Duality

Chern-Simons $\Longleftrightarrow$ Calabi-Yau

and

Integrals on Moduli Spaces

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A. **String Theory** should be the final theory of the world, and should be unique.

But now there are **Five** different looking string theories.

Physicists: these theories should be equivalent, in a way dual to each other: their partition functions should be "equivalent".

Question: How to compute partition functions?

Localizations, modern version of residue

theorem, on infinite dimensional spaces: Path integrals; Reduce to integrals of **Chern classes.**

The identifications of partition functions of different theories have produced many surprisingly beautiful mathematics like the famous mirror formula.

B. The mathematical proofs of such formulas depend on **Localization Techniques** on various finite dimensional moduli spaces. More precisely integrals of Chern classes on moduli spaces:

Combined with various mathematics: Chern-Simons knot invariants, combinatorics of symmetric groups, Kac-Moody algebras' representations, Calabi-Yau, geometry and topology of moduli space of stable maps.

Localization has been **successful** in proving many conjectures from physics, see my ICM2002 lecture for more examples.

This lecture is dedicated to my friends in Peking University.

Papers Containing the Results :

(1). *A Proof of a conjecture of Mariño-Vafa on Hodge Integrals,*

C.-C. Liu, K. Liu and J. Zhou, math.AG/0306434.

(2). *A Formula of Two Partition Hodge Integrals,*

C.-C. Liu, K. Liu and J. Zhou, math.AG/0310273.

(3). *A Mathematical Theory of Topological Vertex,*

J. Li, C.-C. Liu, K. Liu and J. Zhou, in preparation.

Spirit of the Results:

(1). Duality: Chern-Simons $\Longleftrightarrow$ Calabi-Yau.

(2). Differential equation: **cut-and-join** equation from both representation theory and geometry.

The Mariño-Vafa Conjecture.

To compute mirror formula for higher genus, we need to compute Hodge integrals (i.e. intersection numbers of λ classes and ψ classes) on the Deligne-Mumford moduli space of stable curves $\overline{\mathcal{M}}_{g,h}$, the most famous orbifold.

A point in $\overline{\mathcal{M}}_{g,h}$ consists of $(C, x_1, \ldots, x_h)$, a (nodal) curve and n smooth points on C.

The Hodge bundle $\mathbb{E}$ is a rank g vector bundle over $\overline{\mathcal{M}}_{g,h}$ whose fiber over $[(C, x_1, \ldots, x_h)]$ is $H^0(C, \omega_C)$. The λ classes are Chern Classes:

$$\lambda_i = c_i(\mathbb{E}) \in H^{2i}(\overline{\mathcal{M}}_{g,h}; \mathbb{Q}).$$

The cotangent line $T^*_{x_i}C$ of C at the i-th marked point x_i gives a line bundle $\mathbb{L}_i$ over $\overline{\mathcal{M}}_{g,h}$. The ψ classes are also Chern classes:

$$\psi_i = c_1(L_i) \in H^2(\overline{\mathcal{M}}_{g,h}; \mathbb{Q}).$$

Define

$$\Lambda_g^\vee(u) = u^g - \lambda_1 u^{g-1} + \cdots + (-1)^g \lambda_g.$$

Mariño-Vafa formula: Generating series of triple Hodge integrals

$$\int_{\overline{\mathcal{M}}_{g,h}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(\tau)\Lambda_g^\vee(-\tau-1)}{\prod_{i=1}^h (1-\mu_i\psi_i)},$$

can be expressed by close formulas of **finite** expression in terms of representations of symmetric groups, or Chern-Simons knot invariants.

