Recent Results on the Moduli Spaces of Riemann Surfaces

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Twenty years ago, at his Nankai Institute of Mathematics, a lecture of Prof. S.-S. Chern on the Atiyah-Singer index formula introduced me to the beautiful subject of geometry and topology.

He described Chern classes and the AS index formula and its three proofs. That is the first seminar on modern mathematics I had ever attended. That changed my life.

I dedicate this lecture in memory of him.

I would like to give an overview of several results we obtained related to the moduli spaces of Riemann surfaces during the past two years.

We have two types of results:

Geometric aspect: Detailed understanding of two new metrics: the Ricci and the perturbed Ricci metric, as well as all of the classical complete metrics: especially the Kähler-Einstein metric of Cheng-Yau.

Algebraic geometric result: stability of the logarithmic cotangent bundle of the moduli spaces.

Our project on the geometric aspect is to understand the various metrics on moduli spaces, and more importantly to introduce new metrics with good (hope to be best possible) geometry properties and find their applications. The results are contained in

1. *Canonical Metrics in the Moduli Spaces of Riemann Surfaces I*. math.DG/0403068. To appear in JDG.

2. *Canonical Metrics in the Moduli Spaces of Riemann Surfaces II*. math.DG/0409220. To appear in JDG.

3. *Good Metrics on Moduli Spaces of Riemann Surfaces.* In preparation.

by K. Liu, X. Sun, S.-T. Yau.

Remark: For simplicity we state our results for $\mathcal{M}_g$, the moduli of closed Riemann surfaces. All of these results hold for $\mathcal{M}_{g,n}$, the moduli of Riemann surface with n punctures (marked points).

Topological aspect: The proofs of *the Mariño-Vafa formula* which gives closed formulas of the generating series of triple Hodge integrals of all genera and any number of marked points and several consequences; proofs of the two and three partition formulas of this type,

The mathematical theory of *topological vertex* which gives complete closed formulas of the Gromov-Witten invariants of all genera and all degrees for all local toric Calabi-Yau manifolds,

All expressed in terms of *Chern-Simons knot invariants*.

The project is motivated by conjectures of Vafa and his collaborators on the large N duality in Chern-Simons theory and string theory. The results are contained in

(1). *A Proof of a conjecture of Mariño-Vafa on Hodge Integrals*, JDG 2003.

(2). *A Formula of Two Partition Hodge Integrals*, math.AG/0310273.

(3). *Mariño-Vafa Formula and Hodge Integral Identities*, math.AG/0308015.

By C.-C. Liu, K. Liu and J. Zhou.

(4). *A Mathematical Theory of Topological Vertex*, math.AG/0408426.

By J. Li, C.-C. Liu, K. Liu and J. Zhou.

Basics of the Teichmüller and the Moduli Spaces:

Moduli spaces and Teichmüller spaces of Riemann surfaces have been studied for many many years, since Riemann, and by Ahlfors, Bers, Royden, Deligne, Mumford, Yau, Siu, Thurston, Faltings, Witten, Kontsevich, McMullen, Gieseker, Mazur, Harris, Wolpert, Bismut, Sullivan, Madsen and many many others and many in the audience.... A generation of young mathematicians. Still many unsolved problems.

They have appeared in many subjects of mathematics from geometry, topology, algebraic geometry, to number theory.

They have also appeared in theoretical physics like string theory. Many computations of path integrals are reduced to integrals of measures or Chern classes on such moduli spaces.

Fix an orientable surface Σ of genus $g \geq 2$.

• *Uniformization Theorem:* Each Riemann surface of genus $g \geq 2$ can be viewed as a quotient of the hyperbolic plane $\mathbb{H}^2$ by a Fuchsian group. Thus there is a unique Kähler-Einstein metric, or the hyperbolic metric on Σ.

The group $Diff^+(\Sigma)$ of orientation preserving diffeomorphisms acts on the space $\mathcal{C}$ of all complex structures on Σ by pull-back.

• *Teichmüller space:*

$$\mathcal{T}_g = \mathcal{C}/Diff_0^+(\Sigma)$$

where $Diff_0^+(\Sigma)$ is the set of orientation preserving diffeomorphisms which are isotopic to identity.

• *Moduli space:*

$$\mathcal{M}_g = \mathcal{C}/Diff^+(\Sigma) = \mathcal{T}_g/Mod(\Sigma)$$

is the quotient of the Teichmüller space by the mapping class group where

$$Mod(\Sigma) = Diff^{+}(\Sigma)/Diff_{0}^{+}(\Sigma).$$

• *Dimension:*

$$\dim_{\mathbb{C}} \mathcal{T}_g = \dim_{\mathbb{C}} \mathcal{M}_g = 3g - 3.$$

$\mathcal{T}_g$ is a pseudoconvex domain in $\mathbb{C}^{3g-3}$. $\mathcal{M}_g$ is a complex orbifold, quasi-projective.

