Good Geometry of the Moduli Spaces of Riemann Surfaces

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I want to dedicate this lecture to my teacher

Qi-Keng Lu

For his teaching in our 3 persons seminar at his home in 1987 on classical domains.

I chose this topic to report, since it is a topic that is very close to Prof. Lu's research:

Teichmüller space is very similar to classical domains.

His training has played important role in my research.

Previous results related to this lecture are in

1. *Canonical Metrics on the Moduli Spaces of Riemann Surfaces I*. JDG 68.

2. *Canonical Metrics on the Moduli Spaces of Riemann Surfaces II*. JDG 69.

The main topic of today's lecture is in

3. *Good Metrics on the Moduli Spaces of Riemann Surfaces I, II*. In Preprints.

by K. Liu, X. Sun, S.-T. Yau.

Moduli spaces and Teichmüller spaces of Riemann surfaces have been studied for many many years, since Riemann.

They have appeared in many subjects of mathematics, from geometry, topology, algebraic geometry to number theory. They have also appeared in theoretical physics like string theory.

Many computations of path integrals are reduced to integrals of Chern classes and metric forms on such moduli spaces.

The Teichmüller space $\mathcal{T}_g$, $g \geq 2$, is a domain of holomorphy embedded in $\mathbb{C}^n$ with $n = 3g - 3$. The moduli space $\mathcal{M}_g$ is a quasi-projective orbifold, as a quotient of $\mathcal{T}_g$ by mapping class group.

All results hold for $\mathcal{M}_{g,n}$ of moduli with n marked points.

The moduli spaces and their compactification have highly nontrivial topology, and have been actively studied for the past years from many point of views in mathematics and physics.

Marino-Vafa conjecture we proved gives a closed formula for the generating series of triple Hodge integrals of all genera and all possible marked points, in terms of Chern-Simons knot invariants.

Witten conjecture, proved by Kontsevich, ELSV formulas relating Hurwitz numbers to Hodge integrals and many other conjectures related to Hodge integrals can be deduced from Marino-Vafa formula by taking various limits.

Gromov-Witten theory can be viewed as a natural extension of the moduli space theory.

The geometry of the Teichmüller spaces and moduli spaces of Riemann surfaces also have very rich structures.

There are many very famous classical metrics on the Teichmüller and the moduli spaces:

(1). **Finsler metrics:**

Teichmüller metric;
Caratheodory metric; Kobayashi metric.

(2). **Kähler metrics:**

The Weil-Petersson metric, (Incomplete).
Cheng-Yau's Kähler-Einstein metric; McMullen metric; Bergman metric; Asymptotic Poincare metric.

(3). **New**: Ricci metric and perturbed Ricci metric (LSY).

The above seven metrics are complete Kähler metrics.

Our project is to study the geometry of the Teichmüller and the moduli spaces. More precisely to understand the various metrics on these spaces, and more important, to introduce new metrics with good property and find their applications.

The key point is the understanding of the Ricci and **the perturbed Ricci metric**: two new complete Kähler metrics. Their curvatures, boundary behaviors, are studied in great details, and are very well understood.

As easy corollary we have proved all of the above complete metrics are equivalent. Also proved that all of the complete Kähler metrics have strongly bounded geometry in Teichmüller spaces.

From these we have good understanding of the Kähler-Einstein metric on both the moduli and the Teichmüller spaces, and find interesting applications to the geometry.

Proof of the stability of the logarithmic cotangent bundle of the moduli spaces and more will follow.

The new metric we introduced, the perturbed Ricci metric has bounded negative holomorphic sectional and Ricci curvature. It has bounded geometry and Poincáre growth.

So this new metric has almost all of the possible good properties: close to be the best, if holomorphic nonpositive bisetional curvature.

Basics of the Teichmüller and the Moduli Spaces:

Fix an orientable surface Σ of genus $g \geq 2$.

• *Uniformization Theorem.*
Each Riemann surface of genus $g \geq 2$ can be viewed as a quotient of the hyperbolic plane $\mathbb{H}^2$ by a Fuchsian group. Thus there is a unique KE metric, or the hyperbolic metric on Σ.

The group $Diff^+(\Sigma)$ of orientation preserving diffeomorphisms acts on the space $\mathcal{C}$ of all complex structures on Σ by pull-back.