Conjectured from large N duality between Chern-Simons and string theory:

Conifold transition: Resolve singularity in two ways:

Conifold X

$$\left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathbf{C}^4 : xw - yz = 0 \right\}$$

(1). *Deformed conifold* T^*S^3

$$\left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathbf{C}^4 : xw - yz = \epsilon \right\}$$

(ϵ real positive number)

(2). *Resolved conifold* $\tilde{X} = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{P}^1$

$$\left\{ ([Z_0, Z_1], \begin{pmatrix} x & y \\ z & w \end{pmatrix}) \in \mathbf{P}^1 \times \mathbf{C}^4 : \begin{array}{l} (x,y) \in [Z_0, Z_1] \\ (z,w) \in [Z_0, Z_1] \end{array} \right\}$$

$$
\begin{array}{ccc}
\tilde{X} & \subset & \mathbf{P}^1 \times \mathbf{C}^4 \\
\downarrow & & \downarrow \\
X & \subset & \mathbf{C}^4
\end{array}
$$

Witten 92: The open topological string theory on the N D-branes on S^3 of T^*S^3 is equivalent to $U(N)$ Chern-Simons gauge thoery on S^3.

Gopakumar-Vafa 98, Ooguri-Vafa 00: The open topological string theory on the N D-branes on S^3 of the deformed conifold is equivalent to the closed topological string theory on the resolved conifold $\tilde{X}$.

Mathematical Consequence: Chern-Simons Partition function:

$$\langle Z(U,V)\rangle = \exp(-F(\lambda,t,V))$$

U: holonomy of the $U(N)$ Chern-Simons gauge field around the knot $K \subset S^3$; V: $U(M)$ matrix

$\langle Z(U,V)\rangle$: Chern-Simons knot invariants of K.

$F(\lambda,t,V)$: Generating series of the open Gromov-Witten invariants of $(\tilde{X}, L_K)$, where L_K is a Lagrangian submanifold of the resolved conifold $\tilde{X}$ "canonically associated to" the knot K.

t'Hooft large N expansion, and canonical identifications of parameters similar to mirror formula: at level k:

$$\lambda = \frac{2\pi}{k+N}, \quad t = \frac{2\pi i N}{k+N}$$

Special case: When K is the unknot, $\langle Z(U, V)\rangle$ was computed in the zero framing by Ooguri-Vafa and in any framing $\tau \in \mathbb{Z}$ by Mariño-Vafa. Comparing with Katz-Liu's computations of $F(\lambda, t, V)$, Mariño-Vafa conjectured a striking formula about triple Hodge integrals in terms of Chern-Simons: representations and combinatorics of symmetric groups.

The framing in Mariño-Vafa's computations corresponds to choice of the circle action on the pair $(\tilde{X}, L_{\text{unknot}})$ in Katz-Liu's localization computations. Both choices are parametrized by an integer τ.

Mariño-Vafa formula:

Geometric side:

For every partition $\mu = (\mu_1 \geq \cdots \mu_{l(\mu)} \geq 0)$, define triple Hodge integral:

$$G_{g,\mu}(\tau) = A(\tau) \cdot \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(-\tau-1)\Lambda_g^\vee(\tau)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)},$$

with

$$A(\tau) = -\frac{\sqrt{-1}^{|\mu|+l(\mu)}}{|\operatorname{Aut}(\mu)|}[\tau(\tau+1)]^{l(\mu)-1} \prod_{i=1}^{l(\mu)} \frac{\prod_{a=1}^{\mu_i-1}(\mu_i\tau+a)}{(\mu_i-1)!}.$$

Introduce generating series

$$G_\mu(\lambda;\tau) = \sum_{g\geq 0} \lambda^{2g-2+l(\mu)} G_{g,\mu}(\tau).$$

Special case when $g = 0$:

$$\int_{\overline{\mathcal{M}}_{0,l(\mu)}} \frac{\Lambda_0^\vee(1)\Lambda_0^\vee(-\tau-1)\Lambda_0^\vee(\tau)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)} = \int_{\overline{\mathcal{M}}_{0,l(\mu)}} \frac{1}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)}$$

which is equal to $|\mu|^{l(\mu)-3}$ for $l(\mu) \geq 3$, and we use this expression to extend the definition to the case $l(\mu) < 3$.

Introduce formal variables $p = (p_1, p_2, \ldots, p_n, \ldots)$, and define

$$p_\mu = p_{\mu_1} \cdots p_{\mu_{l(\mu)}}$$

for any partition μ.

