Probab. Th. Rel. Fields 88, 269–341 (1991)

Probability Theory and Related Fields

Gaussian fluctuations of connectivities in the subcritical regime of percolation

Massimo Campanino[1,*], **J.T. Chayes**[2,**], **and L. Chayes**[2,**]

[1] Dipartimento di Matematica, Università di Bologna, Piazza di Porta S. Donato 5, I-40126 Bologna, Italy

[2] Department of Mathematics, University of California, Los Angeles, CA 90024, USA

Received November 30, 1988; in revised form June 8, 1990

Summary. We consider the d-dimensional Bernoulli bond percolation model and prove the following results for all $p<p_c$: (1) The leading power-law correction to exponential decay of the connectivity function between the origin and the point $(L, 0, \ldots, 0)$ is $L^{-(d-1)/2}$. (2) The correlation length, $\xi(p)$, is real analytic. (3) Conditioned on the existence of a path between the origin and the point $(L, 0, \ldots, 0)$, the hitting distribution of the cluster in the intermediate planes, $x_1=qL, 0<q<1$, obeys a multidimensional local limit theorem. Furthermore, for the two-dimensional percolation system, we prove the absence of a roughening transition: For all $p>p_c$, the finite-volume conditional measures, defined by requiring the existence of a dual path between opposing faces of the boundary, converge – in the infinite-volume limit – to the standard Bernoulli measure.

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* Work supported in part by G.N.A.F.A. (C.N.R.)
** Work supported in part by NSF Grant No. DMS-88-06552

1. Introduction

Consider the nearest-neighbor Bernoulli bond percolation model on the d-dimensional hypercubic lattice in which bonds are occupied with density p and vacant with density $1-p$. (See Sect. 2 for precise definitions and basic properties of the model.) It has been known for some time that in dimension $d \geqq 2$, the percolation model undergoes a phase transition at some value $p_c(d) \in (0, 1)$, below which the occupied clusters are finite w.p. 1 and above which there is an infinite cluster w.p.1. In this paper, we study the detailed asymptotic properties of occupied paths throughout the subcritical regime, i.e. whenever $p < p_c$.

The properties of occupied paths for $p < p_c$ are typically characterized by the *connectivity function*

(1.1) $\tau_{x,y}(p) = P_p$ (x and y are connected by a path of occupied bonds}.

A special case of this is the on-axis connectivity function $\tau_{0,L}(p)$ between the origin and the site $= (L, 0, \ldots, 0)$. In this case the limit

$$\frac{1}{\xi(p)} = -\lim_{L \to \infty} \frac{1}{L} \log \tau_{0,L}(p) \tag{1.2}$$

exists and defines the *correlation length* $\xi(p)$ of the system. It is known that $\xi(p) < \infty$ for $p < p_c$ and that $\xi(p)$ diverges continuously as $p \uparrow p_c$ (see, e.g., [CC3]). Thus, in the low-density phase, the leading behavior of the (on-axis) connectivity function is exponential decay: $\tau_{0,L}(p) \sim e^{-L/\xi}$. Moreover, $\tau_{0,L}$ obeys the bounds

$$K_1(p) \frac{1}{L^{4(d-1)}} e^{-L/\xi(p)} \leqq \tau_{0,L}(p) \leqq e^{-L/\xi(p)}, \tag{1.3}$$

with $K_1(p) > 0$ (see, e.g., [Gr]).

One of the purposes of this work is to obtain the leading power-law correction to exponential decay of the connectivity function. As we will show, this is in turn related to the typical fluctuations of long paths of occupied bonds. The relationship between these quantities is most easily seen by examining the analytic structure of transforms in $x-y$ of $\tau_{x,y}(p)$. By controlling the p-dependence of these transforms, we also obtain detailed information on the correlation length $\xi(p)$.

Our principal results are: $\forall p < p_c$

(I) $\exists K_2(p) \geqq 1$, $\Delta(p) > 0$ such that

$$\Big| \sum_{x: x_1 = L} \tau_{0,x}(p) e^{+L/\xi(p)} - K_2(p) \Big| \leqq e^{-\Delta(p) L};$$

(II) $\exists \alpha(\mathrm{p})/0$ such that $\forall a \in Z^{d-1}$ satisfying $|a| < CL^{3/4-\varepsilon}$ with $\varepsilon > 0$,

$$\tau_{0,(L,a)} = K_2(p) \frac{1}{[\alpha(p)\tau L]^{(d-1)/2}} e^{-L/\xi(p)} e^{-a^2/[\alpha(p)L]} [1 + O(L^{-1}, L^{-4\varepsilon})]$$

where, as also in the following, we write $O(L^{-a}, L^{-b})$ for $O(L^{-\min(a,b)})$;

(III) $\xi(p)$ is real analytic.

Notice that (I) implies that the *point-to-plane connectivity function*

$$\mathbb{G}_L(p) \equiv \sum_{x: x_1 = L} \tau_{0,x}(p) \tag{1.4}$$

has pure exponential decay in the strongest sense: the correction to exponential decay is itself exponentially small. As in field theory, we will call the exponential correction the upper gap.

Result (II) is the central result of this paper. It is obtained from (I) using an analysis of the transform of τ, combined with some estimates on various connectivity functions. For $\mathbf{a}=0$, (II) says that the correction to exponential decay of $\tau_{0,L}(p)$ is of the form $L^{-(d-1)/2}$; in statistical mechanics, this power law prefactor is known as *Ornstein-Zernike* decay. For $\mathbf{a} \neq 0$, observe that the ratio $\tau_{0,(L,a)}/\mathbb{G}_L$ is essentially the hitting distribution[1] of endpoints in the plane $x_1 = L$. Thus, by (I) and (II), under the scaling $\mathbf{a} \to \mathbf{a}/(\alpha L)^{1/2}$, the hitting distribution tends to a Gaussian; the Ornstein-Zernike prefactor arises naturally as the normalization of this Gaussian.

From (II), it can be shown (see Sect. 6) that conditioned on the existence of a cluster between the origin and $y=(L, 0, \ldots, 0)$, the maximum height of the cluster in the plane $x_1 = qL, 0 < q < 1$, obeys a *multidimensional local limit theorem.* In particular, in each direction, the cluster typically wanders a distance $O(\sqrt{L})$ from the x_1-axis. In dimension $d=2$, this also implies the *absence of a roughening transition:* namely, for all $p > p_c$, the finite-volume conditional measures in which a dual path is required between the midpoints of two opposing faces of the boundary, converge – in the infinite-volume limit – to the standard Bernoulli measure. We also prove an alternative characterization of the absence of roughening in terms of the vanishing of a "roughening order parameter."

Result (III) constitutes a characterization of the subcritical regime.

Behavior similar to that described above is expected to hold in a wide variety of spin systems and field theories. Indeed, analogues of some of the results (I)–(III) have been proved in many systems for extreme values of the parameter (e.g., $p \ll 1$ for percolation) via expansion techniques. For general expansion techniques of this form, see, e.g., [PS1; PS2]. Specific perturbative results on Ornstein-Zernike decay and upper gaps appear in [ACC1; ACC2; AK; BrF1; BrF2; BrF3; CC3; P; O1; O2; OB; OBr; S1; S2; S3; S4; Si2]. Roughness of two-dimensional interfaces at low temperatures has also been proved with expansions ([G; Hi1; Hi2]). To our knowledge, the only other systems in which such results are known nonperturbatively are certain random surfaces ([ACC3; CC1]) and self-avoiding walks [CC2], both defined via generating functions. Absence of a roughening transition (an inherently nonpertubative result) is also known for the two-dimensional Ising magnet ([A; Hi3]).

Our proof is modelled closely on that cited above for self-avoiding walks (SAWs). Indeed, we define analogous quantities and rely on some previous lemmas. Here, however, there are several significant complications: First, percolation clusters can branch, whereas by definition SAWs have no branching points. Second, percolation configurations have a density of connected components, whereas (again by definition) the SAW "configurations" each contain only one

[1] This correspondence is exact for a slightly modified connectivity function: the cylinder connectivity function, as defined in Sect. 3

walk. We therefore expect that our proofs of results (I)–(III) should hold, with no essential modification, for the branching polymer system, which has the first, but not the second complication. Indeed, our proofs should also hold for one parameter lattice animals defined via generating functions. It is worth noting, however, that our methods do not readily extend to two-parameter lattice animals, for which results analogous to (I)–(III) remain an open problem.

We conclude this introduction with an outline of the paper (including a rough scheme of the proof):

In Sect. 2, we set general notation, define the percolation model and review some basic esults which are used in our subsequent analysis.

In Sect. 3, we introduce several other connectivity functions (i.e., analogues of the "free" connectivity functions, $\tau_{\mathbf{x},\mathbf{y}}$ and $\mathbb{G}_L$, in which the occupied paths are required to satisfy additional constraints). Some of these, which we call the *cylinder connectivity functions*, differ from $\tau_{\mathbf{x},\mathbf{y}}$ only by boundary constraints. Others, which we call *direct connectivity functions*, have an additional constraint imposed in each hyperplane perpendicular to the x_1-axis. In fact, we define several direct connectivity functions which differ among themselves only by boundary conditions. It is intuitively clear, but rather difficult to prove that connectivity functions which differ only by boundary conditions are bounded uniformly (in $|\mathbf{x}-\mathbf{y}|$) in terms of one another throughout the subcritical regime. This is established in Sect. 3, with some of the more tedious aspects of the proofs relegated to an appendix.

Our motivation for introducing the cylinder and direct connectivity function is explained in Sect. 4. There we show that these functions are related by a renewal equation. Which (by analogy to the study of correlation functions in fluids) we call the Ornstein-Zernike equation. The utility of the Ornstein-Zernike equation is that the exponential decay rates of the cylinder and direct functions are strictly separated, then the equation provides sufficient analytic control to prove analogues of properties (I)–(III) for the cylinder functions. Thus the proof of these properties is reduced to a proof of strict separation of the decay rates. These consequences of an Ornstein-Zernike equation have been exploited previously in work on other systems, where they formed the basis of both perturbative ([ACC1; ACC2; AK; CC3]) and nonperturbative proofs ([ACC3; CC1; CC2]). However, for completeness, we give a rather detailed proof of these consequences for percolation.

Since the direct connectivity function has a density of additional constraints, it is obvious and quite easy to prove that its exponential decay rate is strictly separated from that of the cylinder function $p \ll 1$. What is less obvious is that the separation prevails throughout the subcritical regime. This is established in Sect. 5 using a renormalization scheme, which will be explained in some detailed at the beginning of that section.

We finally comment on results appeared between the submission and the revision of this paper. Weaker lower bounds (but simpler to obtain) for the connectivities along the axes were given by G. Grimmett and by two of the authors (J.T.C. and L.C.) (see respectively [Gr] and [Al] where the author generalizes these results to the behavior of the connectivities in other directions). In [Ma] the author remarks that the asymptotic behavior and the local limit theorem in the case of self-avoiding random walks, treated in [CC2], can be deduced from the separation of the decay rates by introducing a suitable random walk and exploiting probability theorems. The proofs of these theorems are in any case very similar to those given in [CC2] (see Sect. 6 of this paper for the analogous results in our case), while the essential difficulties lie in the proof of the separation of rates and (in the case of this work) of the independence of the behavior of different connectivities from the boundary conditions.

2. Notation, definitions and preliminaries

Consider the d-dimensional hypercubic lattice $\mathbb{Z}^d$. We will denote a generic site $(x_1, \ldots, x_d) \in \mathbb{Z}^d$ by the vector $\mathbf{x}$, except that we will not use vector notation for the origin 0. The unit vectors will be denoted by $\mathbf{e}_1, \ldots, \mathbf{e}_d$. We will often have occasion to distinguish the x_1-coordinate; in this case, we will write $\mathbf{x} = (L, \mathbf{a})$ with $L \in \mathbb{Z}$ and $\mathbf{a} \in \mathbb{Z}^{d-1}$. Furthermore, for $L \in \mathbb{Z}$,

$$\mathbf{P}(L) = \{\mathbf{x} \in \mathbb{Z}^d \mid x_1 = L\} \tag{2.1}$$

will denote a hyperplane perpendicular to the x_1-axis;

$$\mathbf{H}(L) = \{\mathbf{x} \in \mathbb{Z}^d \mid x_1 \leqq L\} \tag{2.2}$$

will denote the half-space to the left of $\mathbf{P}(L)$; and for $L \geqq 1$,

$$\mathbf{S}(L) = \{\mathbf{x} \in \mathbb{Z}^d \mid \mathbf{x} \in \mathbf{P}(j) \quad \text{for some } 0 \leqq j \leqq L\} \tag{2.3}$$

will denote a slab perpendicular to the x_1-axis. We will measure lattice distances in the L^∞-norm, i.e. for $\mathbf{x} \in \mathbb{Z}^d$

$$|\mathbf{x}| \equiv \|\mathbf{x}\|_\infty = \max\{|x_1|, \ldots, |x_d|\}. \tag{2.4}$$

Finally, for $\mathbf{a} \in \mathbb{Z}^{d-1}$, $\mathbf{a}^2 = \sum_i a_i^2$ will denote the standard vector product.

The set of all *bonds* between nearest-neighbor sites of $\mathbb{Z}^d$, i.e. pairs $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$ with $\sum_i |x_i - y_i| = 1$, will be denoted by $\mathbb{B}_d$. Similarly, given $S \subset \mathbb{Z}^d$, $\mathbb{B}(S)$ will denote the set of bonds between nearest-neighbor sites of S. A *path* $\mathscr{P} \subset \mathbb{B}(S)$ is a sequence (finite or infinite) of bonds $b_1, b_2, \ldots$, with no repetitions, such that b_n and b_{n+1} have a common endpoint. Thus two paths are *disjoint* if they have no bonds in common. Similarly, two paths are *site-disjoint* if they have no endpoints of bonds in common. For $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$, a set $S \subset \mathbb{Z}^d$ is said to *separate* $\mathbf{x}$ from $\mathbf{y}$ if all paths from $\mathbf{x}$ to $\mathbf{y}$ include at least one bond with an endpoint in S.

The nearest-neighbor Bernoulli bond percolation model at density p is defined by independently choosing each bond of $\mathbb{B}_d$ to be *occupied* with probability p or *vacant* with probability $1-p$. Thus the configuration space is $\Omega = \{0, 1\}^{\mathbb{B}_d}$; a generic element of Ω will be denoted by ω. For $(L, \mathbf{a}) \in \mathbb{Z}^d$, we will denote by $T^{(L,\mathbf{a})}\omega$ the translate of ω by the vector $(L, \mathbf{a})$. For $S \subset \mathbb{Z}^d$, we will denote by $\omega|_S$ the restriction of ω to S; thus $\omega|_S \in \{0, 1\}^{\mathbb{B}(S)}$. We let P_p denote the product measure on Ω at density p, and E_p denote expectation with respect to P_p. We will often suppress the subscript p in P_p and E_p.

For S_1, $S_2 \subset \mathbb{Z}^d$, we say that S_1 is *connected* to S_2 in the configuration ω if there is a path of occupied bonds in ω from a site in S_1 to a site in S_2.

If such a path occurs within a set of bonds $B \subset \mathbb{B}_d$, we say that S_1 is *connected* to S_2 in B. Such paths may always be taken to be self-avoiding; henceforth, when we say that two sets are connected by a path, we will always mean a self-avoiding occupied path. The maximal connected subsets of ω are called the *(occupied) clusters* of ω. For $\mathbf{x} \in \mathbb{Z}^d$, we let $C(\mathbf{x}) = C(\mathbf{x}; \omega)$ denote the cluster containing $\mathbf{x}$ in ω. Finally, for $S \subset \mathbb{Z}^d$, we will denote by $C(\mathbf{x})\|_S = C(\mathbf{x})\|_S \equiv C(\mathbf{x}; \omega|_S)$ the connected component of $\mathbf{x}$ in S. Note that $C(\mathbf{x})\|_S$ is generally a strict subset of the restriction of the set $C(\mathbf{x})$ to S, i.e. the latter need not be a single component.

It should be remarked that, as defined, $C(\mathbf{x})$ is a set of sites, not bonds. Thus, for $S \subset \mathbb{Z}^d$, when we write $|C(\mathbf{x}) \cap S|$, we will always mean the cardinality of the set of *sites* $C(\mathbf{x}) \cap S$. However, when no confusion arises, we will often speak of "the cluster of $\mathbf{x}$" when we mean the bonds contained between sites in $C(\mathbf{x})$. Furthermore, unless otherwise specified, when we say that two such clusters are disjoint, we will mean that they are *bond* disjoint.

Let us review a few basic facts about percolation. First, it is well known ([BH]) that in dimension $d > 1$, the model undergoes a phase transition at the socalled *percolation threshold*, $p_c = p_c(d) \in (0, 1)$, below which the occupied clusters are finite w.p.1, and above which there is an infinite occupied cluster w.p.1. The order parameter for this transition is called the *infinite cluster density* (or *percolation probability*):

(2.5) $$P_\infty(p) = P_p(0 \text{ belongs to an infinite occupied cluster}).$$

It is known ([Har; AKN]) that the infinite cluster is unique for $p > p_c$.

There is an a priori smaller critical point defined by divergence of the *susceptibility:*

(2.6) $$\chi(p) = E_p(|C(0)|) = \sum_{\mathbf{x}} \tau_{0,\mathbf{x}}(p).$$

Here $\tau_{0,\mathbf{x}}(p)$ is the connectivity function defined in Eq. (1.1). However, it is now known ([K; AB; M; MMS]) that $\chi(p) < \infty$ for $p < p_c$, from which it follows easily [H] that $\xi(p) < \infty$ for $p < p_c$. Thus the a priori upper bound in Eq. (1.3) is non-trivial for all $p < p_c$. We will make repeated use of this bound and its off-axis generalization:

(2.7) $$\tau_{\mathbf{x},\mathbf{y}}(p) \leqq e^{-|\mathbf{x}-\mathbf{y}|/\xi(p)}.$$

We will also use the following bound [AN] on the tail of the finite cluster distribution:

(2.8) $$P_p(|C(0)| > n) \leqq e^{-\kappa(p) n},$$

with $\kappa(p) > 0$ for $p < p_c$. A final inequality which is quite useful in the low-density phase is the *Hammersley-Simon* ([H; Si1]) *inequality* on $\tau_{\mathbf{x},\mathbf{y}}$. Let $\mathbf{x}$, $\mathbf{y} \in \mathbb{Z}^d$ and let $S \subset \mathbb{Z}^d$ be a surface which separates $\mathbf{x}$ from $\mathbf{y}$ (including the degenerate case of a plane). Then

(2.9)
$$\tau_{\mathbf{x},\mathbf{y}} \leqq \sum_{\mathbf{z} \in S} \tau_{\mathbf{x},\mathbf{z}}\, \tau_{\mathbf{z},\mathbf{y}}.$$

We will also need a few general notions and inequalities.

Definition 2.1. Let $\omega_1, \omega_2 \in \Omega$. There is a natural partial order on Ω defined by $\omega_1 \prec \omega_2$ if all occupied bonds in ω_1 are also occupied in ω_2. An event in Ω is said to be *positive* or *increasing* (respectively, *negative* or *decreasing*) if its indicator function is nondecreasing (respectively, nonincreasing) with respect to this partial order.

The *Harris-FKG inequality* ([Har; FKG]) states that if $A_1, A_2 \subset \Omega$ are both positive (or both negative) events, then

(2.10)
$$P_p(A_1 \cap A_2) \geqq P_p(A_1)\, P_p(A_2).$$

Definition 2.2. Let $\omega \in A \subset \Omega$ and $B \subset \mathbb{B}_d$. The event A is said to *occur on the set* B in configuration ω if A occurs in ω restricted to B, regardless of the configuration in $\mathbb{B}_d/B$. We thus define the event

(2.11)
$$A|_B = \{\omega \in A \mid \tilde{\omega} \in A \text{ for all } \tilde{\omega} \text{ such that } \tilde{\omega} = \omega \text{ on all bonds in } B\}.$$

Two events $A_1, A_2 \subset \Omega$ are said to *occur disjointy*, denoted by $A_1 \circ A_2$, if there are (bond) disjoint sets on which they occur.

(2.12)
$$A_1 \circ A_2 = \{\omega \in A_1 \subset A_2 \mid \exists B_1, B_2 \subset \mathbb{B}_d, B_1 \cap B_2 = \emptyset, \omega \in A_1|_{B_1} \cap A_2|_{B_2}\}.$$

The *van den Berg-Kesten inequality* [BK] states that if $A_1, A_2 \subset \Omega$ are both positive (or both negative) events, then

(2.13)
$$P_p(A_1 \circ A_2) \leqq P_p(A_1)\, P_p(A_2).$$

This inequality was extended to the case of the A_i being intersections of positive and negtive events by van den Berg and Fiebig [BF].

3. The basic connectivity functions

3a. Full and cylinder connectivity functions

Consider the connectivity event

(3.1)
$$\mathcal{G}_{\mathbf{x},\mathbf{y}} = \{\omega \mid \mathbf{y} \in C(\mathbf{x})\}.$$

Our goal is to prove statements (I)–(III) for the "free" connectivity function $\tau_{\mathbf{x},\mathbf{y}}(p) = P(\mathcal{G}_{\mathbf{x},\mathbf{y}})$ and the corresponding point-to-plane connectivity function $\mathbb{G}_L$ (cf. Eq. (1.4)). However, it turns out that we must first prove analogous statements for a slightly constrained connectivity function. To this end, consider the following:

Definition 3.1. For $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$, $\mathbf{x} \in \mathbf{P}(0)$, $\mathbf{y} \in \mathbf{P}(L)$, $L \geqq 1$, the *point-to-point cylinder connectivity function* $h_{\mathbf{x},\mathbf{y}}$ is defined by

(3.2a) $$\mathscr{h}_{\mathbf{x},\mathbf{y}} = \{\omega \mid \mathbf{y} \in C(\mathbf{x})\|_{S(L)}, C(\mathbf{x})\|_{S(L)} \cap \mathbf{P}(0) = \{\mathbf{x}\}, C(\mathbf{x})\|_{S(L)} \cap \mathbf{P}(L) = \{\mathbf{y}\}\}$$

(3.2b) $$h_{\mathbf{x},\mathbf{y}}(p) = P(\mathscr{h}_{\mathbf{x},\mathbf{y}}),$$

while for $L = 0$, we use the convention

(3.2c) $$h_{\mathbf{x},\mathbf{y}} = (1-p)^{2(d-1)} \delta_{\mathbf{x},\mathbf{y}}.$$

For non-negative integers L, the *point-to-plane cylinder connectivity function* is defined by

(3.3) $$\mathbb{H}_L(p) = \sum_{y \in \mathbf{P}(L)} h_{\mathbf{x},\mathbf{y}}.$$

Remark. The term cylinder connectivity is sometimes used simply to denote the probability that two points are connected within the slab between their x_1-coordinates. Note, however, that in the above definition we impose the additional constraint that the intersection of the connected component of these points with the bounding planes consists solely of the two points. In the proof of Proposition 3.1 below, we will show that whenever $p < p_c$, the effective "conditionally expected" number of such intersections is finite, and hence the two types of cylinder connectivities differ inessentially. When we must distinguish the two types of boundary conditions (e.g., in the Appendix), we will refer to those used in (3.2) as *strict* cylinder conditions.