• *Compactification:* Riemann surfaces are algebraic curves. Geometric invariant theory of Mumford gives an algebro-geometric construction of the moduli spaces. By adding stable nodal curves we get the projective Deligne-Mumford compactification $\overline{\mathcal{M}}_g$,

$D = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ is a divisor of normal crossings.

• *Tangent and cotangent space:* By the deformation theory of Kodaira-Spencer and the Hodge theory, for any point $X \in \mathcal{M}_g$,

$$T_X\mathcal{M}_g \cong H^1(X, T_X) = HB(X)$$

where $HB(X)$ is the space of harmonic Beltrami differentials on X.

$$T_X^*\mathcal{M}_g \cong Q(X)$$

where $Q(X) = H^0(X, K_X^2)$ is the space of holomorphic quadratic differentials on X.

For $\mu \in HB(X)$ and $\phi \in Q(X)$, the duality between $T_X\mathcal{M}_g$ and $T_X^*\mathcal{M}_g$ is

$$[\mu : \phi] = \int_X \mu\phi.$$

Teichmüller metric is the L^1 norm. The Weil-Petersson (WP) metric is the L^2 norm. The WP metric is incomplete.

Observation:
The Ricci curvature of the Weil-Petersson metric is bounded above by a negative constant, one can use the negative Ricci curvature of the WP metric to define a new metric.

We call this metric the **Ricci metric**

$$\tau_{i\bar{j}} = -Ric(\omega_{WP})_{i\bar{j}}.$$

It turns out that this is a complete Kähler metric, some of its geometric properties are better than those of WP metric in many aspects, but not good enough.

We perturb the Ricci metric with a large constant multiple of the WP metric to get a new complete Kähler metric, the **perturbed Ricci metric**

$$\omega_{\tilde{\tau}} = \omega_\tau + C\,\omega_{WP}.$$

This metric has some desired good curvature properties we need.

There are many very famous classical metrics on the Teichmüller and the moduli spaces:

(1). **Finsler metrics:** Teichmüller metric; Caratheodory metric; Kobayashi metric.

(2). **Kähler metrics:** The (incomplete) Weil-Petersson metric. Cheng-Yau Kähler-Einstein metric; McMullen metric; Asymptotic Poincare metric; Bergman metric.

New Kähler metrics: Ricci metric and the perturbed Ricci metric.

Several of these metrics have important applications in the study of the geometry and topology of the moduli and the Teichmüller spaces, in algebraic geometry, in string theory and in three manifold topology.

Conventions: For a Kähler manifold (M^n, g) with local holomorphic coordinates $z_1, \cdots, z_n$, the curvature of g is given by

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z_k} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}_l}.$$

In this case, the Ricci curvature is

$$R_{i\bar{j}} = -g^{k\bar{l}} R_{i\bar{j}k\bar{l}} = -\partial_{z_i} \partial_{\bar{z}_j} \log \det(g_{i\bar{j}}).$$

The holomorphic sectional curvature is negative means

$$R(v, \bar{v}, v, \bar{v}) < 0.$$

Two Kähler metrics g_1 and g_2 are equivalent or two norms $\| \cdot \|_1$ and $\| \cdot \|_2$ are equivalent if there is a constant $C > 0$ such that

$$C^{-1} g_1 \leq g_2 \leq C g_1$$

or

$$C^{-1} \| \cdot \|_1 \leq \| \cdot \|_2 \leq C \| \cdot \|_1.$$

We denote this by $g_1 \sim g_2$ or $\| \cdot \|_1 \sim \| \cdot \|_2$.

Our Goal and Results:

(1) Understand the two new complete Kähler metrics, the *Ricci metric* and the *Perturbed Ricci metric*. We have rather complete understanding of these two new metrics.

(2) With the help of the new metrics we have much better understanding of the Kähler-Einstein metric: The curvature of KE metric and all of its covariant derivatives are bounded on the Teichmüller space. Injectivity radius has lower bound.

(3) Algebro-geometric consequences: the logarithmic cotangent bundle of the DM moduli space of stable curves is Mumford stable.

(4) Good understanding of the complete classical metrics: all of the classical complete metrics are proved to be equivalent to the new metrics.

(5) The two new metrics and the Weil-Petersson metric are proved to be good in the sense of Mumford.