Generating series for all genera and all possible marked point:

$G(\lambda; \tau; p) = \sum_{|\mu| \geq 1} G_\mu(\lambda; \tau) p_\mu.$

Representation side:

χ_μ: the character of the irreducible representation of symmetric group $S_{|\mu|}$ indexed by μ with $|\mu| = \sum_j \mu_j$,

$C(\mu)$: the conjugacy class of $S_{|\mu|}$ indexed by μ.

Introduce:

$$\mathcal{W}_\mu(\lambda) = \prod_{1\le a<b\le l(\mu)} \frac{\sin[(\mu_a-\mu_b+b-a)\lambda/2]}{\sin[(b-a)\lambda/2]}$$

$$\cdot \frac{1}{\prod_{i=1}^{l(\nu)} \prod_{v=1}^{\mu_i} 2\sin[(v-i+l(\mu))\lambda/2]}.$$

This has an interpretation in terms of **quantum dimension** in Chern-Simons knot theory.

Define

$$R(\lambda;\tau;p) = \sum_{n\ge 1} \frac{(-1)^{n-1}}{n} \sum_\mu [(\sum_{\cup_{i=1}^n \mu^i=\mu}$$

$$\prod_{i=1}^n \sum_{|\nu^i|=|\mu^i|} \frac{\chi_{\nu^i}(C(\mu^i))}{z_{\mu^i}} e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_{\nu^i}\lambda/2} \mathcal{W}_{\nu^i}(\lambda)] p_\mu$$

where μ^i are sub-partitions of μ, $z_\mu = \prod_j \mu_j! j^{\mu_j}$ and $\kappa_\mu = |\mu| + \sum_i(\mu_i^2 - 2i\mu_i)$ for a partition μ: standard for representations.

Mariño-Vafa Conjecture:

$G(\lambda; \tau; p) = R(\lambda; \tau; p)$.

Remark: (1). This is a formula:

G: Geometry = R: Representations

Representations of symmetric groups are essentially combinatorics.

(2). Each $G_\mu(\lambda, \tau)$ is given by a **finite and closed** expression in terms of representations of symmetric groups:

$$G_\mu(\lambda, \tau) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \Big(\sum_{\cup_{i=1}^n \mu^i = \mu} \prod_{i=1}^n$$

$$\sum_{|\nu^i| = |\mu^i|} \frac{\chi_{\nu^i}(C(\mu^i))}{z_{\mu^i}} e^{\sqrt{-1}(\tau + \frac{1}{2})\kappa_{\nu^i}\lambda/2} \mathcal{W}_{\nu^i}(\lambda)$$

$G_\mu(\lambda,\tau)$ gives triple Hodge integrals for moduli spaces of curves of all genera with $l(\mu)$ marked points.

(3). Equivalent expression:

$G(\lambda;\tau;p)^\bullet = e^{G(\lambda;\tau;p)}$

$\sum_{|\mu|\geq 0}\sum_{|\nu|=|\mu|}\frac{\chi_\nu(C(\mu))}{z_\mu}e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_\nu\lambda/2}\mathcal{W}_\nu(\lambda)$

(4). Mariño-Vafa: this formula gives values for all Hodge integrals up to three Hodge classes.

Taking Taylor expansion in τ on both sides, various Hodge integral identities have been derived by C.-C. Liu, K. Liu and Zhou.

For example, as easy consequences of the MV formula and the cut-and-join equation, we have unified simple proofs of the λ_g conjecture,

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \lambda_g = \binom{2g+n-3}{k_1, \ldots, k_n} \frac{2^{2g-1}-1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!},$$

for $k_1 + \cdots + k_n = 2g-3+n$, and the following identities for Hodge integrals:

$$\int_{\overline{\mathcal{M}}_g} \lambda_{g-1}^3 = \int_{\overline{\mathcal{M}}_g} \lambda_{g-2}\lambda_{g-1}\lambda_g = \frac{1}{2(2g-2)!}\frac{|B_{2g-2}|}{2g-2}\frac{|B_{2g}|}{2g},$$

B_{2g} are Bernoulli numbers.

$$\int_{\overline{\mathcal{M}}_{g,1}} \frac{\lambda_{g-1}}{1-\psi_1} = b_g \sum_{i=1}^{2g-1} \frac{1}{i} -$$

$$\frac{1}{2} \sum_{\substack{g_1+g_2=g \\ g_1,g_2>0}} \frac{(2g_1-1)!(2g_2-1)!}{(2g-1)!} b_{g_1} b_{g_2},$$

where

$$b_g = \begin{cases} 1, & g = 0, \\ \frac{2^{2g-1}-1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!}, & g > 0. \end{cases}$$

Idea of Proof: (with Chiu-Chu Liu, Jian Zhou)

The proof is based on the **Cut-and-Join** equation: a beautiful match of Combinatorics and Geometry.