- *Teichmüller space.*

$$\mathcal{T}_g = \mathcal{C}/Diff_0^+(\Sigma)$$

where $Diff_0^+(\Sigma)$ is the set of orientation preserving diffeomorphisms which are isotopic to identity.

- *Moduli space.*

$$\mathcal{M}_g = \mathcal{C}/Diff^+(\Sigma) = \mathcal{T}_g/Mod(\Sigma)$$

is the quotient of the Teichmüller space by the mapping class group where

$$Mod\,(\Sigma) = Diff^+(\Sigma)/Diff_0^+(\Sigma).$$

- *Dimension.*

$$\dim_{\mathbb{C}} \mathcal{T}_g = \dim_{\mathbb{C}} \mathcal{M}_g = 3g - 3.$$

$\mathcal{T}_g$ is a pseudoconvex domain in $\mathbb{C}^{3g-3}$. $\mathcal{M}_g$ is a complex orbifold, it can be compactified to a projective orbifold by adding normal crossing divisors consisting of stable nodal curves, called the Deligne-Mumford compactification, or DM moduli.

• *Tangent and cotangent space.*

By the deformation theory of Kodaira-Spencer and the Hodge theory, for any point $X \in \mathcal{M}_g$,

$$T_X\mathcal{M}_g \cong H^1(X, T_X) = HB(X)$$

where $HB(X)$ is the space of harmonic Beltrami differentials on X.

$$T_X^*\mathcal{M}_g \cong Q(X)$$

where $Q(X)$ is the space of holomorphic quadratic differentials on X.

For $\mu \in HB(X)$ and $\phi \in Q(X)$, the duality between $T_X\mathcal{M}_g$ and $T_X^*\mathcal{M}_g$ is

$$[\mu : \phi] = \int_X \mu\phi.$$

Teichmüller metric is the L^1 norm. The WP metric is the L^2 norm.

Curvature formulas:

- *Weil-Petersson metric*

Let $\mathfrak{X}$ be the total space over the $\mathcal{M}_g$ and π be the projection map.

Pick $s \in \mathcal{M}_g$, let $\pi^{-1}(s) = X_s$. Let $s_1, \cdots, s_n$ be local holomorphic coordinates on $\mathcal{M}_g$ and let z be local holomorphic coordinate on X_s.

Recall

$$T_s\mathcal{M}_g \cong HB(X_s).$$

The Kodaira-Spencer map is

$$\frac{\partial}{\partial s_i} \mapsto A_i \frac{\partial}{\partial z} \otimes d\bar{z} \in HB(X_s).$$

The Weil-Petersson metric is

$$h_{i\bar{j}} = \int_{X_s} A_i \bar{A}_j \; dv$$

where $dv = \frac{\sqrt{-1}}{2} \lambda dz \wedge d\bar{z}$ is the volume form of the KE metric λ on X_s.

By the work of Royden, Siu and Schumacher, let

$$a_i = -\lambda^{-1}\partial_{s_i}\partial_{\bar{z}}\log\lambda.$$

Then

$$A_i = \partial_{\bar{z}} a_i.$$

Let η be a relative $(1,1)$ form on $\mathfrak{X}$. Then

$$\frac{\partial}{\partial s_i}\int_{X_s}\eta = \int_{X_s} L_{v_i}\eta$$

where

$$v_i = \frac{\partial}{\partial s_i} + a_i\frac{\partial}{\partial z}$$

is called the harmonic lift of $\frac{\partial}{\partial s_i}$.

In the following, we let

$$f_{i\bar{j}} = A_i\bar{A}_j \text{ and } e_{i\bar{j}} = T(f_{i\bar{j}}).$$

Here $T = (\Box+1)^{-1}$ with Laplacian $\Box = -\lambda^{-1}\partial_z\partial_{\bar{z}}$, is the Green operator. The functions $f_{i\bar{j}}$ and $e_{i\bar{j}}$ will be the building blocks of the curvature formula.

• *Curvature formula of the WP metric.*
By the work of Wolpert, Siu and Schumacher, the curvature of the Weil-Petersson metric is

$$R_{i\bar{j}k\bar{l}} = -\int_{X_s} (e_{i\bar{j}} f_{k\bar{l}} + e_{i\bar{l}} f_{k\bar{j}})\ dv.$$

Remark: (1). The sign of the curvature of the WP metric can be seen directly.