Now we give the detailed statements of our results:

Theorem. *Let τ be the Ricci metric on the moduli space $\mathcal{M}_g$. Then*

- *τ is equivalent to the asymptotic Poincaré metric.*
- *The holomorphic sectional curvature of τ is asymptotically negative in the degeneration directions.*
- *The holomorphic bisectional curvature, therefore the holomorphic sectional curvature, and the Ricci curvature of τ are bounded.*

We can explicitly write down the asymptotic behavior of this metric: asymptotic Poincaré:

$$\sum_{i=1}^{m} \frac{C_i\,|dt_i|^2}{|t_i|^2 \log^2|t_i|} + \sum_{i=m+1}^{n} ds_i^2.$$

t_i's are the coordinates in the degeneration directions: $\{t_i = 0\}$ define the divisor.

To get control on the signs of the curvatures, we need to perturb the Ricci metric. Recall that the curvatures of the WP metric are negative.

Theorem. *Let $\omega_{\tilde{\tau}} = \omega_\tau + C\omega_{WP}$ be the perturbed Ricci metric on $\mathcal{M}_g$. Then for suitable choice of the constant C, we have*
- *$\tilde{\tau}$ is a complete Kähler metric equivalent to the asymptotic Poincaré metric.*
- *The holomorphic sectional curvature and the Ricci curvature of $\tilde{\tau}$ are bounded from above and below by negative constants.*
- *The bisectional curvature is bounded.*

Only through the understanding of the new metrics, we were able to get the following result about the KE metric which means strongly bounded geometry:

Theorem. *The curvature of the KE metric and all of its covariant derivatives are uniformly bounded on the Teichmüller spaces, and its injectivity radius has positive lower bound.*

We also have the bounded geometry of the new metrics:

Theorem. *Both the Ricci and the perturbed Ricci metric have bounded geometry on the Teichmüller spaces: the curvatures are uniformly bounded and the injectivity radius has positive lower bounds.*

Remark: The perturbed Ricci metric is the first known complete Kähler metric on the moduli space whose holomorphic sectional and Ricci curvature have negative bounds, and bounded geometry. Note that the WP metric does not have bounded curvature and is incomplete.

The detailed understanding of the boundary behaviors of these metrics gives us geometric corollaries.

Let $\bar{E}$ denote the unique logarithmic extension of the cotangent bundle of $\mathcal{M}_g$. Local sections near compactification divisor has the form:

$$\sum_{i=1}^{m} a_i(t,s)\frac{dt_i}{t_i} + \sum_{i=m+1}^{n} a_i(t,s)ds_i,$$

where recall that the compactification divisor is defined by $\prod_{i=1}^{m} t_i = 0$, with $n = 3g-3$.

Theorem. *The first Chern class $c_1(\bar{E}) = [\omega_\tau] = [\omega_{KE}]$ is positive and $\bar{E}$ is Mumford stable with respect to $c_1(\bar{E})$.*

This means for any proper coherent sub-sheaf F of $\bar{E}$, we have

$$\frac{\deg(F)}{\operatorname{rank}(F)} < \frac{\deg(\bar{E})}{\operatorname{rank}(\bar{E})}$$

where the degree is with respect to $c_1(\bar{E})$:

$$\deg(F) = \int_{\overline{\mathcal{M}_g}} c_1(F) c_1(\bar{E})^{n-1}.$$

This theorem also implies that the moduli spaces are of logarithmic general type.

A corollary of our understanding of the new metrics and the Schwarz lemma of Yau is:

Theorem. *On the moduli space $\mathcal{M}_g$, the Teichmüller metric $\|\cdot\|_T$, the Kobayashi metric, the Carathéodory metric $\|\cdot\|_C$, the Kähler-Einstein metric ω_{KE}, the McMullen metric ω_M, the asymptotic Poincaré metric ω_P, the induced Bergman metric ω_B are equivalent to the Ricci metric ω_τ and the perturbed Ricci metric $\omega_{\tilde{\tau}}$. Namely*

$$\omega_{\tilde{\tau}} \sim \omega_{KE} \sim \omega_P \sim \omega_M \sim \omega_B \sim \omega_\tau$$

and

$$\|\cdot\|_K = \|\cdot\|_T \sim \|\cdot\|_C \sim \|\cdot\|_M \sim \|\cdot\|_\tau.$$

Remark: Wolpert: The question on the relationship between Carathéodory metric and the Bergman metric was raised by Bers in the early 70s. Yau conjectured the equivalences of Teichmüller metric and Bergman metric to the KE metric in early 80s.

The equivalences $\|\cdot\|_K = \|\cdot\|_T \sim \|\cdot\|_M$ were known before by Royden and McMullen.

The following theorem is a trivial corollary, since the metrics are asymptotic Poincaré:

Theorem. *The L^2 cohomology groups of $\mathcal{M}_g$ of the above complete Kähler metrics are all the same as the de Rham cohomology groups of $\overline{\mathcal{M}}_g$.*

Recall that goodness of a Kähler metric (in the sense of Mumford) means that near the boundary the metric tensor has logarithmic growth, the connection form and curvature form matrices have Poincaré type growth. The Chern classes of the logarithmic cotangent bundle can be computed by using the good metric.