Cut-and-Join: The combinatorics and geometry:

Combinatorics: Denote by $[s_1, \cdots, s_k]$ a k-cycle in the permutation group:

Cut: a k-cycle is cut into an i-cycle and a j-cycle:

$$[s,t]\cdot[s,s_2,\cdots,s_i,t,t_2,\cdots t_j]$$

$$= [s,s_2,\cdots,s_i][t,t_2,\cdots t_j].$$

Join: an i-cycle and a j-cycle are joined to an $(i+j)$-cycle:

$$[s,t]\cdot[s,s_2,\cdots,s_i][t,t_2,\cdots t_j]$$

$$= [s,s_2,\cdots,s_i,t,t_2,\cdots t_j].$$

Geometry:

Cut: One curve split into two lower degree or lower genus curves.

Join: Two curves joined together to give a higher genus or higher degree curve.

The combinatorics and geometry of cut-and-join are reflected in the following two differential equations, like heat equation:

proved either by direct computations in combinatorics or by localizations on moduli spaces of relative stable maps:

Combinatorics: Computation:

Theorem 1:

$$\begin{aligned}\frac{\partial R}{\partial \tau} &= \frac{1}{2}\sqrt{-1}\lambda \sum_{i,j=1}^{\infty} \Bigg((i+j)p_i p_j \frac{\partial R}{\partial p_{i+j}} \\ &\quad + ij p_{i+j} \left(\frac{\partial R}{\partial p_i}\frac{\partial R}{\partial p_j} + \frac{\partial^2 R}{\partial p_i \partial p_j} \right) \Bigg)\end{aligned}$$

Geometry: Localization:

Theorem 2:

$$\begin{aligned}\frac{\partial G}{\partial \tau} &= \frac{1}{2}\sqrt{-1}\lambda \sum_{i,j=1}^{\infty} \Bigg((i+j)p_i p_j \frac{\partial G}{\partial p_{i+j}} \\ &\quad + ij p_{i+j} \left(\frac{\partial G}{\partial p_i}\frac{\partial G}{\partial p_j} + \frac{\partial^2 G}{\partial p_i \partial p_j} \right) \Bigg)\end{aligned}$$

Initial Value: $\tau = 0$, Ooguri-Vafa formula:

$$G(\lambda, 0, p) = \sum_{d=1}^{\infty} \frac{p_d}{2d \sin\left(\frac{\lambda d}{2}\right)} = R(\lambda, 0, p).$$

The solution is unique! Series of homogeneous ODE:

$G(\lambda; \tau; p) = R(\lambda; \tau; p)$.

Remark: (1). Cut-and-join equation is more fundamental: encodes both geometry and combinatorics: Vafa: Virasoro operators come out of the cut-and-join.

(2). Witten conjecture is about KdV equations. But the Marinõ-Vafa formula gives **closed formula!**

Same argument gives a simple and geometric proof of the ELSV formula for Hurwitz numbers.

The proof of the combinatorial cut-and-join formula, by Jian Zhou, is based on Burnside formula and various results in symmetric functions.

Taking derivative with respect to τ!

The proof of the geometric cut-and-join formula used **Functorial Localization Formula**:

$f: X \to Y$ equivariant map. $F \subset Y$ a fixed component, $E \subset f^{-1}(F)$ fixed components in $f^{-1}(F)$. Let $f_0 = f|_E$, then

For $\omega \in H_T^*(X)$ an equivariant cohomology class, we have identity on F:

$$f_{0*}[\frac{i_E^*\omega}{e_T(E/X)}] = \frac{i_F^*(f_*\omega)}{e_T(F/Y)}.$$

This formula, which is a generalization of Atiyah-Bott localization to relative setting, has been applied to various settings to prove the conjectures from physics.