(2). The precise upper bound $-\frac{1}{2\pi(g-1)}$ of the holomorphic sectional curvature and the Ricci curvature of the WP metric can be obtained by spectrum decomposition of the operator $(\Box + 1)$.

(3). The curvature of the WP metric is not bounded from below. But *surprisingly* the Ricci and the perturbed Ricci metrics have bounded (negative) curvatures.

The curvature of the Ricci metric has more than 80 terms, since it is fourth order derivatives. Perturbed Ricci even has more.

Observation:
The Ricci curvature of the Weil-Petersson metric is bounded above by a negative constant, one can use the negative Ricci curvature of the WP metric to define a new metric.

We call this metric the **Ricci metric**

$$\tau_{i\bar{j}} = -Ric(\omega_{WP})_{i\bar{j}}.$$

We proved the Ricci metric is complete, Poincare growth, and has bounded geometry.

We perturbed the Ricci metric with a large constant multiple of the WP metric. We define the **perturbed Ricci metric**

$$\omega_{\tilde{\tau}} = \omega_\tau + C\,\omega_{WP}.$$

We proved that the perturbed Ricci metric is complete, Poincare growth and has bounded negative holomorphic sectional, negative Ricci curvature, and bounded geometry.

Selected applications of these metrics:

Example: Royden proved that

Teichmüller metric = Kobayashi metric.

This implies that the isometry group of $\mathcal{T}_g$ is exactly the mapping class group.

Example: Ahlfors: the Weil-Petersson (WP) metric is Kähler, the holomorphic sectional curvature is negative.

Masur: WP metric is incomplete.

Wolpert studied WP metric in great details, found many important applications in topology(relation to Thurston's work) and algebraic geometry(relation to Mumford's work).

Each family of semi-stable curves induces a holomorphic maps into the moduli space.

Yau's Schwarz lemma: very sharp geometric height inequalities in algebraic geometry.

Corollaries include:

1. Kodaira surface X has strict Chern number inequality: $c_1(X)^2 < 3c_2(X)$.

2. Beauville conjecture: the number of singular fibers for a non-isotrivial family of semi-stable curves over $\mathbb{P}^1$ is at least 5.

Geometric Height Inequalities, by K. Liu, MRL 1996.

Example: McMullen proved that the moduli spaces of Riemann surfaces are Kähler hyperbolic, by using his own metric which he obtained by perturbing the WP metric.

This means bounded geometry and the Kähler form on the Teichmüller space is of the form $d\alpha$ with α bounded one form.

The lowest eigenvalue of the Laplacian on the Teichmüller space is positive.

Only middle dimensional L^2 cohomology is nonzero on the Teichmüller space.

Example: We have proved that the complete Kähler metrics all have strongly bounded geometry.

In particular, the curvature of KE metric and all of its covariant derivatives are bounded on the Teichmüller space: strongly bounded geometry.

Algebro-geometric consequences: the log cotangent bundle of the Deligne-Mumford moduli space of stable curves is stable.

The (log) cotangent bundle is ample....

Today I will discuss the goodness of the Weil-Petersson metric, the Ricci and the perturbed Ricci metric in the sense of Mumford, and their applications in understanding the geometry of moduli spaces.

The question that WP metric is good or not has been open for many years, according to Wolpert.

Corollaries include:

Chern classes can be defined on the moduli spaces by using the WP metric, the Ricci metric or the perturbed Ricci metric; the L^2-index theory and fixed point formulas can be applied on the Teichmüller spaces.

The log cotangent bundle is Nakano positive; vanishing theorems of L^2 cohomology; rigidity of the moduli spaces.

Goodness of Hermitian Metrics

For an Hermitian holomorphic vector bundle (F, g) over a closed complex manifold M, the Chern forms of g represent the Chern classes of F. However, this is no longer true if M is not closed since g may be singular.

X: quasi-projective variety of $\dim_{\mathbf{C}} X = k$ by removing a divisor D of normal crossings from a closed smooth projective variety $\bar{X}$.

$\bar{E}$: a holomorphic vector bundle of rank n over $\bar{X}$ and $E = \bar{E}\,|_X$.

h: Hermitian metric on E which may be singular near D.

Mumford introduced conditions on the growth of h, its first and second derivatives near D such that the Chern forms of h, as currents, represent the Chern classes of $\bar{E}$.