Below, we establish basic properties of the connectivity functions $\mathbb{H}_L$ and $\mathbb{G}_L$: namely existence of decay rates and a priori bounds. More importantly, we show that $\mathbb{H}_L$ and $\mathbb{G}_L$ are bounded above and below in terms of one another *uniformly* in L.

Proposition 3.1. *Let $\mathbb{G}_L(p)$ and $\mathbb{H}_L(p)$ be defined as in Eqs.* (1.4) *and* (3.2). *Then for all $p \in (0, p_c)$,*

(i) *the limits*

$$\lim_{L \to \infty} \frac{\log \mathbb{H}_L(p)}{L} = \lim_{L \to \infty} \frac{\log \mathbb{G}_L(p)}{L} = -\frac{1}{\xi(p)}$$

exist, where $\xi(p)$ is the decay rate of $\tau_{0,L}$ as defined in Eq. (1.2);

(ii) *for every L*

$$\mathbb{H}_L(p) \leqq e^{-L/\xi(p)} \leqq \mathbb{G}_L(p);$$

and

(iii) $\exists \beta(p) > 0$ *such that for every L*

$$\beta(p)\, \mathbb{G}_L(p) \leqq \mathbb{H}_L(p) \leqq \mathbb{G}_L(p).$$

Corollary. $\beta(p)\, e^{-L/\xi(p)} \leqq \mathbb{H}(p) \leqq e^{-L/\xi(p)}$.

Proof. For the purposes of this proof, we will suppress the argument p in all quantities.

To prove the existence of a limit for the cylinder function, let L_1, L_2 be positive integers and $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{Z}^{d-1}$. The restriction of a configuration $\omega_1 \varepsilon h_{0,(L_1,\mathbf{a}_1)}$ to $S(L_1)$ can be patched together with the translate of restriction to $S(L_2)$ of an $\omega_2 \varepsilon h_{0,(L_2,\mathbf{a}_2)}$ to obtain a configuration in $S(L_1+L_2)$ that is the restriction of a configuration in $h_{0,(L_1+L_2,\mathbf{a}_1+\mathbf{a}_2)}$. Taking into account the repetition of the boundary conditions in the plane $P(L_1)$ this implies

(3.4)
$$h_{0,(L_1,\mathbf{a}_2)}\, h_{0,(L_2,\mathbf{a}_2)} \leqq h_{0,(L_1+L_2,\mathbf{a}_1+\mathbf{a}_2)}(1-p)^{2(d-1)}.$$

On the other hand, given L_1 and L_2, such an ω *uniquely* determines $\mathbf{a}_1$ and $\mathbf{a}_2$. Thus we have the stronger inequality

(3.5)
$$\sum_{\mathbf{a}_1} h_{0,(L_1,\mathbf{a}_1)}\, h_{0,(L_2,\mathbf{a}-\mathbf{a}_2)} \leqq h_{0,(L_1+L_2,\mathbf{a})}(1-p)^{2(d-1)}.$$

Summing over $\mathbf{a}$ and $\mathbf{a}_1$, we obtain

(3.6)
$$\mathbb{H}_{L_1}, \mathbb{H}_{L_2} \leqq \mathbb{H}_{L_1+L_2}(1-p)^{2(d-1)}.$$

By standard arguments, the subadditive inequality (3.5) implies the existence of the first limit in (i) and the fact that it is reached from below (i.e. the first inequality in (ii)). To prove the corresponding statement for $\mathbb{G}_L$, again let L_1 and L_2 be positive integers and $\mathbf{a} \in \mathbb{Z}^{d-1}$. The Hammersley-Simon inequality (2.9) for the connectivity function $\tau_{0,(L_1+L_2,\mathbf{a})}$ may be applied in the case where the bounding surface is the plane $P(L_1)$:

(3.7)
$$\tau_{0,(L_1+L_2,\mathbf{a})} \leqq \sum_{\mathbf{a}_1} \tau_{0,(L_1,\mathbf{a}_1)}\, \tau_{(L_1,\mathbf{a}_1),(L_1+L_2,\mathbf{a})}.$$

Summing (3.6) over $\mathbf{a}$ and exploiting translational invariance, we get

(3.8)
$$\mathbb{G}_{L_1}, \mathbb{G}_{L_2} \geqq \mathbb{G}_{L_1+L_2}$$

which implies the existence of the second limit in (*i*) and the fact that it is reached from above (i.e. the second inequality in (ii)).

It is clear that (iii) implies the equality of the two limits. Furthermore, that the decay rate of $\mathbb{G}_L$ equals that of $\tau_{0,L}$ (c.f. Eq. (1.2)) is implied by the obvious bounds

(3.9)
$$\tau_{0,L} \leqq \mathbb{G}_L = \sum_{\mathbf{x} \in \mathbf{P}(L)} \tau_{0,\mathbf{x}} \leqq \sum_{\mathbf{x} \in \mathbf{P}(L)} e^{-|\mathbf{x}|/\xi},$$

in which we have used the a priori bound (2.7) on $\tau_{0,\mathbf{x}}$.

It remains to prove (iii). The second inequality in (iii) is trivial since every configuration in Ω whose restriction to S(L) belongs in $h_{0,(L,\mathbf{a})}$ is also in $g_{0,(L,\mathbf{a})}$.

To prove the first inequality in (iii), let us define a modified cylinder connectivity which need not obey the condition that $C(0)$ have only a single point

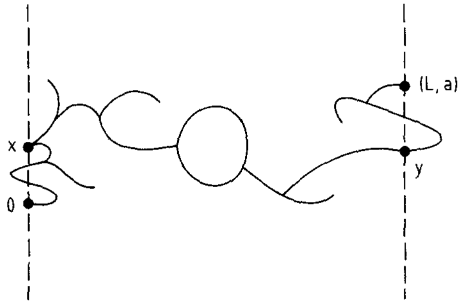

Fig. 1. A configuration in $\mathscr{g}_{0,(L,\mathbf{a})}$

of intersection with each of the bounding planes. That is, for $\mathbf{x} \in \mathbf{P}(0)$, $\mathbf{y} \in \mathbf{P}(L)$, let

(3.10a) $$\tilde{\mathscr{h}}_{\mathbf{x},\mathbf{y}} = \{\omega \mid \mathbf{y} \in C(\mathbf{x}) \|_{\mathbf{S}(L)}\}$$

(3.10b) $$\tilde{h}_{\mathbf{x},\mathbf{y}} = P(\tilde{\mathscr{h}}_{\mathbf{x},\mathbf{y}})$$

(3.10c) $$\tilde{\mathbb{H}}_L = \sum_{\mathbf{y} \in \mathbf{P}(L)} \tilde{h}_{x,y}.$$

First, let us relate $\tilde{\mathbb{H}}_L$ to $\mathbb{G}_L$. If $\omega \in \mathscr{g}_{0,(L,\mathbf{a})}$, then there are two points, $\mathbf{x} \in \mathbf{P}(0)$ and $\mathbf{y} \in \mathbf{P}(L)$, and three disjoint occupied paths: a (possibly trivial) path from 0 to $\mathbf{x}$, a (possibly trivial) path from $\mathbf{y}$ to $(L, \mathbf{a})$, and a non-trivial path from $\mathbf{x}$ to $\mathbf{y}$ in $\mathbf{S}(L)$. (See Fig. 1 for clarification.) It then follows from the van den Berg-Kesten inequality (2.12) that

(3.11) $$\tau_{0,(L,\mathbf{a})} \leqq \sum_{\substack{\mathbf{x} \in \mathbf{P}(0) \\ \mathbf{y} \in \mathbf{P}(L)}} \tau_{0,\mathbf{x}} \tilde{\mathscr{h}}_{\mathbf{x},\mathbf{y}} \tau_{\mathbf{y},(L,\mathbf{a})}.$$

Summing over $\mathbf{a} \in \mathbb{Z}^{d-1}$, we obtain

(3.12) $$\mathbb{G}_L \leqq \chi^2 \tilde{\mathbb{H}}_L,$$

where χ was defined in Eq. (2.6).

We now want to relate $\mathbb{H}_L$ to $\tilde{\mathbb{H}}_L$. To this end, let us (disjointly) partition $\tilde{\mathscr{h}}_{0,(L,\mathbf{a})}$ according to the number of intersections of $C(0)\|_{\mathbf{S}(L)}$ with the planes $\mathbf{P}(0)$ and $\mathbf{P}(L)$:

(3.13a) $$\tilde{\mathscr{h}}_{0,(L,\mathbf{a})}^{(k_1,k_2)} = \{\omega \mid (L, \mathbf{a} \in C(0)\|_{\mathbf{S}(L)}, |C(0)\|_{\mathbf{S}(L)} \cap \mathbf{P}(0)| = k_1, |C(0)\|_{\mathbf{S}(L)} \cap \mathbf{P}(L)| = k_2\},$$

(3.13b) $$\tilde{\mathscr{h}}_{0,(L,\mathbf{a})}^{(k_1)} = \bigcup_{k_2 \geqq 1} \tilde{\mathscr{h}}_{0,(L,\mathbf{a})}^{(k_1,k_2)},$$

and define the connectivity functions

(3.14a) $$\tilde{h}_{0,(L,\mathbf{a})}^{(k_1,k_2)} = P(\tilde{\mathscr{h}}_{0,(L,\mathbf{a})}^{(k_1,k_2)})$$

(3.14b) $$\tilde{h}_{0,(L,\mathbf{a})}^{(k_1)} = \sum_{k_2 \geqq 1} \tilde{h}_{0,(L,\mathbf{a})}^{(k_1,k_2)} = P(\tilde{\mathscr{h}}_{0,(L,\mathbf{a})}^{(k_1)}).$$

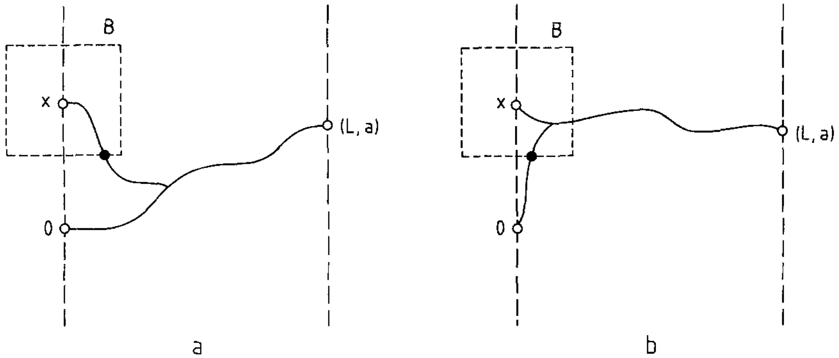

Fig. 2. Configurations in $\tilde{h}^{(k_1)}_{0,(L,\mathbf{a})}$ corresponding to cases (a) and (b). For pictorial clarity, we have omitted the additional k_1-2 connections of $C(0)\|_{\mathbf{S}(L)}$ to the boundary

Then the boundary conditions in the definitions of h and $\tilde{h}$ imply

$$h_{0,(L+2,\mathbf{a})}=\sum_{k_1,k_2\geqq 1}\tilde{h}^{(k_1,k_2)}_{0,(L,\mathbf{a})}[(1-p)^{2(d-1)}p]^2(1-p)^{k_1+k_2-2} \tag{3.15}$$

so that

$$\mathbb{H}_{L+2}=\sum_{k_1,k_2\geqq 1}\tilde{\mathbb{H}}^{(k_1,k_2)}_L[(1-p)^{2(d-1)}p]^2(1-p)^{k_1+k_2-2}, \tag{3.16}$$

where

$$\tilde{\mathbb{H}}^{(k_1,k_2)}_L=\sum_{\mathbf{a}\in\mathbb{Z}^{d-1}}\tilde{h}^{(k_1,k_2)}_{0,(L,\mathbf{a})}. \tag{3.17}$$

By the Jensen inequality

$$\mathbb{H}_{L+2}\geqq\tilde{\mathbb{H}}_L[(1-p)^{2(d-1)}p]^2(1-p)^{2\bar{K}_L-2} \tag{3.19}$$

where

$$\bar{K}_L=\sum_{\substack{k_1\geqq 1\\ k_2\geqq 1}}k_1\frac{\tilde{\mathbb{H}}^{(k_1,k_2)}_L}{\tilde{\mathbb{H}}_L}. \tag{3.20}$$

Let us now consider a configuration in $\tilde{h}^{(k_1)}_{0,(L,\mathbf{a})}$. If there are k_1 points in $C(0)\|_{\mathbf{S}(L)}\cap\mathbf{P}(0)$, at least one of them, say $\mathbf{x}$, is at a distance greater than (const.) $k_1^{1/(d-1)}$ from 0. Let $\mathbf{B}\left(\mathbf{x},\frac{|\mathbf{x}|}{2}\right)$ denote the ball (i.e. cube) of radius $\frac{|\mathbf{x}|}{2}$ about $\mathbf{x}$ and let $\partial\mathbf{B}\left(\mathbf{x},\frac{|\mathbf{x}|}{2}\right)$ denote its boundary. We distinguish two possible cases (see Fig. 2):

(a) There is a connection from 0 to $(L,\mathbf{a})$ outside $\mathbf{B}\left(\mathbf{x},\frac{|\mathbf{x}|}{2}\right)$, and an independent connection from $\mathbf{x}$ to some point of $\partial\mathbf{B}\left(\mathbf{x},\frac{|\mathbf{x}|}{2}\right)$.

(b) There is a connection from $\mathbf{x}$ to $(L, \mathbf{a})$, and an independent connection from 0 to some point of $\partial\mathbf{B}\left(\mathbf{x}, \frac{|\mathbf{x}|}{2}\right)$.

By the van den Berg-Kesten inequality (2.12), we therefore have

$$\tilde{h}^{(k_1)}_{0,(L,\mathbf{a})} \leqq \sum_{\substack{\mathbf{x}\in\mathbf{P}(0) \\ |\mathbf{x}| \geqq (\text{const.}) k_1^{1/(d-1)}}} \tilde{h}_{0,(L,\mathbf{a})} (\text{const.}) |\mathbf{x}|^{d-1} e^{-|\mathbf{x}|/\xi} + \sum_{\substack{\mathbf{x}\in\mathbf{P}(0) \\ |\mathbf{x}| \geqq (\text{const.}) k_1^{1/(d-1)}}} \tilde{h}_{\mathbf{x},(L,\mathbf{a})} |\mathbf{x}|^{d-1} e^{-|\mathbf{x}|/2\xi}. \tag{3.21}$$

Thus

$$\begin{aligned} \bar{K}_L &\leqq \tilde{\mathbb{H}}_L^{-1} \sum_{k_1 \geqq 1} k_1 \sum_{\substack{\mathbf{x}\in\mathbf{P}(0) \\ |\mathbf{x}| \geqq (\text{const.}) k_1^{1/(d-1)}}} \sum_{\mathbf{a}\in\mathbb{Z}^{d-1}} \cdot [\tilde{h}_{0,(L,\mathbf{a})} (\text{const.}) |\mathbf{x}|^{d-1} e^{-|\mathbf{x}|/\xi} + \tilde{h}_{\mathbf{x},(L,\mathbf{a})} |\mathbf{x}|^{d-1} e^{-|\mathbf{x}|/2\xi}] \\ &= \sum_{k_1 \geqq 1} k_1 \sum_{\substack{\mathbf{x}\in\mathbf{P}(0) \\ |\mathbf{x}| \geqq (\text{const.}) k_1^{1/(d-1)}}} (\text{const.}) [|\mathbf{x}|^{d-1} e^{-|\mathbf{x}|/\xi} + |\mathbf{x}|^{d-1} e^{-|\mathbf{x}|/2\xi}] \\ &\leqq K, \end{aligned} \tag{3.22}$$

with $K = K(p)$ a constant independent of L which is finite for $p < p_c$. Inserting (3.22) into (3.19), we obtain

$$\mathbb{H}_{L+2} \geqq \tilde{\mathbb{H}}_L \, e^{-K'(p)} \tag{3.23}$$

with $K'(p)$ independent of L and finite for $p < p_c$. The first inequality of (iii) follows immediately from (3.23) and (3.12). □

3b. The direct connectivity function

In order to control the analytic structure of the transforms of $h_{\mathbf{x},\mathbf{y}}$ (and $\tau_{\mathbf{x},\mathbf{y}}$) we would like to relate $h_{\mathbf{x},\mathbf{y}}$ to another connectivity function via a renewal type equation, and to prove that the latter function has a strictly faster decay rate than $h_{\mathbf{x},\mathbf{y}}$. To this end, we propose the following:

Definition 3.2. For $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$, $\mathbf{x} \in \mathbf{P}(0)$, $\mathbf{y} \in \mathbf{P}(L)$, $L \geqq 1$, the *point-to-point direct connectivity function* $c_{\mathbf{x},\mathbf{y}}$ is defined by

$$\begin{aligned} \mathcal{c}_{\mathbf{x},\mathbf{y}} = \{\omega \mid \mathbf{y} \in C(\mathbf{x})\|_{\mathbf{S}(L)}, C(\mathbf{x})\|_{\mathbf{S}(L)} \cap \mathbf{P}(0) = \{\mathbf{x}\}, C(\mathbf{x})\|_{\mathbf{S}(L)} \cap \mathbf{P}(L) = \{\mathbf{y}\}, \\ |C(\mathbf{x})\|_{\mathbf{S}(L)} \cap \mathbf{P}(j)| \geqq 2 \, \forall 1 \leqq j \leqq L-1\}, \end{aligned} \tag{3.24a}$$

$$c_{\mathbf{x},\mathbf{y}}(p) = P(\mathcal{c}_{\mathbf{x},\mathbf{y}}), \tag{3.24b}$$

while for $L = 0$, we use the convention

$$c_{\mathbf{x},\mathbf{y}}(p) = 0. \tag{3.24c}$$

For non-negative integers, the *point-to-plane direct connectivity function* is defined by

(3.25) $$\mathbb{C}_L(p) = \sum_{\mathbf{y} \in \mathbf{P}(L)} c_{\mathbf{x},\mathbf{y}}(p).$$

Remark. Notice that the configurations in $\mathscr{c}_{\mathbf{x},\mathbf{y}}$ are just the subset of those in $\mathscr{h}_{\mathbf{x},\mathbf{y}}$ which satisfy the constraint

(3.26) $$|C(\mathbf{x})\|_{\mathbf{S}(L)} \cap \mathbf{P}(j)| \geqq 2$$

in every interior plane (i.e. $\forall 1 \leqq j \leqq L-1$). We will occasionally refer to (3.26) as the "$\mathbb{C}$-condition."

It is easy to show that $\mathbb{C}_L$ has a well-defined decay rate:

Proposition 3.2. *Let* $\mathbb{C}_L(p)$ *be defined as in* Eq. (3.25). *Then for all* $p \in (0, p_c)$,

(i) *the limit*

$$\lim_{L \to \infty} \frac{1}{L} \log \mathbb{C}_L(p) = -\frac{1}{\xi_c(p)}$$

exists, and there is a $\lambda(p) < \infty$ *such that for all L*

$$\mathbb{C}_L(p) \leqq \lambda(p)\, e^{-L/\xi_c(p)};$$

(ii) *furthermore*

$$\tfrac{1}{2}\xi(p) \leqq \xi_c(p) \leqq \xi(p),$$

where $\xi(p)$ *is the decay rate of* $\tau_{0,L}$ *as defined in* Eq. (1.2).

Proof. (i) As in the proof of the sharp decay rate for $\mathbb{H}_L$, we claim that $\mathbb{C}_L$ has a subadditive inequality of the form:

(3.27) $$\mathbb{C}_{L_1}\,\mathbb{C}_{L_2} \leqq \lambda(p)\,\mathbb{C}_{L_1+L_2}.$$

Indeed by patching together the restrictions of two configurations $\omega_1 \in \mathscr{C}_{0,(L_1,\mathbf{a}_1)}$ and a translate of $\omega_2 \in \mathscr{C}_{0,(L_2,\mathbf{a}_2)}$, we obtain the restriction of a configuration ω which satisfies all the conditions of the event $\mathscr{C}_{0,(L_1+L_2,\mathbf{a}_1+\mathbf{a}_2)}$ except the $\mathbb{C}$-condition (Eq. (3.26)) in the plane $P(L_1)$. However, if our patching procedure includes occupying one of the $2(d-1)$ vacant bonds incident on $(L_1, \mathbf{a}_1)$ in $P(L_1)$, say $(L_1, \mathbf{a}_1)+e_2$, and vacating the possibly occupied $2d-1$ other bonds incident on $(L_1, \mathbf{a}_1)+e_2$ in $P(L_1)$, then the resulting configuration ω^* is in $\mathscr{C}_{0,(L_1+L_2,\mathbf{a}_1+\mathbf{a}_2)}$. More importantly, given L_1 and L_2, such an ω^* uniquely determines $\mathbf{a}_1$ and $\mathbf{a}_2$. Thus we obtain both an inequality analogous to (3.4) and a strengthened inequality of the form (3.5). Summing over the intermediate points, this yields (3.27), with $\lambda(p)/(1-p)^{2(d-1)}$ the cost of the patching procedure described above. As before, this implies the existence of the limit for $L^{-1} \log \mathbb{C}_L(p)$ and the fact that the limit is reached from below.