It is used to push computations on complicated moduli space to simpler moduli space.

Let $\mathcal{M}_g(\mathbf{P}^1, \mu)$ denote the moduli space of relative stable maps from a genus g curve to $\mathbf{P}^1$ with fixed ramification type μ at ∞, where μ is a partition.

Apply the functorial localization formula to the divisor morphism from the relative stable map moduli space to projective space:

$$\mathrm{Br}: \quad \mathcal{M}_g(\mathbf{P}^1, \mu) \to \mathbf{P}^r,$$

where r denotes the dimension of $\mathcal{M}_g(\mathbf{P}^1, \mu)$.

This is similar to the set-up of mirror principle, with a different linearized moduli.

The fixed points of the target $\mathbf{P}^r$ **precisely labels** the cut-and-join of the triple Hodge integrals.

Applications: Computing GW invariants on Toric Calabi-Yau:

Physical approaches: Aganagic-Mariño-Vafa (2002): BPS numbers for toric Calabi-Yau by using large N dulaity and Chern-Simons invariants.

Mathematical approach (Jian Zhou): Mariño-Vafa formula can be used to compute BPS numbers (which are conjectured to be **integers** by Gopakumar-Vafa) for toric Calabi-Yau 3-fold.

The physical and mathematical approaches should be equivalent:

Bridge: The Mariño-Vafa formula: which can be viewed as a duality:

Chern-Simons $\Longleftrightarrow$ **Calabi-Yau.**

Can we go further with the ideas and methods?

Duality and cut-and-join.

Yes much more!

One, two, three partitions.

Mariño-Vafa: one partition case....

Two Partition.

Let μ^+, μ^- any two partitions. Introduce Hodge integrals:

$$G_{\mu^+,\mu^-}(\lambda;\tau) = B(\tau) \cdot \sum_{g \geq 0} \lambda^{2g-2}$$

$$\int_{\overline{\mathcal{M}}_{g,l(\mu^+)+l(\mu^-)}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(\tau)\Lambda_g^\vee(-\tau-1)}{\prod_{i=1}^{l(\mu^+)} \frac{1}{\mu_i^+}\left(\frac{1}{\mu_i^+}-\psi_i\right) \prod_{j=1}^{l(\mu^-)} \frac{\tau}{\mu_i^-}\left(\frac{\tau}{\mu_j^-}-\psi_{j+l(\mu^+)}\right)}$$

with

$$B(\tau) = -\frac{(\sqrt{-1}\lambda)^{l(\mu^+)+l(\mu^-)}}{z_{\mu^+} \cdot z_{\mu^-}} [\tau(\tau+1)]^{l(\mu^+)+l(\mu^-)-1}.$$

$$\prod_{i=1}^{l(\mu^+)} \frac{\prod_{a=1}^{\mu_i^+-1}\left(\mu_i^+\tau + a\right)}{\mu_i^+!} \cdot \prod_{i=1}^{l(\mu^-)} \frac{\prod_{a=1}^{\mu_i^- -1}\left(\mu_i^- \frac{1}{\tau} + a\right)}{\mu_i^-!}.$$

They **naturally** arise in open string theory.

Introduce notations:

Geometry side:

$G^{\bullet}(\lambda; p^{+}, p^{-}; \tau) =$

$\exp\left(\sum_{(\mu^+,\mu^-)\in\mathcal{P}^2} G_{\mu^+,\mu^-}(\lambda,\tau)p^+_{\mu^+}p^-_{\mu^-}\right),$

$p^{\pm}_{\mu^{\pm}}$ are two sets of formal variables associated to the two partitions.