Theorem. *The Weil-Petersson metric, the Ricci metric and the perturbed Ricci metric are good metrics.*

The L^2 index and fixed point formula may be applied to the Teichmüller spaces to understand the representations of mapping class groups.

The goodness of WP metric has been unknown for a long time. We are double-checking that the KE metric is also good.

The ideas of the proofs: There are 85 terms in the curvature formula of the Ricci metric.

Even more for the perturbed Ricci metric. Too complicated to see any property. We work out its asymptotics near the boundary. Note that the curvature of the WP metric only has two terms.

To compute the asymptotics of the Ricci metric and its curvature, we work on surfaces near the boundary of $\mathcal{M}_g$. The geometry of these surfaces localized on the pinching collars.

We construct approximation solutions on the local model, single out the leading terms and then carefully estimate the error terms one by one. The estimates are long and complicated computations, since the estimates need to be very precise. Even more subtle are the higher order estimates in proving the goodness.

Through careful analysis, we perturb the Ricci metric by the WP metric to control the signs of the curvature in the interior of the moduli, and the non-degenerate direction near the boundary.

The proof of the stability of the logarithmic cotangent bundle needs the detailed understanding of the boundary behaviors of the KE metric to control the convergence of the integrals of the degrees. Also needed is a basic non-splitting property of the mapping class group.

Bounded geometry of KE: The proofs used Ricci flow and the higher order estimates of curvature. The proof of lower bound of injectivity radius used minimal surfaces.

I only list some precise asymptotic estimates. Let $(t_1, \cdots t_m, s_{m+1}, \cdots s_n)$ be the pinching coordinates, $u_i = \frac{l_i}{2\pi}$, $l_i \approx -\frac{2\pi^2}{\log|t_i|}$ short geodesic lengths and $u_0 = \sum u_i + \sum |s_j|$. Recall that $\{t_j = 0\}$ define the boundary:

Theorem. *The Ricci metric τ has the asymptotic behaviors:*
(1) $\tau_{i\bar{i}} = \frac{3}{4\pi^2}\frac{u_i^2}{|t_i|^2}(1+O(u_0))$ *if* $i \le m$*;*
(2) $\tau_{i\bar{j}} = O\left(\frac{u_i^2 u_j^2}{|t_i t_j|}(u_i+u_j)\right)$ *if* $i,j \le m$ *and* $i \ne j$*;*
(3) $\tau_{i\bar{j}} = O\left(\frac{u_i^2}{|t_i|}\right)$ *if* $i \le m < j$*;*
(4) $\tau_{i\bar{j}} = O(1)$ *if* $i,j \ge m+1$*.*

We then derive the curvature asymptotics:

Theorem. *The holomorphic sectional curvature of the Ricci metric τ satisfies*

$$\widetilde{R}_{i\bar{i}i\bar{i}} = \frac{3u_i^4}{8\pi^4|t_i|^4}(1+O(u_0)) < 0$$

if $i \le m$ *and*

$$\widetilde{R}_{i\bar{i}i\bar{i}} = O(1)$$

if $i \ge m+1$*.*

It is important to have the precise estimates of the boundary behaviors and the bounds in the non-degeneration directions, since the leading terms have same order and different signs. More precise higher order estimates and various lifts of local vector fields on $\mathcal{M}_g$ to total space are needed for the goodness of the metrics.

Now we discuss the results on the **topological aspect** of the moduli spaces.

String Theory, as the unified theory of all fundamental forces, should be unique. But now there are *Five* different looking string theories.

Physicists: these theories should be equivalent, in a way dual to each other: their "partition functions" should be "equivalent". The identifications of partition functions of different theories have produced many surprisingly beautiful mathematical formulas.

The mathematical proofs of such formulas depend on *Localization Techniques* on various finite dimensional moduli spaces. More precisely integrals of Chern classes on moduli spaces:

Combined with various mathematics: Chern-Simons knot invariants, combinatorics of symmetric groups, Kac-Moody algebras' representations, Calabi-Yau, geometry and topology of moduli space of stable maps....

A simple technique we use: *Functorial Localization* transfers computations on complicated spaces to simple spaces: Connects computations of mathematicians and physicists.

I will first talk about the Mariño-Vafa formula, and then the other results if have time.

The Mariño-Vafa Conjecture:

For various purposes we need to compute Hodge integrals (i.e. intersection numbers of λ classes and ψ classes) on the Deligne-Mumford moduli space of stable curves $\overline{\mathcal{M}}_{g,h}$.

A point in $\overline{\mathcal{M}}_{g,h}$ consists of $(C, x_1, \ldots, x_h)$, a (nodal) curve and h smooth points on C.