We cover a neighborhood of $D \subset \bar{X}$ by finitely many polydiscs

$$\left\{U_\alpha = \left(\Delta^k, (z_1, \cdots, z_k)\right)\right\}_{\alpha \in A}$$

such that $V_\alpha = U_\alpha \backslash D = (\Delta^*)^m \times \Delta^{k-m}$. Namely, $U_\alpha \cap D = \{z_1 \cdots z_m = 0\}$. We let $U = \bigcup_{\alpha \in A} U_\alpha$ and $V = \bigcup_{\alpha \in A} V_\alpha$. On each V_α we have the local Poincaré metric

$$\omega_{P,\alpha} = \frac{\sqrt{-1}}{2}\left(\sum_{i=1}^{m} \frac{1}{2|z_i|^2 (\log |z_i|)^2} dz_i \wedge d\bar{z}_i + \sum_{i=m+1}^{k} dz_i \wedge d\bar{z}_i\right).$$

Definition: Let η be a smooth local p-form defined on V_α.

(1). We say η has Poincaré growth if there is a constant $C_\alpha > 0$ depending on η such that

$$|\eta(t_1, \cdots, t_p)|^2 \leq C_\alpha \prod_{i=1}^{p} \|t_i\|^2_{\omega_{P,\alpha}}$$

for any point $z \in V_\alpha$ and $t_1, \cdots, t_p \in T_z X$.

(2). η is good if both η and $d\eta$ have Poincaré growth.

Definition: An Hermitian metric h on E is good if for all $z \in V$, assuming $z \in V_\alpha$, and for all basis $(e_1, \cdots, e_n)$ of $\bar{E}$ over U_α, if we let $h_{i\bar{j}} = h(e_i, e_j)$, then

(1). $\left|h_{i\bar{j}}\right|, (\det h)^{-1} \leq C \left(\sum_{i=1}^{m} \log |z_i|\right)^{2n}$ for some $C > 0$;

(2). The local 1-forms $\left(\partial h \cdot h^{-1}\right)_{\alpha\gamma}$ are good on V_α. Namely the local connection and curvature forms of h have Poincaré growth.

• *Properties of Good Metrics*

(1). The definition of Poincaré growth is independent of the choice of U_α or local coordinates on it.

(2). A form $\eta \in A^p(X)$ with Poincaré growth defines a p-current $[\eta]$ on $\bar{X}$. In fact we have

$$\int_X |\eta \wedge \xi| < \infty$$

for any $\xi \in A^{k-p}(\bar{X})$.

(3). If both $\eta \in A^p(X)$ and $\xi \in A^q(X)$ have Poincaré growth, then $\eta \wedge \xi$ has Poincaré growth.

(4). For a good form $\eta \in A^p(X)$, we have $d[\eta] = [d\eta]$.

The importance of a good metric on E is that we can compute the Chern classes of $\bar{E}$ via the Chern forms of h as currents.

Mumford has proved:

Theorem. *Given an Hermitian metric h on E, there is at most one extension $\bar{E}$ of E to $\bar{X}$ such that h is good.*

Theorem. *If h is a good metric on E, the Chern forms $c_i(E, h)$ are good forms. Furthermore, as currents, they represent the corresponding Chern classes $c_i(\bar{E}) \in H^{2i}(\bar{X}, \mathbf{C})$.*

Remark: With the growth assumptions on the metric and its derivatives, we can integrate by part, so Chern-Weil theory still holds.

Good Metrics on Moduli Spaces

Now we consider the metrics induced by the Weil-Petersson metric, the Ricci and perturbed Ricci metrics on the logarithmic extension of the holomorphic tangent bundles over the moduli space of Riemann surfaces.

Our theorems hold for the moduli space of Riemann surfaces with punctures.

Let $\mathcal{M}_g$ be the moduli space of genus g Riemann surfaces with $g \geq 2$ and let $\bar{\mathcal{M}}_g$ be its Deligne-Mumford compactification. Let $n = 3g-3$ be the dimension of $\mathcal{M}_g$ and let $Y = \bar{\mathcal{M}}_g \setminus \mathcal{M}_g$ be the compactification divisor.

Let $\bar{E} = T^*_{\bar{\mathcal{M}}_g}(\log Y)$ be the logarithmic cotangent bundle over $\bar{\mathcal{M}}_g$. For any Kähler metric g

on $\mathcal{M}_g$, let g^* be the induced metric on $\bar{E}$. We know that near the boundary $\{t_1 \cdots t_m = 0\}$,

$$\left(\frac{dt_1}{t_1}, \cdots, \frac{dt_m}{t_m}, dt_{m+1}, \cdots, dt_n\right)$$

is a local holomorphic frame of $\bar{E}$.