(ii) The inequality $\xi_c(p) \leqq \xi(p)$ follows from the bound $\mathbb{C}_L \leqq \mathbb{H}_L$, which holds for all L. To prove the inequality $\xi \leqq 2\xi_c$, let us define the on-axis direct connectivity function $c_{0,L} \equiv c_{0,(L,0,\ldots,0)}$. We claim that this connectivity also obeys the conclusions of (i): namely the limit

(3.28) $$\lim_{L\to\infty} \frac{1}{L} \log - \frac{c_{0,L}(p)}{L}$$

exists, is equal to $1/\xi_c(p)$ and provides the a priori bound

(3.29) $$c_{0,L}(p) \leqq \lambda(p)\, e^{-L/\xi_c(p)}$$

for all L.

Specializing the weak subadditive inequality to the case $a_1 = a_2 = 0$, we obtain the existence of the limit in (3.28), denoted temporarily by ξ_c^*, and the fact that it is reached from below. Furthermore, the analogue of (3.4) with $a_1 = -a_2 = a$, then gives

(3.30) $$c_{0,(L,\mathbf{a})} \leqq \lambda(p)\, e^{-L/\xi_c^*}$$

$\forall \mathbf{a} \in \mathbb{Z}^{d-1}$. This may of course be supplemented with the obvious estimate

(3.31) $$c_{0,(L,\mathbf{a})} = O(e^{-L^2/\xi})$$

for $|\mathbf{a}| > L^2$. Then an analogue of (3.9), implies $\xi_c^* = \xi_c$ and hence (3.29), as claimed.

Next, we may define the double connectivity event

(3.32a) $$\mathscr{J}_{\mathbf{x},\mathbf{y}} = \{\omega \in \mathscr{h}_{\mathbf{x},\mathbf{y}} \mid \exists\, 2 \text{ disjoint connections between } \mathbf{x}+\mathbf{e}_1 \text{ and } \mathbf{y}-\mathbf{e}_1\}$$

and the functions

(3.32b) $$j_{\mathbf{x},\mathbf{y}}(p) = P(\mathscr{J}_{\mathbf{x},\mathbf{y}})$$

and $j_{0,L} \equiv j_{0,(L,0,\ldots,0)}$. By the van den Berg-Kesten inequality, we have

(3.33) $$j_{\mathbf{x},\mathbf{y}} \leqq (\text{const.})\,[\tilde{h}_{\mathbf{x},\mathbf{y}}]^2,$$

and hence that

(3.34) $$\lim_{L\to\infty} -\frac{\log j_{0,L}}{L} \leqq \frac{2}{\xi}.$$

On the other hand, by considering connectivity functions confined, for example, to half spaces, it is straightforward to establish that

(3.35) $$\lim_{L\to\infty} -\frac{\log j_{0,L}}{L} \geqq \frac{2}{\xi}.$$

The desired inequality follows from the observation that

(3.36) $$j_{\mathbf{x},\mathbf{y}} \leqq c_{\mathbf{x},\mathbf{y}}. \quad \square$$

Remark. In light of the above result, it is interesting to note that

$$\lim_{p\to 0} \frac{\xi(p)}{\xi_c(p)} = 2. \tag{3.36}$$

However, we do not, at this time, have any speculation about this ratio in the limit $p \uparrow p_c$.

In our subsequent analysis, we will also require a direct connectivity function with free boundary conditions. Thus we define:

Definition 3.3. For $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$, $\mathbf{x} \in \mathbf{P}(0)$, $\mathbf{y} \in \mathbf{P}(L)$, $L \geqq 1$, the *point-to-point free direct connectivity function* $k_{\mathbf{x},\mathbf{y}}$ is defined by

$$\mathscr{k}_{\mathbf{x},\mathbf{y}} = \{\omega \mid \mathbf{y} \in C(\mathbf{x}), |C(\mathbf{x}) \cap \mathbf{P}(j)| \geqq 2 \,\forall\, 1 \leqq j \leqq L-1\}, \tag{3.37a}$$

$$k_{\mathbf{x},\mathbf{y}}(p) = P(\mathscr{k}_{\mathbf{x},\mathbf{y}}), \tag{3.37b}$$

while for $L=0$, we use the convention

$$k_{\mathbf{x},\mathbf{y}}(p) = 0. \tag{3.37c}$$

For non-negative integers, the *point-to-plane free direct connectivity function* is defined by

$$\mathbb{K}_L(p) = \sum_{\mathbf{y} \in \mathbf{P}(L)} k_{\mathbf{x},\mathbf{y}}(p). \tag{3.38}$$

As with the other connectivity functions, it is easy to show the existence of a decay rate:

Proposition 3.3. *Let* $\mathbb{K}_L(p)$ *be defined as in* Eq. (3.38). *Then for all* $p \in (0, p_c)$ *the limit*

$$\lim_{L\to\infty} \frac{1}{L} \log \mathbb{K}_L(p) = -\frac{1}{\xi_k(p)}$$

exists.

Proof. Let L_1 and L_2 be positive integers and $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{Z}^{d-1}$. We use the Harris-FKG inequality and an easy patching argument to show that

$$k_{0,(L_1,\mathbf{a}_1)}\, k_{0,(L_2,\mathbf{a}_2)} \leqq (\text{const.})\, k_{0,(L_1+L_2,\mathbf{a}_1+\mathbf{a}_2)}. \tag{3.39}$$

Notice that this is an analogue of the weak subadditive inequalities for h and c (e.g., Eq. (3.4)). Thus, following exactly the first half of the proof of Proposition 3.2 (ii), (3.39) establishes existence of a limiting decay rate for the on-axis functions $k_{0,L} \equiv k_{0,(L,0,\ldots,0)}$ and an a priori bound of the form (3.31) for the off-axis functions $k_{0,(L,\mathbf{a})}$. Then we can prove existence of a limiting decay rate for $\mathbb{K}_L$ by "squeezing" with an inequality of the form (3.9). Notive, however, that this does not establish an a priori bound on $\mathbb{K}_L$.

Remark. At several points in our subsequent analysis, we will need to use the fact that the direct connectivity function is essentially independent of boundary conditions. The strongest form of independence is of course uniform bounds analogous to those of Proposition 3.1 (iii). Uniform bounds for the direct connectivities are provided in Proposition 3.4 below, the proof of which appears in an appendix. The proof is substantially more complicated than that for the ordinary connectivities (i.e. $\mathbb{G}_L$ and $\mathbb{H}_L$), since, with free boundary conditions, paths can satisfy the $\mathbb{C}$-condition by rather devious (though improbable) mechanisms.

Proposition 3.4. *Let* $\mathbb{C}_L(p)$ *and* $\mathbb{K}_L(p)$ *be defined as in* (3.25) *and* (3.38), *and let the decay rates* $\xi(p)$, $\xi_c(p)$ *and* $\xi_k(p)$ *be given as in* (1.2), *and Propositions* 3.2 *and* 3.3. *Then for all* $p\in(0,p_c)$, *either* $\xi_c(p)=\xi_k(p)=\frac{1}{2}\xi(p)$ *or* $\exists D(p)<\infty$ *such that*

$$\mathbb{K}_L(p)\leqq D(p)\,\mathbb{C}_L(p)$$

uniformly in L.

4. The Ornstein-Zernike equation and its consequences

In this section, we derive an Ornstein-Zernike equation, i.e. a renewal equation relating the cylinder functions to the direct correlation functions. Then we prove that *if* $\xi_c<\xi$, the Ornstein-Zernike equation implies analogues of results (I)–(III) of the introduction for the *cylinder* functions. The fact that $\xi_c<\xi$ for all $p<p_c$ is established in the next section.

Proposition 4.1. *Let the functions* $h_{0,(L,\mathbf{a})}$, $\mathbb{H}_L(p)$, $c_{0,(L,\mathbf{a})}$ *and* $\mathbb{C}_L(p)$ *be defined as in* Eqs. (3.2), (3.3), (3.24), *and* (3.25), *with the conventions* $h_{0,(0,\mathbf{a})}=f^{-1}(p)\,\delta_{\mathbf{a}=0}$, $f(p)\equiv(1-p)^{-2(d-1)}$, *so that* $\mathbb{H}_0(p)=f^{-1}(p)$ *and* $c_{0,(0,\mathbf{a})}=0$ *so that* $\mathbb{C}_0(p)=0$. *Then* $\forall L\geqq 1$

$$h_{0,(L,\mathbf{a})}=f(p)\sum_{N=0}^{L}\sum_{b\in\mathbf{P}(N)}c_{0,(N,\mathbf{b})}\,h_{0,(L-N,\mathbf{a}-\mathbf{b})}$$

and thus

$$\mathbb{H}_L=f(p)\sum_{N=0}^{L}\mathbb{C}_N\,\mathbb{H}_{L-N}.$$

Proof. First, for $L=1$, the equation is satisfied by noting that $h_{0,(1,\mathbf{a})}=c_{0,(1,\mathbf{a})}\forall\mathbf{a}$. Now take $L\geqq 2$ and consider a configuration $\omega\in\mathscr{h}_{0,(L,\mathbf{a})}$. There are two possibilities: either there is a j with $1\leqq j\leqq L-1$ such that

$$|C(0)|\!|_{\mathbf{S}(L)}\cap\mathbf{P}(j)|=1 \tag{4.1}$$

or no such j exists. In the latter case, $\omega\in\mathscr{c}_{0,(L,\mathbf{a})}$.

In the former case, let the smallest j satisfying (4.1) be denoted by N. Then, it is clear that $\omega\in\mathscr{c}_{0,(N,\mathbf{b})}$ for some $\mathbf{b}\in\mathbb{Z}^{d-1}$, while $T^{(-N,-\mathbf{b})}\omega\in\mathscr{h}_{0,(L-N,\mathbf{a}-\mathbf{b})}$. Note that the only intersection of the regions in which $\mathscr{c}_{0,(N,\mathbf{b})}$ and $\mathscr{h}_{(N,\mathbf{b}),(L,\mathbf{a})}$ are defined is the bounding plane $\mathbf{P}(N)$, where both events require simply that the $2(d-1)$ bonds emanating from $(N,\mathbf{b})$ be vacant.

Conversely, given an $\omega_1|_{\mathbf{S}(N)}\in\mathscr{c}_{0,(N,\mathbf{b})}$ and an $\omega_2|_{\mathbf{S}(L-N)}\in\mathscr{h}_{0,(L-N,a-b)}$, we can "patch" them together to form an $\omega\,\iota_{\mathbf{S}(L)}\subset\mathscr{h}_{0,(L,\mathbf{a})}$. Moreover, this ω satisfies the conditions that that the smallest j obeying (4.1) is $j=N$ and that the unique point in the plane $\mathbf{P}(N)$ belonging to the connected cluster of the origin and

$(L, \mathbf{a})$ is $(N, \mathbf{b})$. Finally, note that these ω_1 and ω_2 separately satisfy the condition that the $2(d-1)$ bonds emanating from $(N, \mathbf{b})$ in $\mathbf{P}(N)$ be unoccupied.

Thus, we have

$$h_{0,(L,\mathbf{a})} = (1-p)^{-2(d-1)} \sum_{N=0}^{L} \sum_{\mathbf{b} \in \mathbf{P}(N)} c_{0,(N,\mathbf{b})} \, h_{0,(L-N,\mathbf{a}-\mathbf{b})} \tag{4.2}$$

as desired. □

In order to derive results (I)–(III), we consider the transforms of the connectivity functions

$$\hat{h}(z, \mathbf{k}; p) = f(p) \sum_{L,\mathbf{a}} h_{0,(L,\mathbf{a})}(p) \, z^L \, e^{i\mathbf{k}\cdot\mathbf{a}} \tag{4.3a}$$

$$\hat{\mathbb{H}}(z; p) = f(p) \sum_{L} \mathbb{H}_L(p) \, z^L = \hat{h}(z, 0; p) \tag{4.3b}$$

and

$$\hat{c}(z, \mathbf{k}; p) = f(p) \sum_{L,\mathbf{a}} c_{0,(L,\mathbf{a})}(p) \, z^L e^{i\mathbf{k}\cdot\mathbf{a}} \tag{4.4a}$$

$$\hat{\mathbb{C}}(z; p) = f(p) \sum_{L} \mathbb{C}_L(p) \, z^L = \hat{c}(z, 0; p). \tag{4.4b}$$

Henceforth, we will occasionally suppress the argument p in the transform functions.

In terms of these transforms, the Ornstein-Zernike equation is simply

$$\hat{h}(z, \mathbf{k}; p) = \frac{1}{1-\hat{c}(z, \mathbf{k}; p)}. \tag{4.5}$$

Information on the connectivity functions can be recovered from (4.5) via contour integration:

$$h_{0,(L,\mathbf{a})}(p) = f^{-1}(p) \frac{1}{2\pi i} \oint \frac{dz}{z^{L+1}} \int_{[-\pi,+\pi]^{d-1}} \frac{1}{1-\hat{c}(z, \mathbf{k}; p)} \cdot e^{-i\mathbf{k}\cdot\mathbf{a}} \frac{d\mathbf{k}}{(2\pi)^{d-1}}. \tag{4.6}$$

The utility of the condition $\xi_c < \xi$ should now be clear: Although $\hat{h}$ is a priori defined only for $|z| < e^{1/\xi}$, if $\xi_c < \xi$, then (4.5) provides a meromorphic extension of $\hat{h}$ to a disk of a larger radius. In order to exploit this, we will need a few properties of $\hat{c}$, as summarized below:

Proposition 4.2. *Let* $c_{0,(L,\mathbf{a})}$ *and* $\mathbb{C}_L(p)$ *be defined as in* Eqs. (3.24) *and* (3.25), *with the decay rate* $\xi_c(p)$ *given by Proposition 3.2. Then the transform functions defined by* Eq. (4.4) *satisfy:*

(i) $\forall p<p_c, \forall |z|<e^{1/\xi_c(p)}$
(a) $\hat{c}(z,\mathbf{k};p)$ *is an even function of each component of* $\mathbf{k}$;
(b) $\forall \mathbf{k}$ *with* $|\mathrm{Re}(k_i)|\leqq\pi, \mathrm{Im}(k_i)=0$
$|\hat{c}(z,\mathbf{k};p)|\leqq\hat{c}(|z|,0;p)\equiv\hat{\mathbb{C}}(|z|;p)$;
(ii) $\forall p<p_c, \forall |z_0|)<e^{1/\xi_c(p)}, \exists\,\delta_k=\delta_k(p,z_0)$
>0 *such that* $\hat{c}(z,\mathbf{k};p)$ *is analytic in the regions* $|z|\leqq|z_0|$ *and* $|\mathbf{k}|\leqq\delta_k$;
(iii) $\forall p_0<p_c, \forall |z_0|<e^{1/\xi_c(p_0)}, \exists\,\delta_p=\delta_p(p_0,z_0)$
>0 *such that* $\hat{\mathbb{C}}(z;p)$ *is analytic in the regions* $|z|\leqq|z_0|$ *and* $|p|\leqq p_0(1+\delta_p)$.

Proof. (i) These properties follow immediately from positivity and symmetry of the coefficients $c_{0,(L,\mathbf{a})}$.

(ii) Let $|z_0|=e^{1/\xi_c-2\varepsilon}$. It suffices to show that $\exists\,\delta_k(\varepsilon)>0$ such that

$$\sum_{L,\mathbf{a}} c_{0,(L,\mathbf{a})}\,z^L e^{i\mathbf{k}\cdot\mathbf{a}} \tag{4.7}$$

converges absolutely in the regions $|z|\leqq|z_0|$ and $|k|\leqq\delta_k(p,\varepsilon)$. Indeed, take z such that $|z|\leqq|z_0|$ and $\mathbf{k}\,|\mathbf{k}|\leqq\varepsilon\xi_c/\xi$. Then

$$|\sum_{L,\mathbf{a}} c_{0,(L,\mathbf{a})}\,z^L e^{i\mathbf{k}\cdot\mathbf{a}}|\leqq\sum_{L,\mathbf{a}} c_{0,(L,\mathbf{a})}\,e^{[1/\xi_c-2\varepsilon]L}e^{[\varepsilon\xi_c/\xi]|\mathbf{a}|}. \tag{4.8}$$

We divide the sum over $\mathbf{a}$ into two regions: $|\mathbf{a}|\leqq[\xi/\xi_c]\,L$ and $|\mathbf{a}|>[\xi/\xi_c]\,L$. Using the a priori upper bound on $\mathbb{C}_L$ from Proposition 3.2, the "small $\mathbf{a}$" sum is bounded by

$$(\text{const.})\sum_L[(\xi/\xi_c)\,L]^{d-1}e^{-[1/\xi_c]L}e^{[1/\xi_c-2\varepsilon]L}e^{\varepsilon L}<\infty. \tag{4.9}$$

Now, using the obvious bound $c_{0,(L,\mathbf{a})}\leqq e^{-|\mathbf{a}|/\xi}$, the "large $\mathbf{a}$" sum is less than

$$(\text{const.})\sum_L e^{-[1/\xi_c]L}e^{[1/\xi_c-2\varepsilon]L}e^{\varepsilon L}<\infty, \tag{4.10}$$

which establishes the absolute convergence of (4.7).

(iii) Let us begin by defining a "lattice animal" generalization of $\mathbb{C}_L(p)$. Thus consider

$$\mathbb{A}_L(r,q)\equiv\sum_{n,m}\Gamma_{nm}(L)\,r^n q^m, \tag{4.11}$$

where $\Gamma_{nm}(L)\geqq 0$ is the number of clusters of volume (number of bonds) n and boundary (number of bonds outside the cluster but connected to it) m which connect the origin and the point $(L,0,\ldots,0)$, which have only a single point of intersection with each of the planes $P(0)$ and $P(L)$, and which obey the $\mathbb{C}$-condition (Eq. (3.26)) in each of the intermediate planes $P(j)$, $1\leqq j\leqq L-1$. Notice that $\Gamma_{nm}(L)=0$ unless $m\leqq 2(2d-1)n$.

Clearly $\mathbb{C}_L(p)=\mathbb{A}_L(p,1-p)$. Thus analyticity of $\hat{\mathbb{A}}(z;r,q)=\sum_L \mathbb{A}_L(r,q)\,z^L$ in z, r and q certainly implies analyticity of $\hat{\mathbb{C}}_L(z;p)$ in z and p. In fact we will prove the stronger result.

For future reference, note that the tail of the sum in (4.11) is bounded by the tail of the finite cluster distribution, which is in turn known (cf. Eq. (2.8)) to be exponentially bounded for $p<p_c$:

$$(4.12)\qquad \sum_{\substack{n>n_0\\ m}} \Gamma_{nm}(L)\,p^n(1-p)^m \leqq \mathrm{Prob}_p(|C(0)|>m_0) \leqq (\text{const.})\,e^{-\kappa(p)n_0},$$

with $\kappa(p)>0$ for $p<p_c$.

Let $p_0<p_c$ and $|z_0|=e^{1/\xi_c(p_0)-2\varepsilon}$. As explained above, it is enough to show that $\exists\,\delta_r(p_0,\varepsilon),\,\delta_q(p_0,\varepsilon)>0$ such that

$$(4.13)\qquad \sum_{L,n,m} z^L\,\Gamma_{nm}(L)\,r^n q^m$$

converges absolutely in the regions $|z|\leqq|z_0|$, $|r|\leqq|p_0|(1+\delta_r)$, $|q|\leqq(1-p_0)(1+\delta_q)$. Indeed, take z such that $|z|\leqq|z_0|$, r such that $|r|\leqq p_0\,e^{\varepsilon_p}$ and q such that $|q|\leqq(1-p_0)\,e^{\varepsilon_p}$ with $\varepsilon_p=[4d-1]^{-1}\varepsilon\kappa(p_0)\,\xi_c(p_0)$. (Recall that $\kappa(p_0)$ is the decay rate of the finite cluster distribution.) Then

$$(4.14)\qquad \sum_{L,n,m} z^L\,\Gamma_{nm}(L)\,r^n q^m \leqq \sum_{L,n,m} e^{[1/\xi_c(p_0)-2\varepsilon]L}e^{\varepsilon_p[n+m]}\cdot\Gamma_{nm}(L)\,p_0^n(1-p_0)^m.$$

As in the proof of (ii), we divide the sum over n into two regions: $n\leqq n_0$ and $n>n_0$, with $n_0=[\kappa(p_0)\,\xi_c(p_0)]^{-1}L$. For each term in the "small n" sum, we bound $e^{\varepsilon_p[n+m]}$ by its maximum value $e^{\varepsilon_p[n_0+2(2d-1)n_0]}=e^{\varepsilon L}$, and then relax the restriction that $n\leqq n_0$, to obtain the upper bound

$$(4.15)\qquad \begin{aligned}&(\text{const.})\sum_L 2(2d-1)(n_0\,L)^2\,e^{[1/\xi_c-2\varepsilon]L}e^{\varepsilon L}\,\mathbb{C}_L(p_0)\\ &\leqq(\text{const.})\sum_L 2(2d-1)(n_0\,L)^2\,e^{[1/\xi_c-2\varepsilon]L}e^{\varepsilon L}e^{-[1/\xi_c]L}<\infty.\end{aligned}$$

On the other hand, according to (4.12), the "large n" sum is also bounded:

$$(4.16)\qquad \begin{aligned}&(\text{const.})\sum_L e^{[1/\xi_c-2\varepsilon]L}e^{\varepsilon L}\sum_{\substack{n>n_0\\ m}}\Gamma_{nm}(L)\,p_0^n(1-p_0)^m\\ &\leqq(\text{const.})\sum_L e^{[1/\xi_c-2\varepsilon]L}e^{\varepsilon L}e^{-\kappa[\kappa\xi_c]^{-1}L}<\infty,\end{aligned}$$

which establishes the absolute convergence of the sum (4.13). □

Next we derive the consequences of the Ornstein-Zernike equation: first, a lemma we will need to control the renormalization scheme of Sect. 5, and then, results (I)–(III) for the cylinder functions. Results (I) and (II) for the free connectivity function are established in Sect. 6. Much of the analysis in the rest of this section parallels previous analyses of Ornstein-Zernike equations in other systems, some of which is repeated here for completeness.

Lemma 4.3. ([CC2], Proposition 4.1). *Let* $\mathbb{H}_L(p)$ *and* $\mathbb{C}_L(p)$ *be defined as in* Eqs. (3.2), (3.3), (3.24), (3.25) *with the decay rate* $\xi(p)$ *of* $\mathbb{H}_L$ *given by Proposition* 3.1 (i). *Remember that* $\mathbb{H}_L$ *satisfies the corollary to Proposition* 3.1, *and that the transform functions* $\hat{\mathbb{H}}(z)$ *and* $\hat{\mathbb{C}}(z)$ *are related by (the* $\mathbf{k}=0$ *form of) equation* (4.5): $\hat{\mathbb{H}}(z)=[1-\hat{\mathbb{C}}(z)]^{-1}$. *Then:*

(i) $\hat{\mathbb{C}}(e^{1/\xi})=1$; *and*

(ii) $\sum_{L=0}^{\infty} L\mathbb{C}_L\, e^{L/\xi}<\infty$.