The Hodge bundle $\mathbb{E}$ is a rank g vector bundle over $\overline{\mathcal{M}}_{g,h}$ whose fiber over $[(C, x_1, \ldots, x_h)]$ is $H^0(C, \omega_C)$. The λ classes are Chern Classes:

$$\lambda_i = c_i(\mathbb{E}) \in H^{2i}(\overline{\mathcal{M}}_{g,h}; \mathbb{Q}).$$

The cotangent line $T^*_{x_i}C$ of C at the i-th marked point x_i gives a line bundle $\mathbb{L}_i$ over $\overline{\mathcal{M}}_{g,h}$. The ψ classes are also Chern classes:

$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,h}; \mathbb{Q}).$$

Define

$$\Lambda_g^\vee(u) = u^g - \lambda_1 u^{g-1} + \cdots + (-1)^g \lambda_g.$$

Mariño-Vafa formula: Generating series over g of triple Hodge integrals

$$\int_{\overline{\mathcal{M}}_{g,h}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(\tau)\Lambda_g^\vee(-\tau-1)}{\prod_{i=1}^h (1-\mu_i\psi_i)},$$

can be expressed by close formulas of **finite** expression in terms of representations of symmetric groups, or Chern-Simons knot invariants. Here τ is a parameter.

Conjectured from large N duality between Chern-Simons and string theory.

Remark: Mumford first computed some low genus intersection numbers in early 80s. Witten conjecture in early 90s is about the integrals of the ψ classes.

Conifold transition: Resolve singularity in two ways:

Conifold X

$$\left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathbf{C}^4 : xw - yz = 0 \right\}$$

(1). Deformed conifold T^*S^3

$$\left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathbf{C}^4 : xw - yz = \epsilon \right\}$$

(ϵ real positive number)

(2). Resolved conifold $\tilde{X} = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{P}^1$

$$\left\{ ([Z_0, Z_1], \begin{pmatrix} x & y \\ z & w \end{pmatrix}) \in \mathbf{P}^1 \times \mathbf{C}^4 : \begin{array}{l} (x,y) \in [Z_0, Z_1] \\ (z,w) \in [Z_0, Z_1] \end{array} \right\}$$

$$\begin{array}{ccc} \tilde{X} & \subset & \mathbf{P}^1 \times \mathbf{C}^4 \\ \downarrow & & \downarrow \\ X & \subset & \mathbf{C}^4 \end{array}$$

Witten 92: The open topological string theory on the deformed conifold T^*S^3 is equivalent to Chern-Simons gauge theory on S^3.

Gopakumar-Vafa 98, Ooguri-Vafa 00: The open topological string theory on the deformed conifold T^*S^3 is equivalent to the closed topological string theory on the resolved conifold $\tilde{X}$.

Vafa and his collaborators 98-04: For the past several years, Vafa et al developed these duality ideas into the most powerful and effective tool to get closed formulas for the Gromov-Witten invariants on all local toric Calabi-Yau manifolds: *Topological Vertex.*

We have a rather complete mathematical theory of topological vertex. Start with Mariño-Vafa formula.

Mathematical Consequence of the Duality:
Chern-Simons Partition function:

$$\langle Z(U,V)\rangle = \exp(-F(\lambda, t, V))$$

U: holonomy of the $U(N)$ Chern-Simons gauge field around the knot $K \subset S^3$; V: $U(M)$ matrix

$\langle Z(U,V)\rangle$: Chern-Simons knot invariants of K.

$F(\lambda, t, V)$: Generating series of the open Gromov-Witten invariants of $(\tilde{X}, L_K)$, where L_K is a Lagrangian submanifold of the resolved conifold $\tilde{X}$ "canonically associated to" the knot K. (Taubes).

t'Hooft large N expansion, and canonical identifications of parameters similar to mirror formula: duality identification at level k:

$$\lambda = \frac{2\pi}{k+N}, \quad t = \frac{2\pi i N}{k+N}.$$

Special case: When K is the unknot, $\langle Z(U,V)\rangle$ was computed in the zero framing by Ooguri-Vafa and in any framing $\tau \in \mathbb{Z}$ by Mariño-Vafa.

Comparing with Katz-Liu's computations of $F(\lambda, t, V)$, Mariño-Vafa conjectured a striking formula about triple Hodge integrals in terms of Chern-Simons: representations and combinatorics of symmetric groups.

The framing in Mariño-Vafa's computations corresponds to choice of the circle action on the pair $(\tilde{X}, L_{\text{unknot}})$ in Katz-Liu's localization computations. Both choices are parametrized by an integer τ.