In these notations, near the boundary the log tangent bundle $F = T_{\bar{\mathcal{M}}_g}(\log Y)$ has local frame

$$\left\{t_1 \frac{\partial}{\partial t_1}, \cdots, t_m \frac{\partial}{\partial t_m}, \frac{\partial}{\partial t_{m+1}}, \cdots, \frac{\partial}{\partial t_n}\right\}.$$

We have proved several results about the goodness of the metrics on moduli spaces. By very subtle analysis on the metric, connection and curvature tensors. We first proved the following theorem:

Theorem. *The metric h^* on the logarithmic cotangent bundle $\bar{E}$ over the DM moduli space induced by the Weil-Petersson metric is good in the sense of Mumford.*

Based on the curvature formulae of the Ricci and perturbed Ricci metrics we have derived before, we have proved the following theorem from much more detailed and harder analysis: estimates over 80 terms.

Theorem. *The metrics on the log tangent bundle $T_{\bar{\mathcal{M}}_g}(\log Y)$ over the DM moduli space induced by the Weil-Petersson metric, the Ricci and perturbed Ricci metrics are good in the sense of Mumford.*

A direct corollary is

Theorem. *The Chern classes $c_k\left(T_{\bar{\mathcal{M}}_g}(\log Y)\right)$ are represented by the Chern forms of the Weil-Petersson, Ricci and perturbed Ricci metrics.*

This in particular means we can use the explicit formulas of Chern forms of the Weil-Petersson metric derived by Wolpert to represent the classes, as well as those Chern forms of the Ricci and the perturbed Ricci metric.

Dual Nakano Negativity of WP Metric

It was shown by Ahlfors, Royden and Wolpert that the Weil-Petersson metric have negative Riemannian sectional curvature.

Schumacher showed that the curvature of the WP metric is strongly negative in the sense of Siu.

In 2005, we showed that the curvature of the WP metric is dual Nakano negative.

Let (E^m, h) be a holomorphic vector bundle with a Hermitian metric over a Kähler manifold (M^n, g). The curvature of E is given by

$$P_{i\bar{j}\alpha\bar{\beta}} = -\partial_\alpha \partial_{\bar{\beta}} h_{i\bar{j}} + h^{p\bar{q}} \partial_\alpha h_{i\bar{q}} \partial_{\bar{\beta}} h_{p\bar{j}}.$$

(E, h) is Nakano positive if the curvature P defines a positive form on the bundle $E \otimes T_M$.

Namely, $P_{i\bar{j}\alpha\bar{\beta}} C^{i\alpha} \overline{C^{j\beta}} > 0$ for all $n \times n$ complex matrix $C \neq 0$.

E is dual Nakano negative if the dual bundle (E^*, h^*) is Nakano positive. Our result is

Theorem. *The Weil-Petersson metric on the tangent bundle $T_{\mathcal{M}_g}$ are dual Nakano negative.*

To prove this theorem, we only need to show that $(T^*\mathcal{M}_g, h^*)$ is Nakano positive. Let $R_{i\bar{j}k\bar{l}}$ be the curvature of $T\mathcal{M}_g$ and $P_{i\bar{j}k\bar{l}}$ be the curvature of the cotangent bundle.

We first have $P_{m\bar{n}k\bar{l}} = -h^{i\bar{n}} h^{m\bar{j}} R_{i\bar{j}k\bar{l}}$.

Thus if we let $a_{kj} = \sum_m h^{m\bar{j}} C^{mk}$, we then have

$$P_{m\bar{n}k\bar{l}} C^{mk} \overline{C^{nl}} = -\sum_{i,j,k,l} R_{i\bar{j}k\bar{l}} a_{ij} \overline{a_{lk}}.$$

Recall that at $X \in \mathcal{M}_g$ we have

$$R_{i\bar{j}k\bar{l}} = -\int_X \left(e_{i\bar{j}} f_{k\bar{l}} + e_{i\bar{l}} f_{k\bar{j}}\right) dv.$$

By combining the above two formulae, to prove that the WP metric is Nakano negative is equivalent to show that

$$\int_X \left(e_{i\bar{j}} f_{k\bar{l}} + e_{i\bar{l}} f_{k\bar{j}}\right) a_{ij} \overline{a_{lk}} \ dv > 0.$$

For simplicity, we first assume that matrix $[a_{ij}]$ is invertible.