By Proposition 3.4, we also have:

Corollary. $\sum_{L=0}^{\infty} L\mathbb{K}_L\, e^{L/\xi}<\infty$.

Proof. The proof given in [CC2] holds without any modification. One simply notes that the left hand sides of (i) and (ii) can be expressed as the $x\uparrow e^{1/\xi}$ limits of $\hat{\mathbb{C}}(x)$ and $x\frac{d}{dx}\hat{\mathbb{C}}(x)$, respectively, for $x\in\mathbb{R}^+$. These quantities can then be bounded, via the Ornstein-Zernike equation, by computing $\hat{\mathbb{H}}(x)=\sum\mathbb{H}_L x^L$ using the upper and lower bounds on $\mathbb{H}_L$ from the corollary to Proposition 3.1. □

Theorem 4.4. *Let the functions* $h_{0,(L,\mathbf{a})}$, $\mathbb{H}_L(p)$, $c_{0,(L,\mathbf{a})}$ *and* $\mathbb{C}_L(p)$ *be defined as in* Eqs. (3.2), (3.3), (3.24), *and* (3.25), *with decay rates* $\xi(p)$ *and* $\xi_c(p)$ *as given by Propositions* 3.1 (i) *and* 3.2 (i). *Remember that* $\mathbb{H}_L(p)$ *obeys the corollary to Proposition* 3.1. *Suppose further that the transformed functions* $\hat{h}(z,\mathbf{k};p)$ *and* $\hat{c}(z,\mathbf{k};p)$ *are related by an Ornstein-Zernike equation of the form* (4.5), *and remember that* $\hat{c}(z,\mathbf{k};p)$ *satisfies the conclusions of Proposition* 4.2 *and Lemma* 4.3 (i). *Then whenever* $\xi_c(p)<\xi(p)$:

(I) $\exists\tilde{K}_2(p)\leqq 1, \Delta(p)>0$ *such that*

$$|\mathbb{H}_L(p)\,e^{+L/\xi(p)}-\tilde{K}_2(p)|\leqq e^{-\Delta(p)L}.$$

(II) $\exists\alpha(p)>0$ *such that* $\forall\mathbf{a}\in\mathbb{Z}^{d-1}$ *satisfying* $|\mathbf{a}|\leqq L^{3/4-\varepsilon}$ *with* $\varepsilon>0$,

$$h_{0,(L,\mathbf{a})}\sim\tilde{K}_2(p)\frac{1}{[\alpha(p)\,\pi L]^{(d-1)/2}}\,e^{-L/\xi(p)}\,e^{-\mathbf{a}^2/[\alpha(p)L]}[1+O(L^{-1},L^{-4\varepsilon})].$$

In particular:

(i) *The expression above represents the first term in an asymptotic expansion: i.e., for any fixed function* $\mathbf{a}(L)$, $[\mathbf{a}(L)|\leqq L^{3/4-\varepsilon}$, *with* $\mathbf{a}(L)$ *tending to infinity (e.g. as a power of L), the* $O(L^{-1},L^{-4\varepsilon})$ *terms can be systematically calculated in an asymptotic series.*

(ii) *The error term is uniform in* **a**: *i.e.* $\exists d_1(p),\ d_2(p)<\infty$ *such that* $\forall \mathbf{a}\in\mathbb{Z}^{d-1}$ *satisfying* $|\mathbf{a}|\leqq L^{3/4-\varepsilon}$

$$|[h_{0,(L,\mathbf{a})}][\alpha(p)\,\pi L]^{(d-1)/2}\,e^{+L/\xi(p)}\,e^{+\mathbf{a}^2/[\alpha(p)L]}-\tilde{K}_2(p)|\leqq\frac{d_1(p)}{L}+\frac{d_2(p)}{L^{4\varepsilon}}.$$

(iii) *The tail of the distribution is uniformly bounded*: *i.e.* $\exists d_3(p)<\infty$ *such that* $\forall \mathbf{a}\in\mathbb{Z}^{d-1}$ *satisfying* $|\mathbf{a}|\leqq L^{3/4-\varepsilon}$

$$\sum_{\mathbf{b}:|b_j|\geqq|a_j|} h_{0,(L,\mathbf{b})}\leqq d_3(p)\,L^{(d-1)/2}\,e^{-L/\xi(p)}\,e^{-\mathbf{a}^2/[\alpha(p)L]}$$

where a_j *and* b_j *are the* j^{th} *components of the vectors* **a** and **b**.

(III) $\xi(p)$ *is real analytic.*

Remark. The hypothesis $\xi_c(p)<\xi(p)$ is verified for all $p<p_c$ in Theorem 5.4; thus (I)–(III) hold throughout the low-density phase.

Proof. Much of the following proof can be found in the union of ([ACC2], Theorems 2.4, 2.7, 4.1 and 5.5; Lemmas B.1 and B.2) and ([CC3], Theorems 5.10 and 5.11). There, however, a complete asymptotic analysis along the lines of equations (4.31)–(4.42) of the following proof was not done explicitly, and hence in [CC2] it was (overenthusiastically) concluded that an analogue of (II) holds $\forall|\mathbf{a}|<\eta L$, for some $\eta>0$, rather than $\forall|\mathbf{a}|\leqq|\mathbf{a}|\leqq L^{3/4-\varepsilon}$. Actually for $L^{3/4}\leqq L^{1-\varepsilon}$, one can derive an analogue of (II); however, it need not have the Gaussian form. Here we present a complete proof.

(I) By positivity of coefficients, $\hat{\mathbb{H}}(z)$ is analytic for $|z|<e^{1/\xi}$ and $\hat{\mathbb{C}}(z)$ is analytic for $|z|<e^{1/\xi_c}$. Thus by the Ornstein-Zernike equation and the asumption $\xi_c(p)<\xi(p)$, $\hat{\mathbb{H}}(z)$ has a meromorphic extension to the larger region $|z|<e^{1/\xi_c}$. This implies that $\exists\Delta\in(0,1/\xi_c-1/\xi)$ such that the only singularity of $\hat{\mathbb{H}}(z)$ in the region $|z|\leqq e^{1/\xi+\Delta}$ occurs at $z=e^{1/\xi}$; furthermore, by the a priori upper bound of Proposition 3.1, this singularity is a simple pole. Thus we have $\hat{\mathbb{H}}(z)=F(z)[1-ze^{-1/\xi}]^{-1}$ with $F(z)$ analytic for $|z|\leqq e^{1/\xi+\Delta}$. In terms of the coefficients of F, $\mathbb{H}_L$ may be written as

$$\mathbb{H}_L=f^{-1}(p)\,e^{-L/\xi}\sum_{n=0}^{L}F_n\,e^{n/\xi}. \tag{4.17}$$

The Cauchy bound: $|F_n|\leqq(\text{const})(e^{1/\xi+\Delta})^{-n}$ now implies the desired result, with $\tilde{K}_2(p)=f^{-1}(p)\,F(e^{1/\xi(p)})$. The fact that $\tilde{K}_2(p)\leqq 1$ follows from the a priori bound $\mathbb{H}_L\leqq e^{-L/\xi}$.

(II) This proof amounts to an analysis of the contour integral in equation (4.6); thus in the following, we will restrict to **k** with $|\mathrm{Re}(k_i)|\leqq\pi$, $\mathrm{Im}(k_i)=0$. First we want to show that the only significant contribution to the integral occurs in the neighborhood of $|\mathbf{k}|=0$. Explicitly, we claim that $\forall\delta\in(0,\pi],\exists t(\delta)>0,\beta_1(\delta)<\infty$ such that

$$\left|f^{-1}(p)\frac{1}{2\pi i}\oint\frac{dz}{z^{L+1}}\int_{|\mathbf{k}|\geqq\delta}\frac{1}{1-\hat{c}(z,\mathbf{k};p)}\,e^{-i\mathbf{k}\cdot\mathbf{a}}\frac{d\mathbf{k}}{(2\pi)^{d-1}}\right| \tag{4.18}$$
$$\leqq\beta_1\,e^{-(1/\xi+t)L}$$

uniformly in L. To prove this, it suffices to show that $\forall|\mathbf{k}|\geqq\delta$, $\exists t(\delta)>0$, $s(\delta)>0$ such that

$$|\hat{c}(z,\mathbf{k};p)|\leqq 1-s(\delta) \tag{4.19}$$

whenever $|z|\leqq e^{1/\xi+t}$. Indeed, given (4.19), one may simply integrate (4.18) about the circle $|z|=e^{1/\xi+t}$, bounding the integrand by $1/s(\delta)$.

To prove (4.19), express $\hat{c}(z,\mathbf{k})$ as a power series in z with coefficients $c_L(\mathbf{k})$, and note that, by Proposition 4.2(ia), each $c_L(\mathbf{k})$ is a cosine series with positive coefficients. Hence, provided that $c_L(\mathbf{k})$ has a non-trivial $\cos k_j$ term for every j (which here occurs for $L\geqq 2$), $c_L(\mathbf{k})$ is *strictly* maximized by $|\mathbf{k}|=0$. Thus we have a strengthened form of Proposition 4.2(ib): $\forall|\mathbf{k}|\geqq\delta$, $\exists v(\delta)>0$ such that

$$|\hat{c}(z,\mathbf{k};p)|\leqq\hat{\mathbb{C}}(|z|)-2v|z|^2. \tag{4.20}$$

Now by Lemma 4.3(i), $\hat{\mathbb{C}}(e^{1/\xi})=1$. Furthermore, $\hat{\mathbb{C}}(|z|)$ is a strictly monotone increasing, analytic function of $|z|$ for $|z|<e^{1/\xi_c}$. Thus (4.20) implies that $\exists t(\delta)>0$, $s(\delta)>0$ such that (4.19) is satisfied for all $|z|\leqq e^{1/\xi+t}$, as claimed.

Now we focus on the "small $\mathbf{k}$" integral. The first step here is to determine the analytic structure of the integrand. Recall from the proof of (I) that the function $1-\hat{\mathbb{C}}(z)=1-\hat{c}(z,0)$ has a simple zero at $z=e^{1/\xi}$ and no other zeroes in the region $|z|\leqq e^{1/\xi+\Delta}$. We claim that the structure is similar for $|\mathbf{k}|$ sufficiently small: namely, $\exists\delta_2(\Delta)>0$ such that $\forall|\mathbf{k}|<\delta_2$, $1-\hat{c}(z,\mathbf{k})$ has a simple zero and no other zeroes in the region $|z|<e^{1/\xi+\Delta}\equiv z_0$. Indeed, by Proposition 4.2(ii), $\exists\delta_1(z_0)>0$ such that $1-\hat{c}(z,\mathbf{k})$ is analytic for $|\mathbf{k}|<\delta_1$, $|z|\leqq|z_0|<e^{1/\xi_c}$. Thus take $|\mathbf{k}|<\delta_1$. Then provided that

$$|1-\hat{\mathbb{C}}(z)|\geqq|\hat{\mathbb{C}}(z)-\hat{c}(z,\mathbf{k})| \tag{4.21}$$

along the contour $|z|=e^{1/\xi+\Delta}<e^{1/\xi_c}$, Rouché's Theorem says that $1-\hat{c}(z,\mathbf{k})$ has the same number and multiplicity of zeroes inside the contour as does $1-\hat{\mathbb{C}}(z)$ (i.e. one simple zero). Now since $\Delta>0$, $|1-\hat{\mathbb{C}}(z)|$ is uniformly positive along the contour $|z|=e^{1/\xi+\Delta}$. Thus, given analyticity in $\mathbf{k}$, it is clear that $\exists\delta_2(\Delta)>0$, $\delta_2\leqq\delta_1$, such that (4.21) is satisfied $\forall|\mathbf{k}|<\delta_2$, as claimed.

Next, we will determine the form of this zero for $|z|<e^{1/\xi+\Delta}$, $|\mathbf{k}|<\delta_2$. Since $\delta_2\leqq\delta_1$, $\hat{c}(z,\mathbf{k})$ is analytic in this region. Thus, by the analytic implicit function theorem, $\exists\delta_3>0$, $\delta_3\leqq\delta_2$, such that for $|\mathbf{k}|<\delta_3$, the equation

$$1-\hat{c}(z,\mathbf{k})=0 \tag{4.22}$$

has a unique, analytic solution, denoted by $e^{1/\xi(\mathbf{k})}$, describing the motion of the zero in this region. By Proposition 4.2(ia), the expansion of $\xi^{-1}(\mathbf{k})$ contains only even powers of $\mathbf{k}$. Furthermore, the coefficients in the expansion are real. Indeed, $\mathbf{k}\in\mathbb{R}^{d-1}$ implies $\hat{c}(z,\mathbf{k})=\overline{\hat{c}(\bar{z},\mathbf{k})}$, which in turn means that the solutions to (4.22) occur in complex conjugate pairs; uniqueness of the solution then implies $\xi^{-1}(\mathbf{k})\in\mathbb{R}$. Finally, it is easy to verify that $\xi^{-1}(\mathbf{k})$ has a nontrivial quadratic minimum. Thus for $|\mathbf{k}|<\delta_3$, we may write

$$\xi^{-1}(\mathbf{k})=\xi^{-1}(0)+(\alpha/4)\,\mathbf{k}^2+R(\mathbf{k}) \tag{4.23}$$

with $\alpha(p)>0$, and $\xi^{-1}(0;p)=\xi^{-1}(p)\equiv\xi^{-1}$, the inverse correlation length. Finally, we may choose a $\delta>0$, $\delta\leqq\delta_3$, such that for $|\mathbf{k}|<\delta$

$$|R(\mathbf{k})|\leqq\gamma\mathbf{k}^4, \tag{4.24}$$

with γ satisfying

$$\gamma\delta^2\leqq\tfrac{1}{8}\alpha. \tag{4.25}$$

Now we can apply (4.18) to the integral for $|\mathbf{k}|\geqq\delta$, and perform the integral for $|\mathbf{k}|<\delta$. By the above reasoning, for $|\mathbf{k}|<\delta$ we can write

$$\hat{h}(z,\mathbf{k})=F(z,\mathbf{k})[1-ze^{-1/\xi(\mathbf{k})}]^{-1} \tag{4.26}$$

with $\xi^{-1}(\mathbf{k})$ given by (4.23)–(4.25), and $F(z,\mathbf{k})$ analytic for $|z|<e^{1/\xi+\Delta}$. Then following exactly the Cauchy bound proof in (I), we see that

$$\begin{aligned} & f^{-1}(p)\frac{1}{2\pi i}\oint\frac{dz}{z^{L+1}}\int_{|\mathbf{k}|<\delta}\frac{1}{1-\hat{c}(z,\mathbf{k};p)}e^{-i\mathbf{k}\cdot\mathbf{a}}\frac{d\mathbf{k}}{(2\pi)^{d-1}} \\ &=[1+O(e^{-\Delta L})]\int_{|\mathbf{k}|<\delta}\tilde{K}_2(\mathbf{k})\,e^{-L/\xi(\mathbf{k})}e^{-\mathbf{k}\cdot\mathbf{a}}\frac{d\mathbf{k}}{(2\pi)^{d-1}} \end{aligned} \tag{4.27}$$

with $\tilde{K}_2(\mathbf{k};p)=f^{-1}(p)\,F(e^{1/\xi(\mathbf{k};p)},\mathbf{k})$ analytic for $|\mathbf{k}|<\delta$, and $\tilde{K}_2(0;p)=\tilde{K}_2(p)\equiv\tilde{K}_2$, the same constant appearing in (I). The remainder of the proof of (II i)–(II iii) amounts to the evaluation of the integral in (4.27).

To establish (II i) and (II ii), suppose that $\mathbf{a}\in\mathbb{Z}^d$ satisfies $|\mathbf{a}|\leqq L^{3/4-\varepsilon}$ for some $\varepsilon>0$. We divide the integral over $|\mathbf{k}|<\delta$ into two regions: $g(L)\leqq|\mathbf{k}|<\delta$ and $|\mathbf{k}|<g(L)$ with $g(L)=L^{-1/4-\varepsilon/2}$. The first integral is easy to bound. Indeed, using (4.23)–(4.25), we have

$$\begin{aligned} & \left|\int_{g(L)\leqq|\mathbf{k}|<\delta}\tilde{K}_2(\mathbf{k})\,e^{-L/\xi(\mathbf{k})}e^{-i\mathbf{k}\cdot\mathbf{a}}\frac{d\mathbf{k}}{(2\pi)^{d-1}}\right| \\ &\leqq e^{-L/\xi}\int_{g(L)\leqq|\mathbf{k}|}|\tilde{K}_2(\mathbf{k})|\,e^{-(\alpha/8)\mathbf{k}^2L}|\frac{d\mathbf{k}}{(2\pi)^{d-1}} \\ &\leqq\beta_2\,e^{-L/\xi}\exp[-(\alpha/8)\,L^{1/2-\varepsilon}], \end{aligned} \tag{4.28}$$

with $\beta_2=\beta_2(p)<\infty$. Notice that, in the worst case, (4.28) is smaller than the "purported answer" by a factor of $\exp[-(\text{const})\,L^{\varepsilon}]$.

We now do the integral in (4.27) for $|\mathbf{k}|<g(L)$. We claim that the major contribution comes from the quadratic piece of $\xi^{-1}(\mathbf{k})$, i.e. the gaussian integral:

$$\begin{aligned} & \tilde{K}_2\,e^{-L/\xi}\int_{|\mathbf{k}|<g(L)}e^{-(\alpha/4)\mathbf{k}^2L}e^{-i\mathbf{k}\cdot\mathbf{a}}\frac{d\mathbf{k}}{(2\pi)^{d-1}} \\ &=[\operatorname{erf}[(\alpha L)^{1/2}g(L)]]^{d-1}\tilde{K}_2\frac{1}{[\pi\alpha L]^{(d-1)/2}}e^{-L/\xi(p)}e^{-\mathbf{a}^2/[\alpha L]}. \end{aligned} \tag{4.29}$$

Here the error is utterly negligible since

$$\operatorname{erf}[(\alpha L)^{1/2}g(L)]=1+O(\exp[-\alpha L^{1/2-\varepsilon}]). \tag{4.30}$$

We must now demonstrate that the non-quadratic piece of $\xi^{-1}(\mathbf{k})$ yields the claimed correction. The remainder in the region $|\mathbf{k}|<g(L)$ may be expressed as:

$$(4.31)\qquad e^{-L/\xi(p)} \int_{|\mathbf{k}|<g(L)} e^{-(\alpha/4)L\mathbf{k}^2} e^{-i\mathbf{k}\cdot\mathbf{a}} [\tilde{K}_2(\mathbf{k})\, e^{LR(\mathbf{k})} - \tilde{K}_2] \frac{d\mathbf{k}}{(2\pi)^{d-1}}$$
$$= \frac{e^{-L/\xi(p)} e^{-\mathbf{a}^2/[\alpha(p)L]}}{[\pi\alpha L]^{(d-1)/2}} \int_{|\mathbf{s}|<f(L)} \exp\left[-\left(\mathbf{s}+\frac{i\mathbf{a}}{[\alpha L]^{1/2}}\right)^2\right]$$
$$\times \left\{\tilde{K}_2\left(\frac{\mathbf{s}}{[(\alpha/4)\,L]^{1/2}}\right) \exp\left[LR\left(\frac{\mathbf{s}}{[(\alpha/4)\,L]^{1/2}}\right)\right] - \tilde{K}_2\right\} d\mathbf{s}$$

with $f(L)=(1/2)\,\alpha^{1/2}L^{1/4-\varepsilon/2}$. In order to evaluate (4.31), for each component s_j of $\mathbf{s}$, we must do the integral about the contours enclosing the region $|\mathrm{Re}(s_j)| <(1/2)\,\alpha^{1/2}L^{1/4-\varepsilon/2}$ and $0< -\mathrm{Im}(s_j)< a_j/[\alpha L]^{1/2} \leqq \alpha^{-1/2}L^{1/4-\varepsilon}$. Clearly, the function R in (4.31) makes sense (i.e., its argument is small), and the integrand in (4.31) contains no poles in this region. Furthermore, since, at the turning points, $\mathrm{Im}(s_j)/\mathrm{Re}(s_j)=O(L^{-\varepsilon/2})$, the relative contribution of the "vertical pieces" of the contours are $O[\exp(L^{-\varepsilon})]$. Thus we need only evaluate the integral along the "horizontal piece" $\mathrm{Im}(s_j)= -a_j/[\alpha L]^{1/2}$, $|\mathrm{Re}(s_j)| \leqq f(L)$.

Here, if desired, the functions e^{-LR} and $\tilde{K}_2$ may be expanded and the terms explicitly evaluated. Note that this expansion is always legitimate since we are within the domain of analyticity of these functions – indeed, as L gets larger, the arguments get smaller. The range of validity of the series depends, of course, on the nature of $\mathbf{a}(L)$ and on how many terms one wishes to obtain. In particular, there are already several "error" terms which are of the order $O(e^{-L^{\varepsilon}})$, which may be comparable to inverse powers of L when L is rather small. However, as $L\to\infty$ (with some specific $\mathbf{a}(L)$, e.g., obeying upper and lower power law bounds), the expansion represents the correct asymptotic behavior. This establishes (IIi).