The Mariño-Vafa Formula:

Geometric side: For every partition $\mu = (\mu_1 \geq \cdots \geq \mu_{l(\mu)} \geq 0)$, define triple Hodge integral:

$$G_{g,\mu}(\tau) = A(\tau) \cdot \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(-\tau-1)\Lambda_g^\vee(\tau)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)},$$

with

$$A(\tau) = -\frac{\sqrt{-1}^{|\mu|+l(\mu)}}{|\,\mathrm{Aut}(\mu)|}[\tau(\tau+1)]^{l(\mu)-1}\prod_{i=1}^{l(\mu)}\frac{\prod_{a=1}^{\mu_i-1}(\mu_i\tau+a)}{(\mu_i-1)!}.$$

Introduce generating series

$$G_\mu(\lambda;\tau) = \sum_{g\geq 0}\lambda^{2g-2+l(\mu)}G_{g,\mu}(\tau).$$

Special case when $g=0$:

$$\int_{\overline{\mathcal{M}}_{0,l(\mu)}} \frac{\Lambda_0^\vee(1)\Lambda_0^\vee(-\tau-1)\Lambda_0^\vee(\tau)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)}$$

$$= \int_{\overline{\mathcal{M}}_{0,l(\mu)}} \frac{1}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)} = |\mu|^{l(\mu)-3}$$

for $l(\mu) \geq 3$, and we use this expression to extend the definition to the case $l(\mu) < 3$.

Introduce formal variables $p = (p_1, p_2, \ldots, p_n, \ldots)$, and define

$$p_\mu = p_{\mu_1} \cdots p_{\mu_{l(\mu)}}$$

for any partition μ. ($\Leftrightarrow \operatorname{Tr} V^{\mu_j}$)

Generating series for all genera and all possible marked points:

$$G(\lambda; \tau; p) = \sum_{|\mu| \geq 1} G_\mu(\lambda; \tau) p_\mu.$$

Representation side: χ_μ: the character of the irreducible representation of symmetric group $S_{|\mu|}$ indexed by μ with $|\mu| = \sum_j \mu_j$,

$C(\mu)$: the conjugacy class of $S_{|\mu|}$ indexed by μ.

Introduce:

$$\mathcal{W}_\mu(\lambda) = \prod_{1\le a<b\le l(\mu)} \frac{\sin[(\mu_a-\mu_b+b-a)\lambda/2]}{\sin[(b-a)\lambda/2]}$$

$$\cdot\frac{1}{\prod_{i=1}^{l(\nu)}\prod_{v=1}^{\mu_i} 2\sin[(v-i+l(\mu))\lambda/2]}.$$

This has an interpretation in terms of *quantum dimension* in Chern-Simons knot theory.

Define:

$$R(\lambda;\tau;p) = \sum_{n\ge 1}\frac{(-1)^{n-1}}{n}\sum_\mu [\sum_{\cup_{i=1}^n \mu^i=\mu}$$

$$\prod_{i=1}^n \sum_{|\nu^i|=|\mu^i|} \frac{\chi_{\nu^i}(C(\mu^i))}{z_{\mu^i}} e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_{\nu^i}\lambda/2}\mathcal{W}_{\nu^i}(\lambda)]p_\mu$$

where μ^i are sub-partitions of μ, $z_\mu = \prod_j \mu_j! j^{\mu_j}$ and $\kappa_\mu = |\mu| + \sum_i(\mu_i^2 - 2i\mu_i)$ for a partition μ: standard for representations of symmetric groups.

Mariño-Vafa Conjecture:

$$G(\lambda;\tau;p) = R(\lambda;\tau;p).$$

Remark: (1). This is a formula:

G: Geometry = R: Representations

Representations of symmetric groups are essentially combinatorics.

(2). Equivalent expression:

$$G(\lambda;\tau;p)^\bullet = \exp\left[G(\lambda;\tau;p)\right] = \sum_{\mu} G(\lambda;\tau)^\bullet p_\mu =$$

$$\sum_{|\mu|\geq 0} \sum_{|\nu|=|\mu|} \frac{\chi_\nu(C(\mu))}{z_\mu} e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_\nu\lambda/2} \mathcal{W}_\nu(\lambda) p_\mu$$

(3). Each $G_\mu(\lambda,\tau)$ is given by a *finite and closed* expression in terms of representations of symmetric groups:

$$G_\mu(\lambda,\tau) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \sum_{\cup_{i=1}^n \mu^i=\mu} \prod_{i=1}^n$$

$$\sum_{|\nu^i|=|\mu^i|} \frac{\chi_{\nu^i}(C(\mu^i))}{z_{\mu^i}} e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_{\nu^i}\lambda/2} \mathcal{W}_{\nu^i}(\lambda).$$

$G_\mu(\lambda,\tau)$ gives triple Hodge integrals for moduli spaces of curves of all genera with $l(\mu)$ marked points.