Write $T = (\Box + 1)^{-1}$ the Green operator. Recall $e_{i\bar{j}} = T\left(f_{i\bar{j}}\right)$ where $f_{i\bar{j}} = A_i \bar{A}_j$ and A_i is the harmonic representative of the Kodaira-Spencer class of $\frac{\partial}{\partial t_i}$.

Let $B_j = \sum_{i=1}^n a_{ij} A_i$. Then the inequality we need to prove is equivalent to

$$-\sum_{j,k} R(B_j, \bar{B}_k, A_k, \bar{A}_j) =$$

$$\sum_{j,k} \int_X \left(T\left(B_j \bar{A}_j\right) A_k \bar{B}_k + T\left(B_j \bar{B}_k\right) A_k \bar{A}_j \right) dv \geq 0.$$

Let $\mu = \sum_j B_j \bar{A}_j$. Then the first term in the above equation is

$$\sum_{j,k} \int_X T\left(B_j \bar{A}_j\right) A_k \bar{B}_k \; dv = \int_X T(\mu)\bar{\mu} \; dv \geq 0.$$

We then let $G(z,w)$ be the Green's function of the operator T. Let

$$H(z,w) = \sum_j \bar{A}_j(z) B_j(w).$$

The second term is

$$\sum_{j,k} \int_X T\left(B_j \bar{B}_k\right) A_k \bar{A}_j \; dv =$$

$$= \int_X \int_X G(z,w) H(z,w) \bar{H}(z,w) dv(w) dv(z) \geq 0$$

where the last inequality follows from the fact that the Green's function G positive.

Applications

As corollaries of goodness and the positivity or negativity of the metrics, first we directly obtain:

Theorem. *The Chern classes of the log cotangent bundle of the moduli spaces of Riemann surfaces are positive.*

We have several corollaries about cohomology groups of the moduli spaces:

Theorem. *The Dolbeault cohomology of the log tangent bundle* $T_{\bar{\mathcal{M}}_g}(\log Y)$ *on* $\bar{\mathcal{M}}_g$ *computed via the singular WP metric* g *is isomorphic to the ordinary cohomology (or Cech cohomology) of the sheaf* $T_{\bar{\mathcal{M}}_g}(\log Y)$*.*

Here we need the goodness of the metric g induced from the WP metric in a substantial way.

Saper proved that the L^2-cohomology of $\mathcal{M}_g$ of the WP metric h (with trivial bundle $\mathbf{C}$) is the same as the ordinary cohomology of $\bar{\mathcal{M}}_g$. Parallel to his result, we have

Theorem.

$$H^*_{(2)}\Big((\mathcal{M}_g, \omega_\tau), (T_{\mathcal{M}_g}, \omega_{WP})\Big) \cong H^*(\bar{\mathcal{M}}_g, F).$$

An important and direct application of the goodness of the WP metric and its dual Nakano negativity is the vanishing theorem of L^2-cohomology group:

Theorem. *The L^2-cohomology groups*

$$H^{0,q}_{(2)}\left((\mathcal{M}_g, \omega_\tau)\,, \left(T_{\bar{\mathcal{M}}_g}(\log Y), \omega_{WP}\right)\right) = 0$$

unless $q = n$. Here ω_τ is the Ricci metric.

We put the Ricci metric on the base manifold to avoid the incompleteness of the WP metric.

This implies a result of Hacking

$$H^q(\bar{\mathcal{M}}_g, T_{\bar{\mathcal{M}}_g}(\log Y)) = 0,\ q \neq n.$$

To prove this theorem, we first consider the Kodaira-Nakano identity

$$\square_{\bar{\partial}} = \square_{\nabla} + \sqrt{-1}\left[\nabla^2, \Lambda\right].$$

We then apply the Nakano negativity of the WP metric to get the vanishing theorem by using the goodness to deal with integration by part. There is no boundary term.

Remark: (1). As corollaries, we also have: the moduli space of Riemann surfaces is rigid: no holomorphic deformation.

(2). We are proving that the KE metric, Bergman metric are also good metrics.

Thank You All!