Let us now attend to (IIii). Here the nature of $\mathbf{a}(L)$ is unspecified other than the power law *upper* bound $|\mathbf{a}| \leqq L^{3/4-\varepsilon}$. The error term may be written in the form

$$(4.32)\qquad \mathscr{E}(a,L) \equiv \frac{[\alpha\pi L]^{(d-1)/2}}{\tilde{K}_2} e^{+\mathbf{a}^2/[\alpha L]}$$
$$\times e^{+L/\xi} \left| h_{0,L,\mathbf{a})} - \frac{\tilde{K}_2}{[\alpha\pi L]^{(d-1)/2}} e^{-L/\xi} e^{-\mathbf{a}^2/[\alpha L]} \right| .$$

Recalling (4.27)–(4.31), we have the estimate

$$(4.33)\qquad \mathscr{E}(a,L) \leqq \beta_3 \left| \int_{|\mathbf{s}|<f(L)} \exp\left[-\left(\mathbf{s}+\frac{i\mathbf{a}}{[\alpha L]^{1/2}}\right)^2\right] \right.$$
$$\left. \times \left\{\tilde{K}_2\left(\frac{\mathbf{s}}{[(\alpha/4)\,L]^{1/2}}\right) \exp\left[LR\left(\frac{\mathbf{s}}{[(\alpha/4)\,L]^{1/2}}\right)\right] - \tilde{K}_2\right\} d\mathbf{s} \right|$$

with β_3 a finite constant. First observe that fore $\mathbf{k}$ near zero, we may estimate $|\tilde{K}_2(\mathbf{k}) - \tilde{K}_2| \leqq \text{const.}|\mathbf{k}|^2$, so that the $\mathbf{k}$ dependence of $\tilde{K}_2(\mathbf{k})$ produces a contribution to $\mathscr{E}(a, L)$ which may bounded by β'_4/L for some finite β'_4. This is within the stated error. Next, we distort the contour as discussed in the paragraph below Eq. (4.31). Neglecting the vertical pieces, thereby incurring an error of $O[\exp(L^{-\varepsilon})]$, we are left with

$$(4.34) \qquad \mathscr{E}(a, L) \leqq \beta_{3'} \left| \int_{-f(L)}^{f(L)} e^{-\mathbf{x}^2} \left\{ \exp\left[LR\left(\frac{2\mathbf{x}}{[\alpha L]^{1/2}} - \frac{2i\mathbf{a}}{\alpha L} \right)\right] - 1 \right\} \times d^{d-1}\mathbf{x} \right| + \frac{\beta_{4'}}{L}.$$

Now we use explicitly the small argument estimate of Eq. (4.24):

$$(4.35) \qquad |R(u+iv)| \leqq \gamma |u+iv|^4 \leqq 2(u^4+v^4).$$

Thus

$$(4.36) \qquad L\left| R\left(\frac{2\mathbf{x}}{[\alpha L]^{1/2}} - \frac{2i\mathbf{a}}{\alpha L} \right) \right| \leqq 32 \left(\frac{\mathbf{x}^4}{\alpha^2 L} + \frac{\mathbf{a}^4}{\alpha^4 L^3} \right),$$

which is also small since $|\mathbf{x}| < f(L) = (1/2)\,\alpha^{1/2} L^{1/4-\varepsilon/2}$ and $|\mathbf{a}| < L^{3/4-\varepsilon}$. We may now bound the integrand in (4.34) using $|e^y - 1| \leqq e^{|y|} - 1 \leqq 2|y|$ for y small, and apply (4.36) to obtain

$$(4.37) \qquad |\mathscr{E}(a, L)| \leqq \beta_4 \frac{1}{L} + \beta_5 \frac{a^4}{L^3}$$

with $\beta_4 = \beta_4(p)$ and $\beta_5 = \beta_5(p) < \infty$. Thus the dominant correction to (4.29) is $O(L^{-1}, L^{-4\varepsilon})$, as claimed.

Finally, let us attend to (IIiii). First observe that when $|\mathbf{b}|$ is large relative to L, we may use the a priori upper bound (2.7) to estimate the tail of the sum. Thus, for example

$$(4.38) \qquad \sum_{\mathbf{b}: |b_j| > |2L|} h_{0,(L,\mathbf{b})} \leqq (\text{const})\, L^{d-1} e^{-2L/\xi}.$$

Next assume for the moment that each $a_j \geqq 0$. Then we may sum the expression in (4.6) over all b_j in the range $a_j \leqq b_j \leqq 2L$, with the result:

$$(4.39) \qquad \sum_{a_j \leqq b_j \leqq 2L} h_{0,(L,\mathbf{b})} \leqq \sum_{a_j \leqq b_j \leqq 2L + a_j} h_{0,(L,\mathbf{b})}$$
$$= f^{-1}(p) \frac{1}{2\pi i} \oint \frac{dz}{z^{L+1}} \int_{[-\pi, +\pi]^{d-1}} \frac{1}{1 - \hat{c}(z, \mathbf{k}; p)} e^{-i\mathbf{k}\cdot\mathbf{a}} \cdot \left(\prod_{j=1}^{d-1} \frac{1 - e^{-2ik_j L}}{1 - e^{-2ik_j}} \right) \times \frac{d\mathbf{k}}{(2\pi)^{d-1}}.$$

Now we may perform, without modification, the steps (4.18)–(4.28) to obtain a bound on the integral for $|\mathbf{k}| > g(L)$. (Observe that no singularity appears at $k_j=0$ because of the regulating terms $1-e^{-2ik_jL}$.) As before, we may distort the contour for the integral over $|\mathbf{k}|<g(L)$ to obtain:

$$\text{(4.40)} \quad \frac{e^{-L/\xi(p)}e^{-\mathbf{a}^2/[\alpha(p)L]}}{[\pi\alpha L]^{(d-1)/2}} \int_{|\mathbf{x}|<f(L)} e^{-\mathbf{x}^2}\tilde{K}_2\left(\frac{2\mathbf{x}}{[\alpha L]^{1/2}}-\frac{2i\mathbf{a}}{\alpha L}\right) \times \exp\left[LR\left(\frac{2\mathbf{x}}{[\alpha L]^{1/2}}-\frac{2i\mathbf{a}}{\alpha L}\right)\right]\left(\prod_{j=1}^{d-1}\frac{1-e^{ix_j[L/\alpha]^{1/2}}e^{-a_j/\alpha}}{1-e^{-ix_j/[2(\alpha L)^{1/2}]}e^{-a_j/[2\alpha L]}}\right)d\mathbf{s},$$

which is identical to the old result except for the presence of the product. However, for $a_j>0$, it is easy to bound the magnitude of this product. Indeed, the magnitude of each factor in the numerator is bounded above by a constant, while the magnitude of each factor in the denominator is bounded below by (const) L^{-1}. Thus the product is bounded above by (const) L^{d-1}, so that the contribution from the range $0<a_j\leqq b_j\leqq 2L$ yields a result of the desired order of magnitude. The terms which come from the range $b_j\leqq -a_j$ are identical due to lattice symmetry.

In case any of the a_j vanish, we save the j^{th} integral for last and use a separate estimate on the corresponding term in the product, namely

$$\text{(4.41)} \quad \frac{|1-\mathrm{e}^{-2i\omega N}|}{|1-\mathrm{e}^{-i\omega}|}\leqq 2N,$$

which is easily verified, e.g., by induction on N. Thus the contribution from these directions yields the same bound as those with $a_j>0$. Overall, we obtain

$$\text{(4.42)} \quad \sum_{\mathbf{b}:|b_j|\geqq|a_j|} h_{0,(L,\mathbf{b})}\leqq(\text{const})\,L^{d-1}\,h_{0,(L,a)},$$

as claimed.

(III) The proof of this is analogous to that of analyticity of $\xi(\mathbf{k})$ in $\mathbf{k}$ (cf. Eq. (4.22)). Let $p_0<p_c$ and choose z_0 such that $e^{1/\xi(p_0)}<|z_0|<e^{1/\xi_c(p_0)}$. By Proposition 4.2(iii), $\exists\,\delta_1(p_0,z_0)>0$ such that the function $\hat{\mathbb{C}}(z;p)$ is analytic in the region $|z|\leqq|z_0|$, $|p|\leqq p_0(1+\delta_1)$. Then, by the analytic implicit function theorem, $\exists\,\delta_2>0$, $\delta_2\leqq\delta_1$, such that for $|p|\leqq p_0(1+\delta_2)$, the equation $1-\hat{\mathbb{C}}(z;p)=0$ has a unique analytic solution, which we identify as $e^{1/\xi(p)}$. Thus $\xi(p)$ is analytic in a region about any real p such that $\xi_c(p)<\xi(p)$. □

5. Separation of the decay rates

In this section, we prove our principal estimate: namely that the exponential decay rates of the cylinder and direct connectivity functions are strictly separated for all $p<p_c$. This verifies the hypothesis of Theorem 4.4 and thus completes our proof of properties (I)–(III) for the cylinder functions.

As in the analogous proof for SAWs [CC2], our strategy is to define block connectivity functions on slabs of some fixed scale L which interpolate between the free and direct connectivity functions at that scale. Explicitly, these block functions are the probabilities of connections which satisfy the "$\mathbb{C}$-condition" (cf. Eq. (3.26)) over only part of the slab (i.e. over the central region of width

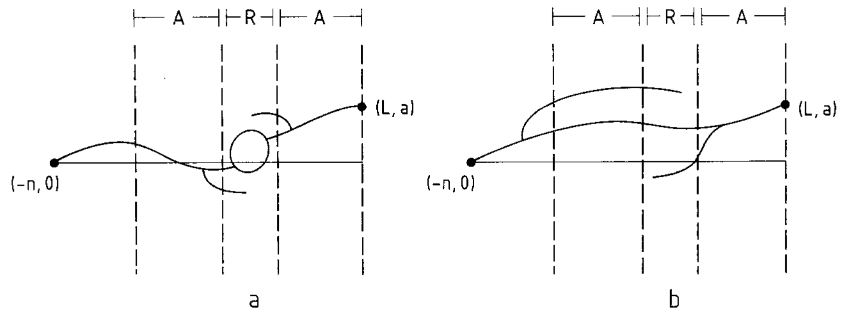

Fig. 3. A "likely" (a) and an "unlikely" (b) configuration in $\mathscr{E}_{1,n}((L, \mathbf{a}); R)$

$R<\frac{1}{3}L$). In Sect. 5a, we will use estimates on the basic connectivity functions to show that the block functions (and hence also the direct functions) decay faster than the cylinder functions, although not necessarily exponentially faster in L.

In order to exploit this extra decay, we require interpolating functions on an even larger scale. Thus, in Sect. 5b, we define rescaled interpolating functions, on scale NL, $N \gg 1$, as the probabilities of connections which obey the $\mathbb{C}$-condition over interior regions (again of width $R<\frac{1}{3}L$) of each slab of scale L. A significant contribution to these rescaled connectivity functions is obtained by simply patching together N block functions. Were this the only contribution, then the extra decay of the block functions in L would imply that the rescaled functions decay *exponentially* faster in N than do the cylinder functions of scale NL. Such an estimate would complete the proof that $\xi_c<\xi$.

Unfortunately, a naive patching argument ignores the contribution of configurations with long branches, e.g., configurations in which the $\mathbb{C}$-condition in one slab is not satisfied until the cluster has ventured into a later slab. These configurations are the analogues of the recurrent walk configurations encountered in the SAW analysis [CC2], although here there are additional possibilities. Our key estimate (Lemma 5.3) shows that whenever $p<p_c$, the probabilities of these "recurrent configurations" is exponentially smaller than that of the configurations obtained by simple patching. Furthermore, the probabilities of these recurrent configurations are related to various rescaled functions via a coupled system of Ornstein-Zernike or renewal *inequalities.* By analyzing the renewal inequalities, using both the bounds on the block functions and the bounds on the probabilities of recurrent configurations, we obtain $\xi_c<\xi$ (Theorem 5.4).

5a. The block functions

As explained above, we will use functions which interpolate between the free and direct functions. We propose (see Fig. 3):

Definition 5.1. Let $A, R \in \mathbb{Z}^+$, $A>R$, $L \equiv 2A+R$. Let $n \in \mathbb{Z}$, $0 \leq n < L$. The *point-to-point block connectivity function* of central scale R from $(-n, 0) \equiv (-n, 0, \ldots, 0)$ to $(L, \mathbf{a})$ is defined by

$$(5.1\,\mathrm{a}) \quad \mathscr{E}_{1,n}((L, \mathbf{a}); R) = \{\omega \,|\, (L, \mathbf{a}) \in C((-n, 0))\|_{\mathbf{H}(L)}, \\ |C((-n, 0))\|_{\mathbf{H}(L)} \cap \mathbf{P}(j)| \geq 2 \ \forall j \text{ with } A \leq j \leq A+R\}$$

(5.1 b) $$b_{1,n}((L,\mathbf{a});R;p)=P(\mathscr{b}_{1,n}((L,\mathbf{a});R)),$$

and the corresponding *point-to-plane block connectivity function is*

(5.1 c) $$\mathbb{B}_{1,n}(L;R;p)=\sum_{\mathbf{a}\in\mathbb{Z}^{d-1}} b_{1,n}((L,\mathbf{a});R;p).$$

We will often suppress the arguments $(L,\mathbf{a})$, R and p in our notation for these functions.

The quantities of principal concern to us here will be the *block connectivities from the origin:*

(5.2) $$\mathscr{b}_1\equiv\mathscr{b}_{1,0},\quad b_1\equiv b_{1,0},\quad \mathbb{B}_1\equiv\mathbb{B}_{1,0};$$

and the *extended block connectivities:*

(5.3) $$\mathscr{a}_1\equiv\bigcup_{n=A}^{A+R}\mathscr{b}_{1,n},\quad a_1\equiv\sum_{n=A}^{A+R} b_{1,n},\quad \mathbb{A}_1\equiv\sum_{n=A}^{A+R}\mathbb{B}_{1,n}.$$

Remark. Since the block functions are required to obey the $\mathbb{C}$-condition over the central region, but are otherwise unconstrained, it is clear that (as promised) the block functions from the origin interpolate between the free and direct functions:

(5.4) $$\mathbb{G}_L(p)\geqq\mathbb{B}_1(L;R;p)\geqq\mathbb{C}_L(p).$$

In particular, the decay rate of $\mathbb{B}_1(L)$ provides a bound on the decay rate of $\mathbb{C}_L$. Were this bound exponentially faster in L than $\xi(p)$, we would be done. Although we cannot establish such an exponential bound, we make the following modest start:

Lemma 5.1. *Let* $\mathbb{B}_{1,n}(L;R;p)$ *be defined as in* Eq. (5.1 c) *with* $L>3R$. *Suppose that* $\mathbb{G}_L$ *satisfies Proposition* 3.1 *with decay rate* $\xi(p)$, *and that* $\mathbb{K}_L$ *satisfies the corollary to Lemma* 4.3. *Then* $\exists\,\delta(R;p)$ *with* $\lim_{R\to\infty}\delta(R;p)=0\,\forall p<p_c$, *such that*

$$\mathbb{B}_{1,n}(L;R;p)\leqq\delta(R;p)\,e^{-(L+n)/\xi(p)}$$

uniformly in L and n.

Proof. As in Definition 5.1, we let $L=2A+R$. By an argument analogous to that in the proof of Proposition 4.1, it is easy to see that given a configuration $\omega\in\mathscr{b}_{1,n}((L,\mathbf{a});R)$, there must exist a rightmost plane to the left of $\mathbf{P}(A)$ and a leftmost plane to the right of $\mathbf{P}(A+R)$ in which the $\mathbb{C}$-condition is not satisfied by $C((-n,0))\|_{\mathbf{H}(L)}$. We denote these planes by $\mathbf{P}(N_1)$, $-n\leqq N_1\leqq A$, and $\mathbf{P}(N_2)$, $A+R\leqq N_2\leqq L$. This gives us disjoint realizations of the events $\mathscr{g}_{(-n,0),(N_1,\mathbf{a}_1)}$, $\mathscr{k}_{(N_1,\mathbf{a}_1),(N_2,\mathbf{a}_2)}$ and $\mathscr{g}_{(N_2,\mathbf{a}_2),(L-N_2,\mathbf{a}-\mathbf{a}_1-\mathbf{a}_2)}$. Applying the van den Berg-Kesten inequality (2.12), summing over points in $\mathbf{P}(N_1)$, $\mathbf{P}(N_2)$ and $\mathbf{P}(L)$, and shifting indices, we obtain

(5.5) $$\mathbb{B}_{1,n}(L;R)\leqq\sum_{L_1=0}^{A+n}\sum_{L_2=0}^{A}\mathbb{G}_{A+n-L_1}\mathbb{K}_{L_1+L_2+R}\mathbb{G}_{A-L_2}.$$

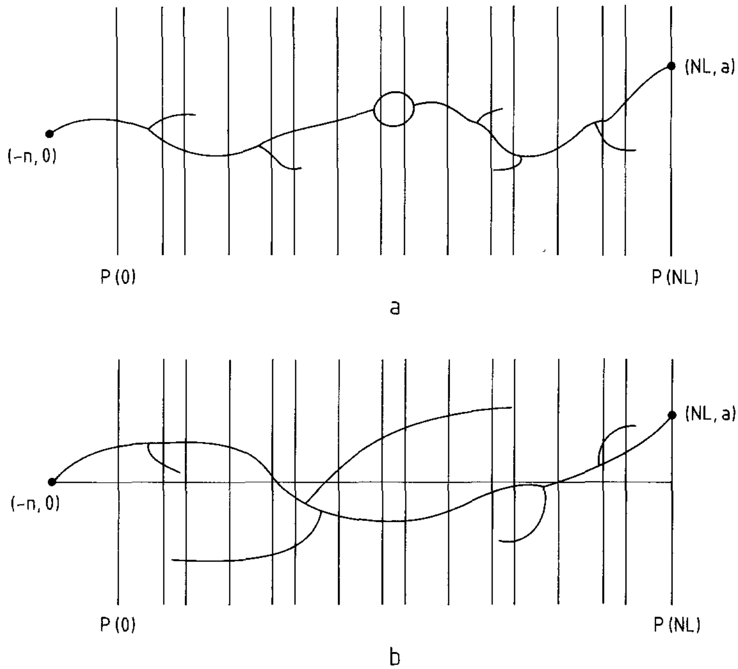

Fig. 4. A "likely" (**a**) and an "unlikely" (**b**) configuration in $\mathscr{E}_{N,n}(NL, \mathbf{a}); R)$

Using the a priori upper bound: $\mathbb{G}_L \leqq \beta^{-1}(p)\, e^{-L/\xi}$ from Proposition 3.1, relaxing the upper bound on the remaining sum, and using the fact that for any function F defined on the positive integers, $\sum_{L_1}\sum_{L_2} F(L_1+L_2) = \sum_L LF(L)$, we have

$$\text{(5.6)} \qquad \mathbb{B}_{1,n}(L;R) \leqq \beta^{-2} e^{-(L+n)/\xi} \sum_{L=0}^{\infty} L\mathbb{K}_{L+R}\, e^{(L+R)/\xi}$$

$$\leqq \beta^{-2} e^{-(L+n)/\xi} \sum_{L \geqq R}^{\infty} L\mathbb{K}_L\, e^{L/\xi}.$$

However, by the corollary to Lemma 4.3, the coefficient of $\beta^{-2} e^{-(L+n)/\xi}$ in (5.6) is the tail of a convergent sum, and thus tends to zero with R. □

5b. *The rescaled functions*

We now want to take advantage of the extra decay provided by Lemma 5.1. Thus we propose (see Fig. 4):

Definition 5.2. Let $A, R \in \mathbb{Z}^+$, $A > R$, $L \equiv 2A + R$. Let $n \in \mathbb{Z}$, $0 \leqq n < L$. Let $N \in \mathbb{Z}^+$. The *point-to-point rescaled connectivity function* of central scale R from $(-n, 0)$ to $(NL, \mathbf{a})$ is defined by:

$$\text{(5.7a)} \qquad \mathscr{E}_{N,n}((NL,\mathbf{a});R) = \{\omega \,|\, (L,\mathbf{a}) \in C((-n,0))\|_{\mathbf{H}(NL)}, \\ \cdot |C((-n,0))\|_{\mathbf{H}(NL)} \cap \mathbf{P}((M-1)L+j)| \\ \geqq 2\ \forall M \text{ with } 1 \leqq M \leqq N, \forall j \text{ with } A \leqq j \leqq A+R\}$$

(5.7b) $$b_{N,n}((NL,\mathbf{a});R;p)=P(\mathscr{b}_{N,n}((NL,\mathbf{a});R)),$$

and the corresponding *point-to-plane rescaled connectivity function* is

(5.7c) $$\mathbb{B}_{N,n}(L;R;p)=\sum_{\mathbf{a}\in\mathbb{Z}^{d-1}} b_{N,n}((NL,\mathbf{a});R;p).$$

As before, the connectivities of principal concern to us will be the *rescaled connectivities from the origin:*

(5.8) $$\mathscr{b}_N\equiv\mathscr{b}_{N,0},\qquad b_N\equiv b_{N,0},\qquad \mathbb{B}_N\equiv\mathbb{B}_{N,0};$$

and the *extended rescaled connectivities:*

(5.9) $$\mathscr{a}_N\equiv\bigcup_{n=A}^{A+R}\mathscr{b}_{N,n},\qquad a_N\equiv\sum_{n=A}^{A+R} b_{N,n},\qquad \mathbb{A}_N\equiv\sum_{n=A}^{A+R}\mathbb{B}_{N,n}.$$

Remarks. (1) Notice that $\mathbb{B}_0$ is not defined by the above equations; we will use the convention

(5.10) $$\mathbb{B}_0\equiv 1.$$

On the other hand, $\mathbb{A}_0$ makes perfect sense; it is just a sum of free connectivities. By Proposition 3.1

(5.11) $$\mathbb{A}_0=\sum_{n=A}^{A+R}\mathbb{G}_N\leqq\lambda_1(p)\,e^{-A/\xi(p)},$$

with $\lambda_1(p)<\infty$ uniformly in $A\,\forall p<p_c$.

(2) It is seen that, for $N=1$, the above definition is entirely consistent with Definition 5.1 of the block functions. Indeed, let us denote the "central region" of the M^{th} slab by $S_c(M)$:

(5.12) $$S_c(M)=\{\mathbf{x}\in\mathbb{Z}^d\,|\,x_1=(M-1)\,L+j \text{ for some } j \text{ with } A\leqq j\leqq A+R\}.$$

While the configurations in $\mathscr{b}_{1,n}$ are required to satisfy the $\mathbb{C}$-condition in $S_c(1)$, those in $\mathscr{b}_{N,n}$ are required to satisfy the $\mathbb{C}$-condition in $S_c(M)$ $\forall\, 1\leqq M\leqq N$. We will often say that a configuration *double-covers* the region $S_c(M)$ if it satisfies the $\mathbb{C}$-condition in every plane of that region. It will also be useful to have explicit notation for the entire M^{th} slab; thus we define

(5.13) $$S(M)=\{\mathbf{x}\in\mathbb{Z}^d\,|\,x_1=(M-1)\,L+j \text{ for some } j \text{ with } 0\leqq j<L\}.$$

(3) As before, the rescaled functions (from the origin) interpolate between the free and direct functions of the appropriate scale:

(5.14) $$\mathbb{G}_{NL}(p)\geqq\mathbb{B}_N(L;R;p)\geqq\mathbb{C}_{NL}(p).$$

Thus, if we can show that *for fixed* L, $\mathbb{B}_N(L)$ has an exponentially faster decay rate in N than does the free function $\mathbb{G}_{NL}$, we are done. We begin by establishing the existence of a decay rate for $\mathbb{B}_N$:

Proposition 5.2. *Let* $\mathbb{B}_{N,n}(L;R;p)$ *be defined as in* Eq. (5.7). *Then for all* $p\in(0,p_c)$, *the limit*

$$\lim_{N\to\infty}\frac{1}{N}\log\mathbb{B}_{N,n}(L;R;p)\equiv-\frac{1}{\xi_b(L;R;p)}$$

exists and is independent of n.