(4). Mariño-Vafa formula gives explicit values of many interesting Hodge integrals up to three Hodge classes:

• Taking limit $\tau \longrightarrow 0$ we get the λ_g conjecture (Faber-Pandhripande),

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1}\cdots\psi_n^{k_n}\lambda_g = \binom{2g+n-3}{k_1,\ldots,k_n} \frac{2^{2g-1}-1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!},$$

for $k_1+\cdots+k_n=2g-3+n$, and the following identity for Hodge integrals:

$$\int_{\overline{\mathcal{M}}_g}\lambda_{g-1}^3=\int_{\overline{\mathcal{M}}_g}\lambda_{g-2}\lambda_{g-1}\lambda_g$$

$$=\frac{1}{2(2g-2)!}\frac{|B_{2g-2}|}{2g-2}\frac{|B_{2g}|}{2g},$$

B_{2g} are Bernoulli numbers. And other identities.

• Taking limit $\tau\longrightarrow\infty$, we get the famous ELSV formula which relates the generating series of Hurwitz numbers to Hodge integrals.

• Taking limit $\mu_i\longrightarrow\infty$ again in ELSV, Okounkov-Pandhripande derived the Witten conjecture proved by Kontsevich.

The idea to prove the Mariño-Vafa formula is to prove that both sides satisfy the *Cut-and-Join* equation:

Theorem 1:

$$\begin{aligned}\frac{\partial R}{\partial \tau} &= \frac{1}{2}\sqrt{-1}\lambda \sum_{i,j=1}^{\infty}\Bigg((i+j)p_i p_j \frac{\partial R}{\partial p_{i+j}} \\ &\quad + ijp_{i+j}\left(\frac{\partial R}{\partial p_i}\frac{\partial R}{\partial p_j} + \frac{\partial^2 R}{\partial p_i \partial p_j}\right)\Bigg)\end{aligned}$$

Theorem 2:

$$\begin{aligned}\frac{\partial G}{\partial \tau} &= \frac{1}{2}\sqrt{-1}\lambda \sum_{i,j=1}^{\infty}\Bigg((i+j)p_i p_j \frac{\partial G}{\partial p_{i+j}} \\ &\quad + ijp_{i+j}\left(\frac{\partial G}{\partial p_i}\frac{\partial G}{\partial p_j} + \frac{\partial^2 G}{\partial p_i \partial p_j}\right)\Bigg)\end{aligned}$$

Initial Value: $\tau = 0$, Ooguri-Vafa formula:

$$G(\lambda, 0, p) = \sum_{d=1}^{\infty} \frac{p_d}{2d \sin\left(\frac{\lambda d}{2}\right)} = R(\lambda, 0, p).$$

Linear systems of ODE: *The solution is unique!*

$$G(\lambda; \tau; p) = R(\lambda; \tau; p).$$

Remark: (1). Cut-and-join equation is encoded in the geometry of the moduli spaces of stable maps: convolution formula of the form: (disconnected version: $G^\bullet = \exp G$)

$$G^\bullet_\mu(\lambda,\tau) = \sum_{|\nu|=|\mu|} \Phi^\bullet_{\mu,\nu}(-\sqrt{-1}\tau\lambda) z_\nu K^\bullet_\nu(\lambda)$$

where $\Phi^\bullet_{\mu,\nu}$ is series of double Hurwitz numbers, z_ν the combinatorial constants. Equivalently this gives the explicit solution of the cut-and-join equation, with initial value $K^\bullet(\lambda)$, the integrals of Euler classes on moduli of relative stable maps.

(2). Witten conjecture is about KdV equations. But the Mariño-Vafa formula gives *closed formula!*

The proof of the geometric cut-and-join equation used *Functorial Localization Formula*:

$f: X \to Y$ equivariant map. $F \subset Y$ a fixed component, $E \subset f^{-1}(F)$ fixed components in $f^{-1}(F)$. Let $f_0 = f|_E$, then

For $\omega \in H_T^*(X)$ an equivariant cohomology class, we have identity on F:

$$f_{0*}\{\frac{i_E^*\omega}{e_T(E/X)}\} = \frac{i_F^*(f_*\omega)}{e_T(F/Y)}.$$

This formula, similar to Riemann-Roch, a generalization of Atiyah-Bott localization to relative setting, has been applied to various settings to prove the conjectures from physics.

It is used to push computations on complicated moduli space to simpler moduli space: the proof of the mirror formula; the proof of the Hori-Vafa formula; the proof of the ELSV formula.... In each case we have natural equivariant maps from the moduli spaces to much simpler spaces which the physicists use.

In our first proof of the Mariño-Vafa formula we used the moduli spaces of relative stable maps to $\mathbf{P}^1$ as introduced by J. Li and natural maps to projective spaces.

Remarks: (1). The cut-and-join equation is closely related to the Virasoro algebra.