Representation side:

$R^{\bullet}(\lambda; p^{+}, p^{-}; \tau) = \sum_{|\nu^{\pm}|=|\mu^{\pm}|\geq 0} \frac{\chi_{\nu^+}(C(\mu^+))}{z_{\mu^+}} \frac{\chi_{\nu^-}(C(\mu^-))}{z_{\mu^-}} \cdot$

$e^{\sqrt{-1}(\kappa_{\nu^+}\tau+\kappa_{\nu^-}\tau^{-1})\lambda/2}\mathcal{W}_{\nu^+,\nu^-}p^+_{\mu^+}p^-_{\mu^-}.$

Here

$\mathcal{W}_{\mu,\nu} = q^{l(\nu)/2}\mathcal{W}_{\mu} \cdot s_{\nu}(\mathcal{E}_{\mu}(t))$

$$= (-1)^{|\mu|+|\nu|} q^{\frac{\kappa_\mu+\kappa_\nu+|\mu|+|\nu|}{2}}$$

$$\sum_\rho q^{-|\rho|} s_{\mu/\rho}(1, q, \dots) s_{\nu/\rho}(1, q, \dots)$$

in terms of Schur functions: Chern-Simons invariant of **Hopf link.**

Theorem: (Zhou conjecture),

$$G^\bullet(\lambda; p^+, p^-; \tau) = R^\bullet(\lambda; p^+, p^-; \tau).$$

Idea of Proof: (With C.-C. Liu and J. Zhou)

Both sides satisfies same cut-and-join equation:

$$\frac{\partial}{\partial \tau} H^\bullet = \frac{1}{2}(CJ)^+ H^\bullet - \frac{1}{2\tau^2}(CJ)^- H^\bullet,$$

where

$(CJ)^{\pm}$ cut-and-join operator: differential with respect to $p^{\pm}$.

and the same initial value at $\tau = -1$:

$$G^{\bullet}(\lambda; p^{+}, p^{-}; -1) = R^{\bullet}(\lambda; p^{+}, p^{-}; -1),$$

Ooguri-Vafa formula.

The cut-and-join equation can be written in a linear matrix form, naturally through localization technique.

The proof of the geometric cut-and-join equation: Reorganize the generating series from localizations on moduli space of stable maps into $\mathbf{P}^1 \times \mathbf{P}^1$ blown up two lines at ∞, in terms of the two-Hurwitz numbers

Generating series of two Hurwitz numbers has a nice cut-and-join equation, which is "transmitted" to the cut-and-join of G. This proof is more geometric and natural.

Iqbal and Vafa et al Conjecture:

Aganagic-Klemm-Mariño-Vafa (2003): Topological vertex. Complete formula for computations of GW invariants and BPS numbers for all degree and all genus in terms of Chern-Simons. (BPS numbers are related to GW invariants by Gopakumar-Vafa formula.)

GW invariants here are integrals of Chern classes on stable map moduli spaces.

Iqbal's instanton counting in terms of Chern-Simons invariants.

Jian Zhou: Introduced "chemistry": a beautiful trick: Re-organize contributions of fixed points as combinations of the two partition Hodge integral formulas and using the above two partition formula transform the expressions into Chern-Simons invariants.

Proved the formula of Iqbal and Vafa et al: amazing formulas for all toric Calabi-Yau three folds.

Three Partition.

Topological Vertex introduced by Vafa et al can be deduced from a three partition analogue of such formulas.

Given any three partitions μ^1, μ^2, μ^3, the cut-and-join equation in this case, for both the geometry and representation sides, is

$$\frac{\partial}{\partial \tau} F^{\bullet}_{g,\mu^1,\mu^2,\mu^3}(\tau) = (CJ)^1 F^{\bullet}_{g,\mu^1,\mu^2,\mu^3}(\tau) +$$

$$\frac{1}{\tau^2}(CJ)^2 F^{\bullet}_{g,\mu^1,\mu^2,\mu^3}(\tau) + \frac{1}{(\tau+1)^2}(CJ)^3 F^{\bullet}_{g,\mu^1,\mu^2,\mu^3}(\tau).$$

Initial value at $\tau = 1$, given by the formula of two partition case.

Chern-Simons invariant side is given by $\mathcal{W}_{\mu^1,\mu^2,\mu^3}$.

$\mathcal{W}_{\mu^1,\mu^2,\mu^3}$ is a combination of $\mathcal{W}_{\mu,\nu}$.

Joint with J. Li, C.-C. Liu and J. Zhou: Mathematical definition of topological vertex.