Proof. That the limit is independent of n is obvious, since the $\mathbb{B}_{N,n}$ differ from each other by factors which depend on $n<L$, not on N. Thus we may confine attention to $n=0$. For $\mathbb{B}_{N,0}$, we can simply repeat the arguments in the proofs of Propositions 3.2(ii) and 3.3: As usual, we have a weak subadditive inequality analogous to (3.4) or (3.39), which implies the existence of a limiting decay rate for the on-axis point-to-point functions $b_{N,0}((L,0))$ and an a priori upper bound for the off-axis functions $b_{N,0}((L,a))$. Then $\mathbb{B}_{N,0}$ is shown to have the same limiting decay rate by "squeezing" with an inequality of the form (3.9). □

As is apparent from the above discussions, our strategy is to show that, for some choice of R, $\xi_b^{-1}(L;R)>\xi^{-1}L$ uniformly in L. This, together with (5.14), would give us $\xi_c^{-1}>\xi^{-1}$. In order to prove this, we will of course use the extra decay from Lemma 5.1. Indeed, if the $\mathbb{B}_N$ basically decoupled into N copies of $\mathbb{B}_1$ (i.e. if $\mathbb{B}_N\sim\mathbb{B}_1^N$), Lemma 5.1 would directly imply $\xi_b^{-1}(L;R)\geq\xi^{-1}L+|\log\delta(R)|$. In fact, we will ultimately obtain an estimate which is almost this strong. However, in order to do this, we will have to bound the probabilities of those configurations which do not decouple (e.g., the unlikely configuration in Fig. 4) by even faster exponential terms. This is the subject of our principal estimate:

Lemma 5.3. *Let the functions* $\mathbb{A}_N(L;R;p)$ *and* $\mathbb{B}_N(L;R;p)$ *be as given in Definition 5.2 with* $L=2A+R$, $A>R$, *and with* $\mathbb{A}_0$ *and* $\mathfrak{B}_0$ *given by* Eqs. (5.11) *and* (5.10). *Suppose that* $\mathbb{G}_L$ *satisfies Proposition* 3.1, *with decay rate* $\xi(p)$. *Then* $\forall N\geq 1$, $\mathbb{A}_N$ *and* $\mathbb{B}_N$ *obey the coupled renewal inequalities:*

(i) $$\mathbb{B}_N\leq\sum_{K=0}^{N}\mathbb{V}_K\,\mathbb{B}_{N-K}+\sum_{J=0}^{N}\mathbb{U}_J\,\mathbb{A}_{N-J}$$

(ii) $$\mathbb{A}_N\leq\sum_{K=0}^{N}\tilde{\mathbb{V}}_K\,\mathbb{B}_{N-K}+\sum_{J=0}^{N}\tilde{\mathbb{U}}_J\,\mathbb{A}_{N-J}$$

where the functions $\mathbb{V}_K(p)$, $\mathbb{U}_J(p)$, $\tilde{\mathbb{V}}_K(p)$, $\tilde{\mathbb{U}}_J(p)\geq 0$ *satisfy*

$$\mathbb{V}_0=\mathbb{U}_0=\tilde{\mathbb{V}}_0=\tilde{\mathbb{U}}_0=\mathbb{U}_1=\tilde{\mathbb{U}}_1=0$$
$$\mathbb{V}_1=\mathbb{B}_1,\qquad \tilde{\mathbb{V}}_1=\mathbb{A}_1$$
$$\mathbb{U}_2\leq\lambda_2(p)\,e^{+R/\xi(p)}\,e^{-2L/\xi(p)},\qquad \tilde{\mathbb{U}}_2\leq\tilde{\lambda}_2(p)\,e^{-A/\xi(p)}e^{+R/\xi(p)}e^{-2L/\xi(p)}$$

$$\mathbb{V}_K\leq\lambda_3(p)\,e^{-A/\xi(p)}e^{-2(K-1)L/\xi(p)}\qquad (K\geq 2)$$
$$\tilde{\mathbb{V}}_K\leq\tilde{\lambda}_3(p)\,e^{-2A/\xi(p)}e^{-2(K-1)L/\xi(p)}\qquad (K\geq 2)$$
$$\mathbb{U}_J\leq\lambda_4(p)\,e^{-3A/\xi(p)}(J-2)\,e^{-2(J-2)L/\xi(p)}\qquad (J\geq 3)$$
$$\tilde{\mathbb{U}}_J\leq\tilde{\lambda}_4(p)\,e^{-4A/\xi(p)}(J-2)\,e^{-2(J-2)L/\xi(p)}\qquad (J\geq 3)$$

with $\lambda_i(p)$, $\tilde{\lambda}_i(p)<\infty$ *uniformly in A, R, L, K and J* $\forall p<p_c$.

Remark. As will become apparent in the following proof, the functions $\mathbb{V}_K(p)$, $\mathbb{U}_J(p)$, $\tilde{\mathbb{V}}_K(p)$ and $\tilde{\mathbb{U}}_J(p)$ represent the weights of various unlikely ("recurrent") configurations contributing to the rescaled functions.

Proof. Our strategy is to decompose $\mathscr{E}_{N,n}$ into a union of disjoint events:

$$\mathscr{E}_{N,n}((NL, \mathbf{a}); R) = \bigcup_{K=1}^{N} \mathscr{E}_{N,n}^{(K)}((NL, \mathbf{a}); R) = \bigcup_{K=1}^{N} \bigcup_{J=K}^{N} \mathscr{E}_{N,n}^{(K,J)}((NL, \mathbf{a}); R). \tag{5.15}$$

Let us first describe the K-decomposition. The configurations in $\mathscr{E}_{N,n}^{(K)}$ are those configurations in $\mathscr{E}_{N,n}$ for which K is the smallest integer in $[1, \ldots, N]$ such that $|C((-n,0))\|_{\mathbf{H}(KL)} \cap \mathbf{P}(j)| \geqq 2 \, \forall j$ with $A \leqq j \leqq A+R$, i.e. such that $C((-n,0))\|_{\mathbf{H}(KL)}$ double-covers $S_c(1)$. (See Eq. (5.12) for a definition of $S_c(1)$.) For example, the configuration depicted in Fig. 4b is in $\mathscr{E}_{N,n}^{(2)}$, since the cluster "ventures" into the second slab before satisfying the $\mathbb{C}$-condition in every plane of the central region of the first slab.
As usual, we can define the connectivities

$$b_{N,n}^{(K)}((NL, \mathbf{a}); R; p) = P(\mathscr{E}_{N,n}^{(K)}((NL, \mathbf{a}); R)) \tag{5.16a}$$

$$\mathbb{B}_{N,n}^{(k)}(L; R; p) = \sum_{\mathbf{a} \in \mathbb{Z}^{d-1}} b_{N,n}^{(K)}((NL, \mathbf{a}); R; p). \tag{5.16b}$$

Next, let us describe how J is chosen for a given K. Let $\omega \in \mathscr{E}_{N,n}^{(K)}$ and consider the set of points

$$Q(K; \omega) \equiv Q(K) = C((-n,0))\|_{\mathbf{H}(KL)} \cap \mathbf{P}(KL). \tag{5.17}$$

By the definition of $\mathscr{E}_{N,n}^{(K)}$, $Q(K)$ must contain a non-empty subset

(5.18)
$$q(K; \omega) \equiv q(K) = \{\mathbf{x} \in Q(K) \mid \mathbf{x} \text{ is connected to } (NL, \mathbf{a}) \text{ by a path of occupied bonds in } \mathbf{H}(NL) \text{ in the complement of the sites of } C((-n,0))\|_{\mathbf{H}(KL)} \setminus \{\mathbf{x}\}\},$$

i.e, $q(K)$ is the set of sites in $Q(K)$ which actually reach $(NL, \mathbf{a})$. Furthermore, since all points $\mathbf{x} \in q(K)$ are connected to $(NL, \mathbf{a})$, the cluster that accomplishes this connection must be the same for all these points, i.e. the set

$$C^N(q(K)) = C(\mathbf{x})\|_{\mathbf{H}(NL)} \setminus \{C((-n,0))\|_{\mathbf{H}(KL)} \setminus \{\mathbf{x}\}\} \tag{5.19}$$

is independent of the site $\mathbf{x} \in q(K)$. If $C^N(q(K))$ double-covers $S_c(M)$ for every $M = K+1, \ldots, N$, we define $J = K$. Otherwise, we define J to be the largest integer in $[K+1, \ldots, N]$ such that $S_c(J)$ is *not* double-covered by $C^N(q(K))$. For example, the configuration depicted in Fig. 4b is in $\mathscr{E}_{N,n}^{(2,4)}$ since the component from the plane $\mathbf{P}(2L)$ which eventually reaches $(NL, \mathbf{a})$ does not double-cover $S_c(4)$, although it does double-cover all later central regions.

Again, we can define the connectivities

(5.20a) $$b_{N,n}^{(K,J)}((NL, \mathbf{a}); R; p) = P(\mathscr{E}_{N,n}^{(K,J)}((NL, \mathbf{a}); R))$$

(5.20b) $$\mathbb{B}_{N,n}^{(K,J)}(L; R; p) = \sum_{\mathbf{a}\in\mathbb{Z}^{d-1}} b_{N,n}^{(K,J)}((NL, \mathbf{a}); R; p).$$

We will also use the rotation

(5.21a) $$\mathbb{A}_N^{(K,J)}(L; R; p) = \sum_{n=A}^{A+R} \mathbb{B}_{N,n}^{(K,J)}(L; R; p)$$

(5.21b) $$\mathbb{B}_N^{(K,J)}(L; R; p) = \mathbb{B}_{N,0}^{(K,J)}(L; R; p).$$

Due to the disjoint union, we have

(5.22) $$\mathbb{B}_{N,n} = \sum_{J=1}^{N} \sum_{K=1}^{J} \mathbb{B}_{N,n}^{(K,J)},$$

with similar expressions for $\mathbb{A}_N$ and $\mathbb{B}_N$. We claim that the first sums in the renewal inequalities come from the terms with $K=J$, and that the second sums come from the $K<J$ terms. Thus we will treat these cases separately. Furthermore, in each case, we will treat $K=1$ separately from $K\geqq 2$.

Let us begin with $K=J$. Thus take $\omega\in\mathscr{E}_{N,n}^{(K,K)}((NL, \mathbf{a}); R)$. First, by the definition of K, there is a point $(KL, \mathbf{b})\in q(K)$ (defined in Eq. (5.18)) such that $C((-n, 0))\|_{\mathbf{H}(KL)}$ provides a realization of $\mathscr{E}_{K,n}^{(K,K)}((KL, \mathbf{b}); R)$. Furthermore, by the definition of J, $C^N(q(K))$ (defined in Eq. (5.19)) gives a realization of $T^{(KL,\mathbf{b})}\mathscr{E}_{N-K,0}(((N-K)L, \mathbf{a}-\mathbf{b}); R)$ disjointy from $C((-n, 0))\|_{\mathbf{H}(KL)}$. Notice that this second event begins at $n=0$, i.e. is a translate of $\mathscr{E}_{N-K}$. Thus by subadditivity, the van den Berg-Kesten inequality (2.12) and translation invariance, we have

(5.23) $$b_{N,n}^{(K,K)}((NL, \mathbf{a}); R; p) \leqq \sum_{\mathbf{b}\in\mathbb{Z}^{d-1}} P(\mathscr{E}_{K,n}^{(K,K)}((KL, \mathbf{b}); R))) \times b_{N-K}(((N-K)L, \mathbf{a}-\mathbf{b}); R; p).$$

Now suppose $K=1$ (and still assume $K=J$). Then $\mathscr{E}_{1,n}^{(1,1)}((L, \mathbf{b}); R) = \mathscr{E}_{1,n}((L, \mathbf{b}); R)$, i.e. there is no further decomposition. Summing (5.23) over $\mathbf{a}$ and $\mathbf{b}$, we obtain

(5.24) $$\mathbb{B}_{N,n}^{(1,1)} \leqq \mathbb{B}_{1,n}\, \mathbb{B}_{N-1}.$$

This immediately gives the first nontrivial terms in the first sums of the renewal inequalities. Indeed, for $n=0$, (5.24) becomes

(5.25b) $$\mathbb{B}_N^{(1,1)} \leqq \mathbb{B}_1\, \mathbb{B}_{N-1};$$

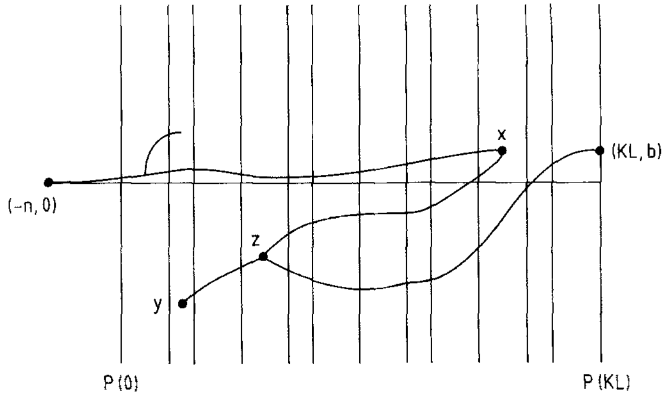

Fig. 5. A configuration in $\mathscr{E}^{(K,K)}_{K,n}((KL,\mathbf{b});R))$

the r.h.s. of (5.25b) is the $K=1$ term of the inequality (i) with $\mathbb{V}_1=\mathbb{B}_1$. Similarly, summing n from A to $A+R$, (5.24) gives

(5.25a) $$\mathbb{A}^{(1,1)}_N \leqq \mathbb{A}_1\, \mathbb{B}_{N-1},$$

the r.h.s. of which is the $K=1$ term of the inequality (ii) with $\tilde{\mathbb{V}}_1=\mathbb{A}_1$.

Now suppose $K\geqq 2$ and $\omega\in\mathscr{E}^{(K,K)}_{K,n}((KL,\mathbf{b});R))$. (See Fig. 5.) Then, by the definition of K, there must exist two points $\mathbf{x}\in S(K)$ and $\mathbf{y}\in S_c(1)$ and two occupied paths:

$$\mathscr{P}_{[1]}:\ \text{from } (-n,0) \text{ to } \mathbf{x}$$

$$\mathscr{P}_{[2]}:\ \text{from } \mathbf{x} \text{ to } \mathbf{y}$$

with $\mathscr{P}_{[1]}$ and $\mathscr{P}_{[2]}$ site-disjoint except for the point $\mathbf{x}$. Furthermore, by the definition of $\mathscr{E}_{K,n}((KL,\mathbf{b});R))$, there must exist a point $\mathbf{z}\in\mathscr{P}_{[1]}\cup\mathscr{P}_{[2]}$ and an occupied path

$$\mathscr{P}_{[3]}:\ \text{from } \mathbf{z} \text{ to } (KL,\mathbf{b})$$

which is site-disjoint from $\mathscr{P}_{[1]}$ and $\mathscr{P}_{[2]}$, except for the single point $\mathbf{z}$. There are two topologically distinct cases: $\mathbf{z}\in\mathscr{P}_{[1]}$ or $\mathbf{z}\in\mathscr{P}_{[2]}$; however, these lead to the same final estimate, so we will explicitly treat only the latter. Again, using the van den Berg-Kesten inequality, Eq. (5.23) and the above reasoning imply

(5.26) $$b^{(K,K)}_{N,n}((NL,\mathbf{a}))\leqq \sum_{\substack{\mathbf{b}\in\mathbb{Z}^{d-1}\\ \mathbf{x}\in S(K)\\ \mathbf{y}\in S_c(1)\\ \mathbf{z}\in\mathbf{H}(KL)}} \tau_{(-n,0),\mathbf{x}}\,\tau_{\mathbf{x},\mathbf{z}}\,\tau_{\mathbf{z},\mathbf{y}}\,\tau_{\mathbf{z},(KL,\mathbf{b})}\,b_{N-K}(((N-K)L,\mathbf{a}-\mathbf{b}))).$$

Summing over $\mathbf{a}$ and $\mathbf{b}$ and the transverse coordinates, and using the notation $x_1\equiv X$, $y_1\equiv Y$ and $\mathbf{z}_1\equiv Z$, we have

(5.27) $$\mathbb{B}^{(K,K)}_{N,n}\leqq \sum_{\substack{(K-1)L\leqq X\leqq KL\\ A\leqq Y\leqq A+R\\ Z\leqq KL}} \mathbb{G}_{n+X}\,\mathbb{G}_{|X-Z|}\,\mathbb{G}_{|Z-Y|}\,\mathbb{G}_{|KL-Z|}\,\mathbb{B}_{N-K}.$$

Finally, using the a priori bounds $\mathbb{G}_L \leqq \beta^{-1}(p)\, e^{-L/\xi}$ from Proposition 3.1 and performing the remaining sums, we obtain

$$\mathbb{B}_{N,n}^{(K,K)} \leqq \lambda_3(p)\, e^{-n/\xi(p)}\, e^{-A/\xi(p)}\, e^{-2(K-1)L/\xi(p)}\, \mathbb{B}_{N-K} \tag{5.28}$$

with $\lambda_3(p) < \infty$ for $p < p_c$. The contribution from the case $\mathbf{z} \in \mathscr{P}_{[1]}$ only modifies the constant $\lambda_3(p)$.

It is easy to see that (5.28) provides the $K \geqq 2$ terms in the first sums of the inequalities (i) and (ii). Indeed, for $n=0$ we obtain

$$\mathbb{B}_N^{(K,K)} \leqq \mathbb{V}_K\, \mathbb{B}_{N-K} \tag{5.29 b}$$

with $\mathbb{V}_K$ of the stated form, while if we sum n from A to $A+R$, we have

$$\mathbb{A}_N^{(K,K)} \leqq \tilde{\mathbb{V}}_K\, \mathbb{B}_{N-K} \tag{5.29 a}$$

with

$$\begin{aligned} \tilde{\mathbb{V}}_K &\leqq \lambda_3(p)\, e^{-A/\xi(p)}\, e^{-2(K-1)L/\xi(p)} \sum_{n=A}^{A+R} e^{-n/\xi(p)} \\ &\leqq \tilde{\lambda}_3(p)\, e^{-2A/\xi(p)}\, e^{-2(K-1)L/\xi(p)} \end{aligned} \tag{5.30}$$

and $\tilde{\lambda}_3(p) < \infty$ for $p < p_c$.

Now we will treat the case $K < J$. Here we will first assume $K \geqq 2$ (so that $J \geqq 3$), and then explicitly consider the case $K=1$, $J=2$. Thus we take $\omega \in \mathscr{E}_{N,n}^{(K,J)}((NL, \mathbf{a}); R)$, $J > K \geqq 2$. Then, by exactly the same reasoning as in the case $J = K \geqq 2$, the definition of K implies that there exist points $(KL, \mathbf{b}) \in q(K)$, $\mathbf{x} \in S(K)$ and $\mathbf{y} \in S_c(1)$, and two occupied paths

$$\mathscr{P}_{[1]}: \text{ from } (-n, 0) \text{ to } \mathbf{x}$$
$$\mathscr{P}_{[2]}: \text{ from } \mathbf{x} \text{ to } \mathbf{y}$$

with $\mathscr{P}_{[1]}$ and $\mathscr{P}_{[2]}$ site-disjoint except for $\mathbf{x}$, as well as a point $\mathbf{z} \in \mathscr{P}_{[1]} \cup \mathscr{P}_{[2]}$ and an occupied path

$$\mathscr{P}_{[3]}: \text{ from } \mathbf{z} \text{ to } (KL, \mathbf{b})$$

which is site-disjoint from $\mathscr{P}_{[1]}$ and $\mathscr{P}_{[2]}$ except for $\mathbf{z}$. Again, we will explicitly consider only the case $\mathbf{z} \in \mathscr{P}_{[2]}$, since the other case leads to an upper bound of the same form.

Now, by the definition of J, we know that $C^N(q(K))$ (defined in (5.19)) does not double-cover the region $S_c(J)$, although it does double-cover $S_c(M)$ $\forall M > J$. Thus there must exist a rightmost plane in $S_c(J)$ in which the $\mathbb{C}$-condition is not satisfied by $C^N(q(K))$, i.e. there must be a largest integer W in $[((J-1)L$

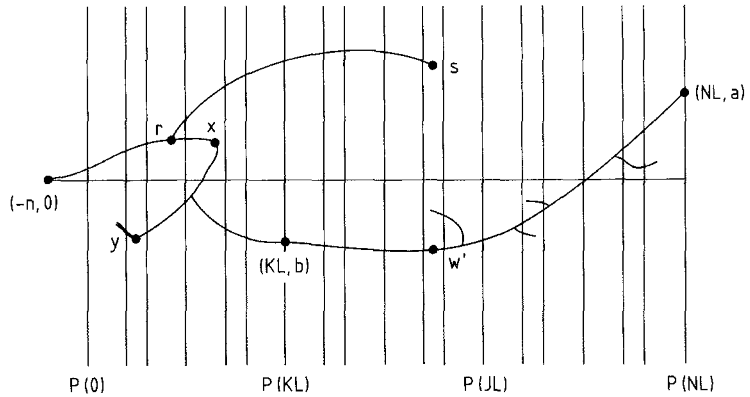

Fig. 6. A configuration in $\mathscr{E}_{N,n}^{(K,J)}((NL, \mathbf{a}); R)$ with $J>K$

$+A, \ldots, (J-1)L+A+R]$ such that $\mathbf{P}(W) \cap C^N(q(K))$ contains only a single point. Let us denote this point by $\mathbf{w}^* = (W, \mathbf{w})$. (See Fig. 6.)

