moduli space: the proof of the mirror formula; the proof of the Hori-Vafa formula; the proof of the ELSV formula....

Let $\mathcal{M}_g(\mathbf{P}^1, \mu)$ denote the moduli space of relative stable maps from a genus g curve to $\mathbf{P}^1$ with fixed ramification type μ at ∞, where μ is a partition.

Apply the functorial localization formula to the divisor morphism from the relative stable map moduli space to projective space:

Br : $\quad \mathcal{M}_g(\mathbf{P}^1, \mu) \to \mathbf{P}^r,$

where r denotes the dimension of $\mathcal{M}_g(\mathbf{P}^1, \mu)$.

This is similar to the set-up of mirror principle, with a different linearized moduli.

The fixed points of the target $\mathbf{P}^r$ *precisely labels* the cut-and-join of the triple Hodge integrals. Reduce the the study of polynomials in the equivariant cohomology group of $\mathbf{P}^r$.

Remarks: The cut-and-join equation is closely related to the Virasoro algebra.

Other approaches:

(1) Direct derivation of convolution formula. (Y.-S. Kim)

(2) Okounkov-Pandhrapande: using ELSV.

The Mariño-Vafa formula can be viewed as a duality:

Chern-Simons $\Longleftrightarrow$ Calabi-Yau.

Can we go further with the ideas and methods?

Duality and cut-and-join.

Yes much more!

One, two, three partitions.

Mariño-Vafa: one partition case....

Topological vertex: three partition case.

Two Partition:

Let μ^+, μ^- any two partitions. Introduce Hodge integrals:

$$G_{\mu^+,\mu^-}(\lambda;\tau) = B(\tau) \cdot \sum_{g\geq 0} \lambda^{2g-2}$$

$$\int_{\overline{\mathcal{M}}_{g,l(\mu^+)+l(\mu^-)}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(\tau)\Lambda_g^\vee(-\tau-1)}{\prod_{i=1}^{l(\mu^+)} \frac{1}{\mu_i^+}\left(\frac{1}{\mu_i^+}-\psi_i\right) \prod_{j=1}^{l(\mu^-)} \frac{\tau}{\mu_i^-}\left(\frac{\tau}{\mu_j^-}-\psi_{j+l(\mu^+)}\right)}$$

with

$$B(\tau) = -\frac{(\sqrt{-1}\lambda)^{l(\mu^+)+l(\mu^-)}}{z_{\mu^+}\cdot z_{\mu^-}} [\tau(\tau+1)]^{l(\mu^+)+l(\mu^-)-1}.$$

$$\prod_{i=1}^{l(\mu^+)} \frac{\prod_{a=1}^{\mu_i^+-1}\left(\mu_i^+\tau+a\right)}{\mu_i^+!} \cdot \prod_{i=1}^{l(\mu^-)} \frac{\prod_{a=1}^{\mu_i^--1}\left(\mu_i^-\frac{1}{\tau}+a\right)}{\mu_i^-!}.$$

They *naturally* arise in open string theory.

Introduce notations:

Geometry side:

$G^\bullet(\lambda; p^+, p^-; \tau) =$

$\exp\left(\sum_{(\mu^+,\mu^-)\in\mathcal{P}^2} G_{\mu^+,\mu^-}(\lambda,\tau) p^+_{\mu^+} p^-_{\mu^-}\right),$

$p^\pm_{\mu^\pm}$ are two sets of formal variables associated to the two partitions.

Representation side:

$R^\bullet(\lambda; p^+, p^-; \tau) = \sum_{|\nu^\pm|=|\mu^\pm|\geq 0} \frac{\chi_{\nu^+}(C(\mu^+))}{z_{\mu^+}} \frac{\chi_{\nu^-}(C(\mu^-))}{z_{\mu^-}}.$

$e^{\sqrt{-1}(\kappa_{\nu^+}\tau + \kappa_{\nu^-}\tau^{-1})\lambda/2} \mathcal{W}_{\nu^+,\nu^-} p^+_{\mu^+} p^-_{\mu^-}.$

Here

$\mathcal{W}_{\mu,\nu} = q^{l(\nu)/2}\mathcal{W}_{\mu} \cdot s_{\nu}(\mathcal{E}_{\mu}(t))$

$= (-1)^{|\mu|+|\nu|} q^{\frac{\kappa_\mu + \kappa_\nu + |\mu| + |\nu|}{2}}$

$\sum_{\rho} q^{-|\rho|} s_{\mu/\rho}(1, q, \dots) s_{\nu/\rho}(1, q, \dots)$

in terms of Schur functions s: Chern-Simons invariant of *Hopf link.*

Theorem: We have the equality:

$G^{\bullet}(\lambda; p^{+}, p^{-}; \tau) = R^{\bullet}(\lambda; p^{+}, p^{-}; \tau)$.

Idea of Proof: (with C.-C. Liu and J. Zhou)

Both sides satisfies the same equation (follows from convolution formula):

$\frac{\partial}{\partial\tau} H^{\bullet} = \frac{1}{2}(CJ)^{+} H^{\bullet} - \frac{1}{2\tau^2}(CJ)^{-} H^{\bullet}$, where $(CJ)^{\pm}$ cut-and-join operator: differential with respect to $p^{\pm}$.

and the same initial value at $\tau = -1$:

$$G^{\bullet}(\lambda; p^{+}, p^{-}; -1) = R^{\bullet}(\lambda; p^{+}, p^{-}; -1),$$

Ooguri-Vafa formula.

The cut-and-join equation can be written in a linear matrix form, follows from the convolution formula, naturally from localization technique on moduli.