(2). Other later approaches:

(a). Direct derivation of convolution formula (Liu-Liu-Zhou).

(b). Okounkov-Pandhripande: use ELSV formula and λ_g conjecture.

The Mariño-Vafa formula can be viewed as a duality:

Chern-Simons $\Longleftrightarrow$ **Calabi-Yau.**

Can we go further with the ideas and methods?

Duality $\Leftrightarrow$ convolution and cut-and-join.

Yes much more!

One, two, three partitions.

Mariño-Vafa: one partition case....

Topological vertex: three partition case.

Generating series of Hodge integrals with more partitions $\Leftrightarrow$ closed formulas in terms of Chern-Simons invariants of Hopf link and more....

Mirror symmetry used periods and holomorphic anomaly to compute GW series, difficult for higher genera. Topological vertex gives complete answers for all genera and all degrees in the local Calabi-Yau cases in terms of Chern-Simons knot invariants.

Topological Vertex is related to a three partition analogue of the Mariño-Vafa formula. This formula gives closed formula for the generating series of the Hodge integrals involving three partitions. The cut-and-join equation has the form:

$$\frac{\partial}{\partial\tau}F^{\bullet} = (CJ)^1 F^{\bullet} + \frac{1}{\tau^2}(CJ)^2 F^{\bullet} + \frac{1}{(\tau+1)^2}(CJ)^3 F^{\bullet}$$

where (CJ) denotes the cut-and-join operator with respect to the three groups of infinite variables associated to the three partitions.

We first derive the convolution formulas both in combinatorics and in geometry. Then we need to prove the identity of initial values at $\tau = 1$. Much more complicated in both geometry and in combinatorics.

By using gluing formula of the topological vertex, we can derive closed formulas for generating series of GW invariants, all genera and all degrees, open or closed, for all local toric Calabi-Yau, in terms Chern-Simons invariants, by simply looking at their moment map graphs.

Let $N_{g,d}$ denote the GW invariants of a local toric CY, total space of canonical bundle on a toric surface S: the Euler number of the obstruction bundle on the moduli space $\overline{\mathcal{M}}_g(S,d)$ of stable maps of degree $d \in H_2(S,\mathbb{Z})$ from genus g curve into the surface S:

$$N_{g,d} = \int_{[\overline{\mathcal{M}}_g(S,d)]^v} e(V_{g,d})$$

with $V_{g,d}$ a vector bundle induced by the canonical bundle K_S: at point $(\Sigma; f) \in \overline{\mathcal{M}}_g(S,d)$, its fiber is $H^1(\Sigma, f^*K_S)$. Write

$$F_g(t) = \sum_d N_{g,d}\, e^{-d\cdot t}.$$

Example: Topological vertex formula of GW generating series in terms of CS invariants. For the total space of canonical bundle $\mathcal{O}(-3)$ on $\mathbf{P}^2$:

$$\exp\big(\sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t)\big) = \sum_{\nu_1,\nu_2,\nu_3} \mathcal{W}_{\nu_1,\nu_2}\mathcal{W}_{\nu_2,\nu_3}\mathcal{W}_{\nu_3,\nu_1}\cdot$$

$$(-1)^{|\nu_1|+|\nu_2|+|\nu_3|} q^{\frac{1}{2}\sum_{i=1}^{3}\kappa_{\nu_i}} e^{t(|\nu_1|+|\nu_2|+|\nu_3|)}.$$

Here $q = e^{\sqrt{-1}\lambda}$, and $\mathcal{W}_{\mu,\nu}$ are from the Chern-Simons knot invariants of Hopf link.

Three vertices of moment map graph $\leftrightarrow$ three $\mathcal{W}_{\mu,\nu}$'s.

For general local toric Calabi-Yau, the expressions are just similar, closed formulas.

GV conjecture: *There exists expression:*

$$\sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t) = \sum_{k=1}^{\infty} \sum_{g,d} n_d^g \frac{1}{d} (2\sin\frac{d\lambda}{2})^{2g-2} e^{-kd\cdot t},$$

such that n_d^g are integers, called instanton numbers.

For some interesting cases we can interpret the n_d^g as equivariant indices of twisted Dirac operators on moduli spaces of anti-self-dual connections on $\mathbb{C}^2$, related to the Nekrasov conjecture.

By using the Chern-Simons knot expressions from topological vertex, the following theorem was first proved by Peng Pan:

Theorem. *The Gopakumar-Vafa conjecture is true for all (formal) local toric Calabi-Yau for all degree and all genera.*

There should be a more interesting and grand duality picture between Chern-Simons invariants for three dimensional manifolds and Gromov-Witten invariants for toric CY. Real dimension and complex dimension 3.

We expect that our studies of the geometric and topological aspects of the moduli spaces will merge together very soon.

Thank You Very Much!