By the definition of $C^N(q(K))$, $\mathbf{w}^*$ is connected to $(KL, \mathbf{b})$ in the complement of $C((-n, 0))\|_{\mathbf{H}(KL)} \backslash Q(K)$. Thus there is an occupied path

$$\mathscr{P}_{[4]}: \text{ from } (KL, \mathbf{b}) \text{ to } \mathbf{w}^*$$

which is site-disjoint from $\mathscr{P}_{[1]}$, $\mathscr{P}_{[2]}$ and $\mathscr{P}_{[3]}$ (except for the point $(KL, \mathbf{b})$). Furthermore, since $S_c(J+1), \ldots, S_c(N)$ are double-covered in $\mathscr{E}_{N,n}^{(K,J)}$, but are not double-covered by $C^N(q(K))$, there must exist points $\mathbf{r} \in \mathscr{P}_{[1]} \cup \mathscr{P}_{[2]} \cup \mathscr{P}_{[3]}$ and $\mathbf{s} \in S_c(J)$, $s_1 = W-1$, and a path in $C^N(q(K))$

$$\mathscr{P}_{[5]}: \text{from } \mathbf{r} \text{ to } \mathbf{s}$$

which, except for the point $\mathbf{r}$, is site-disjoint from $\mathscr{P}_{[1]}$, $\mathscr{P}_{[2]}$ and $\mathscr{P}_{[3]}$, and which is also site-disjoint from $\mathscr{P}_{[4]}$. Indeed, if $\mathscr{P}_{[5]}$ were not disjoint from $\mathscr{P}_{[4]}$, then we would have had $\mathscr{P}_{[5]} \subset C^N(q(K))$, which would mean that $C^N(q(K))$ double-covered $S_c(M)$ for every $M \geqq K+1$, contradicting our asumption $J>K$.

As with our placement of $\mathbf{z}$, there are several topologically distinct choices for our placement of $\mathbf{r}$: given $\mathbf{z} \in \mathscr{P}_{[2]}$, we could have $\mathbf{r} \in \mathscr{P}_{[1]}$, $\mathbf{r} \in \mathscr{P}_{[3]}$, $\mathbf{r} \in \{$the segment of $\mathscr{P}_{[2]}$ between $\mathbf{y}$ and $\mathbf{z}\}$ or $\mathbf{r} \in \{$the segment of $\mathscr{P}_{[2]}$ between $\mathbf{z}$ and $\mathbf{x}\}$. However, as before, each of these choices will lead to an upper bound of the same form, so we explicitly treat only $\mathbf{r} \in \mathscr{P}_{[1]}$.

Finally, and most importantly, since every connection from $q(K)$ to $(NL, \mathbf{a})$ must pass through the point $\mathbf{w}^* = (W, \mathbf{w}) \in S_c(J)$, and since by the definition of J, $C^N(q(K))$ double-covers $S_c((J+1), \ldots, S_c(N)$, there must be a realization of $T^{(W,\mathbf{w})} \mathscr{E}_{N-J, JL-W}((N-J)L, \mathbf{a}-\mathbf{w}); R)$ in $C^N(q(K))$. This realization is clearly site-disjoint from $\mathscr{P}_{[1]}$, $\mathscr{P}_{[2]}$ and $\mathscr{P}_{[3]}$, since it is a subset of $C^N(q(K))$ which is by definition disjoint from these paths. The realization is also site-disjoint from $\mathscr{P}_{[5]}$; indeed, were this not the case, we would have had $\mathscr{P}_{[5]} \subset C^N(q(K))$, again contradicting our assumption $J>K$. Finally, the realization is also site-disjoint from $\mathscr{P}_{[4]}$, except for the single point $\mathbf{w}^*$. Indeed, this follows immediately from the fact that both the realization and $\mathscr{P}_{[4]}$ are subsets of $C^N(q(K))$, and that

$\mathbf{w}^*$ is by definition the only point of intersection of $C^N(q(K))$ with the plane $\mathbf{P}(W)$.

By subadditivity, the van den Berg-Kesten inequality (2.12) and translation invariance, we have

$$b_{N,n}^{(K,J)}((NL,\mathbf{a})) \leqq \sum_{\substack{b\in\mathbb{Z}^{d-1}\\ \mathbf{x}\in S(K)\\ \mathbf{y}\in S_c(1)\\ \mathbf{z}\in\mathbf{H}(KL)\\ \mathbf{w}^*\in S_c(J)\\ \mathbf{r}\in\mathbf{H}(KL)\\ \mathbf{s}\in S_c(J)}} \tau_{(-n,0),r}\,\tau_{r,\mathbf{x}}\,\tau_{\mathbf{x},\mathbf{z}}\,\tau_{\mathbf{z},\mathbf{y}}\,\tau_{\mathbf{z},(KL,\mathbf{b})}\,\tau_{\mathbf{r},\mathbf{s}}\,\tau_{(KL,\mathbf{b}),\mathbf{w}^*} \cdot b_{N-J,JL-W}(((N-J)L,\mathbf{a}-\mathbf{w}))). \tag{5.31}$$

Summing over $\mathbf{a}$, $\mathbf{b}$ and the transverse coordinates, and using the notation $x_1\equiv X$, $y_1\equiv Y$, $z_1\equiv Z$ and $w_1\equiv W$, we have

$$\mathbb{B}_{N,n}^{(K,J)} \leqq \sum_{\substack{(K-1)L\leqq X\leqq KL\\ A\leqq Y\leqq A+R\\ Z\leqq KL\\ r_1\leqq KL\\ (J-1)L+A\leqq W\leqq (J-1)L+A+R\\ s_1=W-1}} \mathbb{G}_{|n-r_1|}\,\mathbb{G}_{|r_1-X|}\,\mathbb{G}_{|X-Z|}\,\mathbb{G}_{|Z-Y|} \cdot \mathbb{G}_{|KL-Z|}\,\mathbb{G}_{|r_1-s_1|}\,\mathbb{G}_{|KL-W|}\,\mathbb{B}_{N-J,JL-W}. \tag{5.32}$$

Using the a priori bound on $\mathbb{G}_L$ and performing the final sums, we have

$$\mathbb{B}_{N,n}^{(K,J)} \leqq \lambda_4(p)\, e^{-n/\xi(p)}\, e^{-3A/\xi(p)}\, e^{-2(J-2)L/\xi(p)}\, \mathbb{A}_{N-J} \tag{5.33}$$

with $\lambda_4(p)<\infty$ for $p<p_c$. Here the factor of $\mathbb{A}_{N-J}$ came from summing $\mathbb{B}_{N-K,JL-W}$ over W in the allowed range. As before, the other possible positions of $\mathbf{z}$ and $\mathbf{r}$ yield estimates of the same form, and thus only modify $\lambda_4(p)$. Finally, noting that the expression on the r.h.s. of (5.33) is independent of K (and recalling that we have restricted to the case $J>K\geqq 2$), we obtain

$$\sum_{K=2}^{J-1} \mathbb{B}_{N,n}^{(K,J)} \leqq \lambda_4(p)\, e^{-n/\xi(p)}(J-2)\, e^{-3A/\xi(p)}\, e^{-2(J-2)L/\xi(p)}\, \mathbb{A}_{N-J}. \tag{5.34}$$

Again, it is easy to see that this produces all but the first nontrivial terms in the second sums of inequalities (i) and (ii). Indeed, for $n=0$, we get

$$\sum_{K=2}^{J-1} \mathbb{B}_{N}^{(K,J)} \leqq \mathbb{U}_J\, \mathbb{A}_{N-J} \tag{5.35b}$$

with $\mathbb{U}_j$ of the stated form, while performing the sum over n: A to $A+R$, we get

$$\sum_{K=2}^{J-1} \mathbb{A}_{N}^{(K,J)} \leqq \tilde{\mathbb{U}}_J\, \mathbb{A}_{N-J} \tag{5.35a}$$

with

$$(5.36)\qquad \tilde{\mathbb{U}}_J \leqq \lambda_4(p)(J-2)\, e^{-3A/\xi(p)} e^{-2(J-2)L/\xi(p)} \sum_{n=A}^{A+R} e^{-n/\xi(p)}$$
$$\leqq \tilde{\lambda}_4(p)\, e^{-4A/\xi(p)} (J-2)\, e^{-2(J-2)L/\xi(p)}$$

and $\tilde{\lambda}_4(p) < \infty$ for $p < p_c$.

Let us now attend to the single remaining case $K=1$, $J=2$, and prove that it yields the remaining nontrivial term $\mathbb{U}_2 \mathbb{A}_{N-2}$ in (i) and $\tilde{\mathbb{U}}_2 \mathbb{A}_{N-2}$ in (ii). To this end, take $\omega \in \mathscr{E}^{(1,2)}_{N,n}((NL, \mathbf{a}); R)$. Then, by the definition of K and the fact that $K=1$, there must exist a point $(L, \mathbf{b}) \in q(1)$ and a path

$$\mathscr{P}_{[1]}: \text{from } (-n, 0) \text{ to } (L, \mathbf{b})$$

in $C((-n, 0))\|_{\mathbf{H}(L)} \setminus Q(1)$. Furthermore, following exactly the reasoning in the case $J > K \geqq 2$, there must exist points $\mathbf{w}^* = (W, \mathbf{w}) \in S_c(2)$, $\mathbf{r} \in \mathscr{P}_{[1]}$ and $\mathbf{s} \in S_c(2)$, $s_1 = W-1$, defined as before, and two occupied paths

$$\mathscr{P}_{[4]}: \text{ from } (L, \mathbf{b}) \text{ to } \mathbf{w}^*$$
$$\mathscr{P}_{[5]}: \text{ from } \mathbf{r} \text{ to } \mathbf{s}$$

which are site-disjoint from each other and from $\mathscr{P}_{[1]}$ except at the specified points of intersection. Finally, as before, there must be a realization of $T^{(W,\mathbf{w})} \mathscr{E}_{N-2, 2L-W}(((N-2)L, \mathbf{a}-\mathbf{w}); R)$ which is site-disjoint from $\mathscr{P}_{[1]}$, $\mathscr{P}_{[4]}$ and $\mathscr{P}_{[5]}$, except at $\mathbf{w}^*$. Thus we obtain

$$(5.37)\quad b^{(1,2)}_{N,n}((NL, \mathbf{a})) \leqq \sum_{\substack{\mathbf{b} \in \mathbb{Z}^{d-1} \\ \mathbf{w}^* \in S_c(2) \\ \mathbf{r} \in \mathbf{H}(L) \\ \mathbf{s} \in S_c(2)}} \tau_{(-n,0),\mathbf{r}}\, \tau_{\mathbf{r},(L,\mathbf{b})}\, \tau_{\mathbf{r},\mathbf{s}}\, \tau_{(L,\mathbf{b}),\mathbf{w}^*}\, b_{N-2, 2L-W}(((N-2)L, \mathbf{a}-\mathbf{w}))).$$

which yields

$$(5.38)\qquad \mathbb{B}^{(1,2)}_{N,n} \leqq \lambda_2(p)\, e^{-nR/\xi(p)} e^{+R/\xi(p)} e^{-2L/\xi(p)} \mathbb{A}_{N-2}$$

with $\lambda_2(p) < \infty$ for $p < p_c$. Setting $n=0$, we get

$$(5.39\,\mathrm{b})\qquad \mathbb{B}^{(1,2)}_N \leqq \mathbb{U}_2 \mathbb{A}_{N-2},$$

while summing over n gives

$$(5.39\,\mathrm{a})\qquad \mathbb{A}^{(1,2)}_N \leqq \tilde{\mathbb{U}}_2 \mathbb{A}_{N-2}$$

with $\mathbb{U}_2$ and $\tilde{\mathbb{U}}_2$ of the stated forms. □

Theorem 5.4. *Let* $\xi(p)$ *and* $\xi_c(p)$ *be the decay rates of* $\tau_{0,L}(p)$ *and* $c_{0,L}(p)$ *as defined in* (1.2) *and Proposition* 3.2. *Then* $\forall p < p_c$, $\xi_c(p) < \xi(p)$.

Proof. As explained earlier (cf. Eq. (5.14) and the discussion following the proof of Proposition 5.2), the decay rate ξ_b provides a bound on the decay rate of the $\mathbb{C}$-functions: $\xi_c^{-1} L \geqq \xi_b^{-1}(L; R)$. Thus it suffices to show that, for some choice of R and L, $\xi_b^{-1}(L;R) > \xi^{-1}L$, for which we need an upper bound on the $\mathbb{B}_N$. To this end, let us examine the system of coupled renewal inequalities derived in Lemma 5.3:

$$\mathbb{B}_N \leqq \sum_{K=0}^{N} \mathbb{V}_K \mathbb{B}_{N-K} + \sum_{J=0}^{N} \mathbb{U}_J \mathbb{A}_{N-J}; \quad N \geqq 1 \tag{5.40a}$$

$$\mathbb{A}_N \leqq \sum_{K=0}^{N} \tilde{\mathbb{V}}_K \mathbb{B}_{N-K} + \sum_{J=0}^{N} \tilde{\mathbb{U}}_J \mathbb{A}_{N-J}; \quad N \geqq 1. \tag{5.40b}$$

Given these inequalities (and the bounds on the coefficients as derived in Lemma 5.3), we may prove the desired result either by induction in N, or by studying the transforms of $\mathbb{B}_N$ and $\mathbb{A}_N$. Here we give a proof based on the transforms.

First, it is clear that were (5.40a) and (5.40b) equalities, then the sequences $(\mathbb{B}_N)$ and $(\mathbb{A}_N)$ could be generated, recursively, via knowledge of $\mathbb{B}_0$, $\mathbb{B}_1$, $\mathbb{A}_0$, $\mathbb{A}_1$, $(\mathbb{V}_N)$, $(\mathbb{U}_N)$, $(\tilde{\mathbb{V}}_N)$ and $(\tilde{\mathbb{U}}_N)$. Similarly, provided all of the quantities are non-negative, upper bounds on the these eight (sets of) quantities systematically generate upper bounds on the $\mathbb{B}_2$, $\mathbb{B}_3$, ...,; $\mathbb{A}_2$, $\mathbb{A}_3$, ... by substituting into (5.40a) and (5.40b) as though they were equalities. We may therefore consider the system

$$(\mathbf{B}_N = \sum_{K=0}^{N} \mathbf{V}_K \mathbf{B}_{N-K} + \sum_{J=0}^{N} \mathbf{U}_J \mathbf{A}_{N-J} \tag{5.41a}$$

$$\mathbf{A}_N = \sum_{K=0}^{N} \tilde{\mathbf{V}}_K \mathbf{B}_{N-K} + \sum_{J=0}^{N} \tilde{\mathbf{U}}_J \mathbf{A}_{N-J} \tag{5.41b}$$

$$\mathbf{B}_0 = 1 \tag{5.42a}$$

$$\mathbf{A}_0 = \lambda e^{-A/\xi(p)} \tag{5.42b}$$

$$\mathbf{V}_0 = \mathbf{U}_0 = \tilde{\mathbf{V}}_0 = \tilde{\mathbf{U}}_0 = 0 \tag{5.42c}$$

$$\mathbf{B}_1 = \mathbf{V}_1 = \lambda \delta(R)\, e^{-L/\xi(p)} \tag{5.42d}$$

$$\mathbf{A}_1 = \tilde{\mathbf{V}}_1 = \lambda e^{-A/\xi(p)} \delta(R)\, e^{-L/\xi(p)} \tag{5.42e}$$

$$\mathbf{U}_1 = \tilde{\mathbf{U}}_1 = 0 \tag{5.42f}$$

$$\mathbf{U}_2 = \lambda e^{+R/\xi(p)} e^{-2L/\xi(p)}, \quad \tilde{\mathbf{U}}_2 = \lambda e^{-A/\xi(p)} e^{+R/\xi(p)} e^{-2L/\xi(p)} \tag{5.42g}$$

$$\mathbf{V}_K = \lambda e^{-A/\xi(p)} e^{-2(K-1)L/\xi(p)} \quad (K \geqq 2) \tag{5.42h}$$

$$\tilde{\mathbf{V}}_K = \lambda e^{-2A/\xi(p)} e^{-2(K-1)L/\xi(p)} \quad (K \geqq 2) \tag{5.42i}$$

$$\mathbf{U}_J = \lambda e^{-3A/\xi(p)} (J-2)\, e^{-2(J-2)L/\xi(p)} \quad (J \geqq 3) \tag{5.42j}$$

$$\tilde{\mathbf{U}}_J = \lambda e^{-4A/\xi(p)} (J-2)\, e^{-2(J-2)L/\xi(p)} \quad (J \geqq 3) \tag{5.42k}$$

and use the solution $\mathbf{B}_N \geqq \mathbb{B}_N$ to derive our bound on ξ_b. Here (5.42a) and (5.42b) come from (5.10) and (5.11); (5.42d) and (5.42e) follow from Lemma 5.1; and the remaining equations follow from Lemma 5.3. Note that we have replaced all the various λ's – which are A and R independent – by $\lambda \equiv \max\{1, \lambda_2(p),$

$\tilde{\lambda}_2(p), \lambda_3(p), \tilde{\lambda}_3(p), \lambda_4(p), \tilde{\lambda}_4(p)\}$, and (for reasons which will shortly become clear) we have degraded the bound of Lemma 5.1 by inserting an extra factor of λ into (5.42d).

Transforming the system (5.41) we have

$$\hat{\mathbf{B}}(z) = 1 + \hat{\mathbf{V}}(z)\,\hat{\mathbf{B}}(z) + \hat{\mathbf{U}}(z)\,\hat{\mathbf{A}}(z) \tag{5.43a}$$

$$\hat{\mathbf{A}}(z) = \mathbf{A}_0 + \hat{\hat{\mathbf{V}}}(z)\,\hat{\mathbf{B}}(z) + \hat{\hat{\mathbf{U}}}(z)\,\hat{\mathbf{A}}(z); \tag{5.43b}$$

or, after a bit of algebra,

$$\hat{\mathbf{B}}(z) = [1 - \hat{\hat{\mathbf{U}}}(z) + \mathbf{A}_0\,\hat{\mathbf{U}}(z)]/[1 - (\hat{\mathbf{V}}(z) + \hat{\hat{\mathbf{U}}}(z)) + (\hat{\mathbf{V}}(z)\,\hat{\hat{\mathbf{U}}}(z) - \hat{\mathbf{U}}(z)\,\hat{\hat{\mathbf{V}}}(z))] \tag{5.44a}$$

$$\hat{\mathbf{A}}(z) = [\mathbf{A}_0(1 - \hat{\mathbf{V}}(z)) + \hat{\hat{\mathbf{V}}}(z)]/[1 - (\hat{\mathbf{V}}(z) + \hat{\hat{\mathbf{U}}}(z)) + (\hat{\mathbf{V}}(z)\,\hat{\hat{\mathbf{U}}}(z) - \hat{\mathbf{U}}(z)\,\hat{\hat{\mathbf{V}}}(z))]. \tag{5.44b}$$

The leading large-N behavior of $\mathbf{B}_N$ (or $\mathbf{A}_N$) is determined by the earliest pole of (5.44). Thus the solutions to the equation

$$1 = \hat{\mathbf{V}}(z) + \hat{\hat{\mathbf{U}}}(z) + (\hat{\mathbf{V}}(z)\,\hat{\hat{\mathbf{U}}}(z) - \hat{\mathbf{U}}(z)\,\hat{\hat{\mathbf{V}}}(z)) \tag{5.45}$$

will provide a lower bound on e^{1/ξ_b} and hence an upper bound on ξ_c. Will not actually solve (5.45).

Instead we will choose values of R and A such that we obtain a lower bound on the solution of (5.45) which separates ξ_c from ξ.

First, it is observed that since we have arranged $\hat{\hat{\mathbf{U}}}(z) = e^{-A/\xi(p)}\hat{\mathbf{U}}(z)$ and $\hat{\hat{\mathbf{V}}}(z) = e^{-A/\xi(p)}\hat{\mathbf{V}}(z)$, the cross term vanishes identically. This renders the r.h.s. of (5.45) a (strictly) monotone increasing function for $z \in \mathbb{R}^+$. From Eqs. (5.42), we have:

$$\hat{\mathbf{V}}(z) + \hat{\hat{\mathbf{U}}}(z) = \lambda\delta(R)\,\kappa z + \lambda e^{-A/\xi(p)} e^{+R/\xi(p)} \kappa^2 z^2 + \mathbf{T}_1(z) + \mathbf{T}_2(z) \tag{5.46}$$

where

$$\mathbf{T}_1(z) = \frac{\lambda e^{-A/\xi(p)} \kappa^2 z^2}{1 - \kappa^2 z} \tag{5.47a}$$

and

$$\mathbf{T}_2(z) = \frac{\lambda e^{-4A/\xi(p)} \kappa^2 z^3}{[1 - \kappa^2 z]^2}, \tag{5.47b}$$

and, for algebraic ease, we have temporarily adopted the notation $\kappa \equiv \kappa(R, A) = e^{-L/\xi}$.

Our scheme is as follows: Let $z^* = z^*(R, A)$ denote the solution to (5.45). We pick R_0 large enough so that

$$\lambda\delta(R_0) \leqq \tfrac{1}{4}. \tag{5.48}$$

Next, define $z_0(R_0, A)$ to undershoot a crude version of (5.45), i.e.

$$\lambda\delta(R_0)\,\kappa z_0 = \tfrac{1}{2}. \tag{5.49}$$

Then, we claim that we can find $A(R_0)$ large enough so that

$$\hat{\mathbf{V}}(z_0)+\hat{\mathbf{U}}(z_0)<1, \tag{5.50}$$

and hence, by monotonicity,

$$z_0<z^*. \tag{5.51}$$

By the above reasoning and Eqs. (5.48), (5.49) and (5.51), we have

$$e^{L/\xi_c}\geqq e^{1/\xi_b}\geqq z^*\geqq 2e^{L/\xi} \tag{5.52}$$

and hence, since $L=2A+R$,

$$\frac{1}{\xi_c}\geqq\frac{\log 2}{2A(R_0)+R_0}+\frac{1}{\xi} \tag{5.53}$$

which is the desired result.