The proof of the geometric side: Reorganize the generating series from localizations on moduli space of stable maps into $\mathbf{P}^1 \times \mathbf{P}^1$ blown up two lines at ∞, in terms of the two-Hurwitz numbers

Three Partition:

Topological Vertex introduced by Vafa et al can be deduced from a three partition analogue of such formulas. (LLLZ)

Given any three partitions μ^1, μ^2, μ^3, the cut-and-join equation in this case, for both the geometry and representation sides, is

$$\frac{\partial}{\partial \tau} F^{\bullet}_{g,\mu^1,\mu^2,\mu^3}(\tau) = (CJ)^1 F^{\bullet}_{g,\mu^1,\mu^2,\mu^3}(\tau) +$$

$$\frac{1}{\tau^2}(CJ)^2 F^{\bullet}_{g,\mu^1,\mu^2,\mu^3}(\tau) + \frac{1}{(\tau+1)^2}(CJ)^3 F^{\bullet}_{g,\mu^1,\mu^2,\mu^3}(\tau).$$

Initial value at $\tau = 1$, given by the formula of two partition case.

Chern-Simons invariant side is given by $\mathcal{W}_{\mu^1,\mu^2,\mu^3}$.

$\mathcal{W}_{\mu^1,\mu^2,\mu^3}$ is a combination of $\mathcal{W}_{\mu,\nu}$.

$F^{\bullet}_{g,\mu^1,\mu^2,\mu^3}(\tau)$ is the generating series of all genera and all marked points of the triple Hodge integral:

$$A \int_{\overline{\mathcal{M}}_{g,l_1+l_2+l_3}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(\tau)\Lambda_g^\vee(-\tau-1)}{\prod_{j=1}^{l_1}(1-\mu_j^1\psi_j)\prod_{j=1}^{l_2}\tau(\tau-\mu_j^2\psi_{l_1+j})}\cdot$$

$$\frac{(\tau(\tau+1))^{l_1+l_2+l_3-1}}{\prod_{j=1}^{l_3}(-\tau-1)(-\tau-1-\mu_j^3\psi_{l_1+l_2+j})}.$$

$$A = \frac{-(\sqrt{-1}\lambda)^{l_1+l_2+l_3}}{|\operatorname{Aut}(\mu^1)||\operatorname{Aut}(\mu^2)||\operatorname{Aut}(\mu^3)|}\prod_{j=1}^{l_1}\frac{\prod_{a=1}^{\mu_j^1-1}(\tau\mu_j^1+a)}{(\mu_j^1-1)!}\cdot$$

$$\prod_{j=1}^{l_2}\frac{\prod_{a=1}^{\mu_j^1-1}((-1-1/\tau)\mu_j^2+a)}{(\mu_j^2-1)!}\prod_{j=1}^{l_3}\frac{\prod_{a=1}^{\mu_j^1-1}(-\mu_j^3/(\tau+1)+a)}{(\mu_j^3-1)!}$$

In the above expression, $l_i = l(\mu^i)$, $i = 1, 2, 3$. Complicated coefficients but natural from localization on relative moduli.

$F^{\bullet}_{g,\mu^1,\mu^2,\mu^3}(\tau)$ has an expression in terms of the Chern-Simons invariants: the $\mathcal{W}_{\mu^1,\mu^2,\mu^3}$, a closed finite expression.

Closed Formulas for GW Invariants in terms of Chern-Simons Invariants:

Topological vertex gives the *most effective way* to compute GW invariants (i.e. Euler numbers) of toric CY: Both open and closed, by gluing the vertices.

Vafa and his group derived the topological vertex by using string duality, Chern-Simons and Calabi-Yau, a physical theory.

We established the *mathematical theory* for the topological vertex, and derived mathematical corollaries: Knot invariants $\Leftrightarrow$ GW invariants:

By using gluing formula of the topological vertex, we can derive closed formulas for generating series of GW invariants, all genera and all degrees, open or closed, for all toric Calabi-Yau, in terms Chern-Simons invariants: finite sum of products of those $\mathcal{W}$'s.

Gopakumar-Vafa Conjecture and Equivariant Indices of Elliptic Operators :

Let $N_{g,d}$ denote the GW invariant: the Euler number of the obstruction bundle on the moduli space of stable maps of degree $d \in H_2(S, \mathbb{Z})$ from genus g curve into the surface S:

$$N_{g,d} = \int_{[\overline{\mathcal{M}}_g(S,d)]^v} e(V_{g,d})$$

with $V_{g,d} = R^1\pi_*\mu^* K_S$ a vector bundle on the moduli induced by the canonical bundle K_S, where $\pi : U \to \overline{\mathcal{M}}_g(S, d)$ denotes the universal curve and μ can be considered as the evaluation or universal map. Write

$$F_g(t) = \sum_d N_{g,d}\, e^{-d\cdot t}.$$

GV conjecture: There exists expression:

$$\sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t) = \sum_{k=1}^{\infty} \sum_{g,d} n_d^g \frac{1}{d} (2\sin \frac{d\lambda}{2})^{2g-2} e^{-kd\cdot t},$$

such that n_d^g are integers, called instanton numbers.

Theorem: (LLLZ) For many interesting cases, these n_d^g's can be written as equivariant indices on the moduli spaces of Anti-Self-Dual connections on $\mathbb{C}^2$.

Remarks: The proof is to compare fixed point formula expressions for equivariant indices of elliptic operators on the ASD moduli and the combinatorial expressions of GW invariants on stable curve moduli. They agree up to non-trivial "mirror transformation".

First explicit complete examples for all genera and all degrees.

There is a more interesting and grand duality picture between CS invariants for 3-folds and GW invariants for toric CY.

Thank You Very Much!