It remains to satisfy (5.50). Setting $\lambda\kappa z_0=1/2\delta$, the second term on the r.h.s. of (5.46) becomes

$$\frac{1}{4\lambda\delta^2}e^{-A/\xi(p)}e^{+R_0/\xi(p)} \tag{5.54a}$$

while $\mathbf{T}_1$ becomes

$$\frac{1}{4\lambda\delta^2}e^{-A/\xi(p)}\left[1-\frac{1}{2\delta}e^{-2A/\xi(p)}e^{-R_0/\xi(p)}\right]^{-1} \tag{5.54b}$$

and $\mathbf{T}_2$ is given by

$$\frac{1}{8\lambda^2\delta^3}e^{-2A/\xi(p)}e^{+R_0/\xi(p)}\left[1-\frac{1}{2\delta}e^{-2A/\xi(p)}e^{-R_0/\xi(p)}\right]^{-2}. \tag{5.54c}$$

It is manifest that the quantities in (5.54) tend to zero as $A\to\infty$. □

6. Fluctuations of paths in percolation

In this section, we establish our principal results: First, in Sect. 6a, we prove results (I)–(III) of the introduction for the standard connectivities (Theorem 6.2). Next, in Sect. 6b, we use these results to establish a multidimensional local limit theorem for percolation clusters (Theorem 6.3). Finally, in Sect. 6c, we prove absence of a roughening transition in two-dimensional percolation (Theorem 6.4).

6a. Asymptotic properties of the standard connectivity functions

Using the knowledge that $\forall p<p_c$, ξ_c is strictly less than ξ (Theorem 5.4), results (I)–(III) of Theorem 4.4 now hold for the quantities $h_{0,(L,\mathbf{a})}$ and $\mathbb{H}_L$ throughout

the subcritical regime. However, the purpose of this investigation was, of course, to establish such results for the "real" connectivity functions, namely $\tau_{0,(L,\mathbf{a})}$ and $\mathbb{G}_L$. This has already been done for result (III) since we have proved that the decay rates for the g's and h's are identical (Proposition 3.1). The analogues of (I) and (II) will follow by analytically relating the transforms of $\tau_{0,(L,\mathbf{a})}$ and $h_{0,(L,\mathbf{a})}$.

Proposition 6.1. *Let the direct connectivities* $c_{\mathbf{x},\mathbf{y}}$ *and* $k_{\mathbf{x},\mathbf{y}}(p)$ *be defined as in* Eqs. (3.24) *and* (3.37) *(and recall that they are related by the uniform bounds of Proposition* 4.3, *with common decay rate* $\xi_c(p)$*). Let* $h_{\mathbf{x},\mathbf{y}}(p)$ *and* $\tau_{\mathbf{x},\mathbf{y}}(p)$ *be the cylinder and free connectivities, defined in* Eqs. (3.2) *and* (1.1), *and let* $\hat{h}(z,\mathbf{k};p)$ *and* $\hat{\tau}(z,\mathbf{k};p)$ *denote their Laplace-Fourier transforms. Then* $\hat{\tau}(z,\mathbf{k};p)$ *may be expressed as*

$$\hat{\tau}(z,\mathbf{k};p)=A_1(z,\mathbf{k};p)+A_2(z,\mathbf{k};p)\,\hat{h}(z,\mathbf{k};p)$$

where $A_1(z,\mathbf{k})$ *and* $A_2(z,\mathbf{k})$ *have the same joint analyticity that was established for* $\hat{c}(z,\mathbf{k})$ *in Proposition* 4.2 (ii). *Explicitly:*

$\forall p<p_c$, $\forall |z_0|<e^{1/\xi_c(p)}$; $\exists\,\delta_k=\delta_k(p,z_0)>0$ *such that* $A_1(z,\mathbf{k})$ *and* $A_2(z,\mathbf{k})$ *are analytic in the regions* $|z|\leqq|z_0|$ *and* $|\mathbf{k}|\leqq\delta_k$.

Moreover, $A_1(z,\mathbf{k})$ *is (a constant plus) the transform of* $k_{0,(L,\mathbf{a})}$, *while* $A_2(z,\mathbf{k})$ *is the (square of the) transform of another modification of the direct correlation function.*

Proof. Let us define direct correlation events $\mathscr{d}_{0,(L,\mathbf{a})}$ which behave like the $\mathscr{c}$-events at the left boundary ($\mathbf{P}(0)$), but which are permitted the freedom of $\mathscr{k}$-events at the right boundary ($\mathbf{P}(L)$):

(6.1 a) $$\mathscr{d}_{0,(L,\mathbf{a})}=\{\omega\in\mathscr{k}_{0,(L,\mathbf{a})}\,|\,C(0)\|_{\mathbf{S}(L)}\cap\mathbf{P}(0)=\{0\}\}$$

(6.1 b) $$d_{0,(L\mathbf{a})}=P(\mathscr{d}_{0,(L,\mathbf{a})}).$$

(Note that these d-functions do not coincide with any of the additional direct connectivities introduced in the Appendix, since the connections contributing to the latter functions were always required to occur in the cylinder.) Now, the d-functions obviously satisfy

(6.2) $$c_{0,(L,\mathbf{a})}\leqq d_{0,(L,\mathbf{a})}\leqq k_{0,(L,\mathbf{a})}.$$

Thus ξ_c is the decay rate for all of these direct correlation functions.

Given this, we can follow exactly the derivation in Eqs. (4.7)–(4.10) to prove that $\hat{d}(\mathbf{k},z)$ and $\hat{k}(\mathbf{k},z)$ are both jointly analytic in z and k, for $|z|\leqq z_0|\mathbf{k}|\leqq\delta_k(p,z_0)$, $\forall z_0$, $0<z_0<e^{1/\xi(p)}$ for some $\delta_k(p,z_0)>0$.

Let us now show that $\hat{d}(\mathbf{k},z)$ and $\hat{k}(\mathbf{k},z)$ can be used to relate the transform functions $\hat{\tau}(z,\mathbf{k};p)$ and $\hat{h}(z,\mathbf{k};p)$. To this end, let $\omega\in\mathscr{g}_{0,(L,\mathbf{a})}$ and consider the intersection of $C(0;\omega)\equiv C(0)$ with the planes $\mathbf{P}(j)$, $1\leqq j\leqq N-1$. Obviously there are only three possibilities:

(a) All of the planes satisfy $|\mathbf{P}(j)\cap C(0)|\geqq 2$.
(b) Exactly one of the planes, say $j=N$, satisfies $|\mathbf{P}(j)\cap C(0)|=1$.
(c) More than one of the planes satisfies $|\mathbf{P}(j)\cap C(0)|=1$. In this case, denote the leftmost such plane by $j=N$, and the rightmost by $j=M>N$.

In case (a), it is clear that

$$\omega \in k_{0,(L,\mathbf{a})}. \tag{6.3a}$$

In the second instance, we let $\mathbf{b} \in \mathbb{Z}^{d-1}$ denote the transverse coordinate of the unique point in $\mathbf{P}(N)$ belonging to $C(0)$. Then one has

$$\omega \in d_{0,(N,\mathbf{b})} \cap d_{(N,\mathbf{b}),(L,\mathbf{a})}. \tag{6.3b}$$

Finally, letting $(N, \mathbf{b})$ and $(M, \mathbf{c})$ denote the unique points in $\mathbf{P}(N)$ and $\mathbf{P}(M)$ belonging to $C(0)$, case (c) implies that

$$\omega \in d_{0,(N,\mathbf{b})} \cap h_{(N,\mathbf{b}),(M,\mathbf{c})} \cap d_{(M,\mathbf{c}),(L,\mathbf{a})}. \tag{6.3c}$$

Using the (near) factorization property of the various events and the fact that these events represent disjoint possibilities which exhaust $g_{0,(L,\mathbf{a})}$, we have

$$\begin{aligned} \tau_{0,(L,\mathbf{a})} = k_{0,(L,\mathbf{a})} &+ f(p) \sum_{N,\mathbf{b}} d_{0,(N,\mathbf{b})}\, d_{(N,\mathbf{b}),(L,\mathbf{a})} \\ &+ f^2(p) \sum_{\substack{N,\mathbf{b} \\ M,\mathbf{c} \\ M>N}} d_{0,(N,\mathbf{b})}\, h_{(N,\mathbf{b}),(M,\mathbf{c})}\, d_{(M,\mathbf{c}),(L,\mathbf{a})}, \end{aligned} \tag{6.4}$$

where $f(p)$ is the "patching factor" defined in the statement of Proposition 4.1. Recalling that $h_{0,(0,\mathbf{a})} = f^{-1}(p)\,\delta_{0,\mathbf{a}}$, it is seen that the second term in (6.4) may be absorbed into the third provided that we relax the requirement of strict separation of M and N. Defining the various transform functions (see also (4.3a)):

$$\hat{t}(z,\mathbf{k};p) = f(p) \sum_{L,\mathbf{a}} \tau_{0,(L,\mathbf{a})}(p)\, z^L e^{i\mathbf{k}\cdot\mathbf{a}} \tag{6.5a}$$

$$\hat{k}(z,\mathbf{k};p) = f(p) \sum_{L,\mathbf{a}} k_{0,(L,\mathbf{a})}(p)\, z^L e^{i\mathbf{k}\cdot\mathbf{a}} \tag{6.5b}$$

$$\hat{d}(z,\mathbf{k};p) = f(p) \sum_{L,\mathbf{a}} d_{0,(L,\mathbf{a})}(p)\, z^L e^{i\mathbf{k}\cdot\mathbf{a}}, \tag{6.5c}$$

with the stipulation that $\tau_{0,(0,\mathbf{a})} = \delta_{0,\mathbf{a}}$ and $k_{0,(0,\mathbf{a})} = d_{0,(0,\mathbf{a})} \equiv 0$, the relationship

$$\hat{t}(z,\mathbf{k};p) = 1 + \hat{k}(z,\mathbf{k};p) + \hat{d}^2(z,\mathbf{k};p)\,\hat{h}(z,\mathbf{k};p) \tag{6.6}$$

is easily verified. □

Theorem 6.2. *Let $\tau_{\mathbf{x},\mathbf{y}}(p)$ and $\mathbb{G}_L(p)$ denote the standard percolation connectivity functions as defined in* Eqs. (1.1) *and* (1.4), *with common decay rate $\xi(p)$ given by Proposition* 3.1. *Then $\forall p < p_c$:*

(I) $\exists K_2(p) \geqq 1$, $\Delta(p) > 0$ *such that*

$$|\mathbb{G}_L(p)\, e^{+L/\xi(p)} - K_2(p)| \leqq e^{-\Delta(p)L}.$$

(II) $\exists \alpha(p) > 0$ *such that* $\forall \mathbf{a} \in \mathbb{Z}^{d-1}$ *satisfying* $|\mathbf{a}| \leqq L^{3/4-\varepsilon}$ *with* $\varepsilon > 0$,

$$\tau_{0,(L,\mathbf{a})} \sim K_2(p) \frac{1}{[\alpha(p)\,\pi L]^{(d-1)/2}}\, e^{-L/\xi(p)}\, e^{-\mathbf{a}^2/[\alpha(p)L]}\,[1 + O(L^{-1}, L^{-4\varepsilon})].$$

In particular:

(i) *The expression above represents the first term in an asymptotic expansion: i.e., for any fixed function* $\mathbf{a}(L)$, $|\mathbf{a}(L)| \leqq L^{3/4-\varepsilon}$, *with* $\mathbf{a}(L)$ *tending to infinity (e.g., as a power of L), the* $O(L^{-1}, L^{-4\varepsilon})$ *terms can be systematically calculated in an asymptotic series.*

(ii) *The error term is uniform in* $\mathbf{a}$: *i.e.* $\exists d_1'(p)$, $d_2'(p) < \infty$ *such that* $\forall \mathbf{a} \in \mathbb{Z}^{d-1}$ *satisfying* $|\mathbf{a}| \leqq L^{3/4-\varepsilon}$

$$|\tau_{0,(L,\mathbf{a})}][\alpha(p)\,\pi L]^{(d-1)/2} e^{+L/\xi(p)} e^{+\mathbf{a}^2/[\alpha(p)L]} - K_2(p)| \leqq \frac{d_1'(p)}{L} + \frac{d_2'(p)}{L^{4\varepsilon}}.$$

(iii) *The tail of the distribution is uniformly bounded: i.e.,* $\exists d_3'(p) < \infty$ *such that* $\forall \mathbf{a} \in \mathbb{Z}^{d-1}$ *satisfying* $|\mathbf{a}| \leqq L^{3/4-\varepsilon}$

$$\sum_{\mathbf{b}:|b_j| \geqq |a_j|} \tau_{0,(L,\mathbf{b})} \leqq d_3'(p)\, L^{(d-1)/2} e^{-L/\xi(p)} e^{-\mathbf{a}^2/[\alpha(p)L]}$$

where a_j *and* b_j *are the* j^{th} *components of the vectors* $\mathbf{a}$ *and* $\mathbf{b}$.

(III) $\xi(p)$ *is real analytic.*

Proof. As previously remarked, (III) has already been established in Theorem 4.4 (III). To establish (I) and (II), we use the fact that $\hat{h}(z, \mathbf{k}; p)$ and $\hat{\tau}(z, \mathbf{k}; p)$ are analytically related by Proposition 6.1. Thus the manipulations performed on $\hat{h}(z, \mathbf{k}; p)$ in the proof of Theorem 4.4 may be directly carried out on $\hat{\tau}(z, \mathbf{k}; p)$. The only change is that the function $F(z, \mathbf{k})$ introduced in Eq. (4.26) is replaced by another function, computable via Proposition 6.1, which modifies the constant $\tilde{K}_2(p)$. The fact that $K_2(p) \geqq 1$ follows from the a priori lower bound on $\mathbb{G}_L$, given in Proposition 3.1. □

6b. A multidimensional local limit theorem

Using the results of the previous subsection (and of previous sections), we will establish a local limit theorem for the transverse fluctuations of subcritical paths. Here we will consider a situation in which the system contains a long path – in particular from 0 to $(L, 0, \ldots, 0)$ – and obtain detailed information on the location of the cluster in the intervening planes. Since the cluster will often have multiple intersections with any given intermediate plane, we will employ the following device:

Definition 6.1. Let $(L, \mathbf{a}) \in \mathbb{Z}^d$ and denote by $\Phi_{\mathbf{a}}^L$ the event that the origin is connected to $\mathbf{P}(L)$ and that the maximum extent of the cluster in $\mathbf{P}(L)$ in the i^{th} coordinate direction is a_i, $i = 2, \ldots, d$.

$$\Phi_{\mathbf{a}}^L = \{\omega \mid \exists \mathbf{x} \in \mathbf{P}(L) \text{ such that } \mathbf{x} \in C(0); \quad \text{and} \quad \max\{y_i \mid \mathbf{y} \in \mathbf{P}(L) \cap C(0)\} = a_i, i = 2, \ldots, d\}. \tag{6.7}$$

Note that, for $d > 2$, $\Phi_{\mathbf{a}}^L$ does not necessarily imply that $\omega \in \mathcal{g}_{0,(L,\mathbf{a})}$.

Our local limit result is as follows:

Theorem 6.3. *Suppose $p<p_c$. Let $q\in(0,1)$, $v\in\mathbb{R}^{d-1}$ and $L\equiv\mathbb{Z}$, and denote by $Q\in\mathbb{Z}$ the smallest integer larger than qL, and by $\mathbf{v}\in\mathbb{Z}^{d-1}$ the vector whose i^{th} component is the smallest integer larger than $\sqrt{L}v_i$. Let the events $g_{0,(L,0)}$ and $\Phi^Q_{\mathbf{v}}$, be defined as in* Eqs. (3.1) *and* (6.7), *and from these events define*

$$\rho_q(v)\equiv\rho_q(v;p)=L^{(d-1)/2}P(\Phi^Q_{\mathbf{v}}\,|\,g_{0,(L,0)}).$$

Then, as $L\to\infty$, for every $\delta>0$,

$$\rho_q(v)=\frac{1}{[q(1-q)\,\pi\alpha(p)]^{(d-1)/2}}\,e^{-v^2/[q(1-q)\alpha(p)]}(1+O(L^{-(1-\delta)}),$$

where $\alpha(p)>0$ is defined in Eq. (4.23), *and is the same constant appearing in Theorems* 4.4 *and* 6.2.

Proof. Let Q and $\mathbf{v}$ be as defined in the statement of the proposition, and suppose $\omega\in\Phi^Q_{\mathbf{v}}\cap g_{0,(L,0)}$. We will distinguish two cases:

(1) $|C(0)\cap\mathbf{P}(Q)|=1$

(2) $|C(0)\cap\mathbf{P}(Q)|>1$.

Before doing detailed estimates on these two cases, let us dispense with an unlikely set of possibilities. Given that $\omega\in\Phi^Q_{\mathbf{v}}\cap g_{0,(L,0)}$, it is of course possible that $\omega\in\Phi^Q_{\mathbf{v}}\cap k_{0,(L,0)}$, or at least that ω contains a (translate of a) realization of $k_{0,(\lambda L,\mathbf{b})}$ or $d_{0,(\lambda L,\mathbf{b})}$ for some $\mathbf{b}\in\mathbb{Z}^{d-1}$ and some $\lambda>0$. However, the fact that $\xi_c<\xi$ (Theorem 5.4) implies that these configurations are (relatively) exponentially unlikely.[2] Thus in our subsequent estimates, we can neglect terms containing k-functions or d-functions of order $O(L)$ (i.e. configurations in which there are $O(L)$ consecutive planes $\mathbf{P}(j)$ with $|C(0)\cap\mathbf{P}(j)|>1$) at a cost of no more than $e^{-L/\xi}e^{-O(L^\varepsilon)}$ for any $\varepsilon\in(0,1)$.

Let us first consider case (1): i.e., the case in which $C(0)$ has only a single point of intersection with $\mathbf{P}(Q)$. Then, by definition, $(Q,\mathbf{v})\in C(0)$. Assuming they exist, let $N>0$ be the leftmost plane and $M<Q$ be the rightmost plane in $\mathbf{S}(Q)\setminus\mathbf{P}(Q)$ in which $C(0)$ contains a unique point. Denote these points by $(N,\mathbf{a})$ and $(M,\mathbf{b})$, respectively. To the right of $\mathbf{P}(Q)$, we denote the similarly distinguished points by $(R,\mathbf{c})$, $R>Q$, and $(P,\mathbf{d})$, $P<L$. Of course, some of these points will not exist if ω contains a realization of $d_{0,(Q,\mathbf{v})}$ or $d_{(Q,\mathbf{v}),(L,0)}$; however, as discussed above, the probability of either event is easily bounded. Now, if e.g. $(M,\mathbf{b})$ does exist, then it is easy to see that in $\mathbf{S}(Q)$, ω contains realizations of $c_{(M,\mathbf{b}),(Q,\mathbf{v})}$, $h_{(N,\mathbf{a}),(M,\mathbf{b})}$ and $d_{0,(N,\mathbf{a})}$ which are disjoint except for the bounding planes $\mathbf{P}(N)$ and $\mathbf{P}(M)$. Thus, except for configurations containing $d_{0,(Q,\mathbf{v})}$ or $d_{(Q,\mathbf{v}),(L,0)}$, the probability of case (1) is given by

$$f^5(p)\sum_{\substack{0<N\leq M<Q<R\leq P<L\\ \mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\in\mathbb{Z}^{d-1}}} d_{0,(N,\mathbf{a})}\,h_{(N,\mathbf{a}),(M,\mathbf{b})}\,c_{(M,\mathbf{b}),(Q,\mathbf{v})}\,c_{(Q,\mathbf{v}),(R,\mathbf{c})}\,h_{(R,\mathbf{c}),(P,\mathbf{d})}\,d_{(P,\mathbf{d}),(L,0)}, \tag{6.8}$$

where $f(p)$ is the "patching factor" defined in the statement of Proposition 4.1.

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[2] Indeed, the result $\xi_c<\xi$ implies that the only configurations of significance are those with $O(L)$ planes in which $C(0)$ has a unique point of intersection

Let us now consider case (2): i.e. when $C(0)$ has multiple intersections with $\mathbf{P}(Q)$. To handle these situations, let us define, for $0<N<L$,

$$c_{0,(L,\mathbf{a})}^{N|\mathbf{b}}=P(\mathscr{C}_{0,(L,\mathbf{a})}\cap\Phi_{\mathbf{b}}^{N}), \tag{6.9}$$

and, in general,

$$c_{(P,\mathbf{c}),(L,\mathbf{a})}^{N|\mathbf{b}}=P(\mathscr{C}_{0,(L,\mathbf{a})}\cap T^{(P,\mathbf{c})}(\Phi_{\mathbf{b}-\mathbf{c}}^{N-P})) \tag{6.10}$$

where $T^{(-)}(-)$ denotes the translation operator. As in case (1), we can define the points $(N,\mathbf{a})$, $(M,\mathbf{b})$, $(R,\mathbf{c})$ and $(P,\mathbf{d})$. Then, provided that $\mathscr{d}_{0,(Q,\mathbf{v})}$ or $\mathscr{d}_{(Q,\mathbf{v}),(L,0)}$ do not occur, we see that the probability of case (2) is given by:

$$f^4(p)\sum_{\substack{0<N\leqq M<Q<R\leqq P<L\\ \mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\in\mathbb{Z}^{d-1}}} d_{0,(N,\mathbf{a})}\,h_{(N,\mathbf{a}),(M,\mathbf{b})}\,c_{(M,\mathbf{b}),(R,\mathbf{c})}^{Q|\mathbf{v}}\,h_{(R,\mathbf{c}),(P,\mathbf{d})}\,d_{(P,\mathbf{d}),(L,0)}. \tag{6.11}$$

Let $\varepsilon\in(0,\frac{1}{4})$. By the above reasoning, up to terms smaller than the order of $e^{-L/\xi}e^{-O(L^\varepsilon)}$, $P(\Phi_{\mathbf{v}}^{Q}\cap\mathscr{g}_{0,(L,0)})$ is given by the sum of (6.8) and (6.11). Furthermore, we claim that outside of the ranges

(i) $N,(L-R),(R-M)<L^\varepsilon$
(ii) $|\mathbf{a}|,|\mathbf{d}|<2L^\varepsilon$
(iii) $|\mathbf{b}-\mathbf{v}|,|\mathbf{c}-\mathbf{v}|<3L^\varepsilon$,
(iv) $|\mathbf{b}|,|\mathbf{v}|,|\mathbf{c}|<L^{1/2+\varepsilon}$

the contribution of (6.8) and (6.11) is also of order $e^{-L/\xi}e^{-O(L^\varepsilon)}$.

Indeed, suppose that $N\geqq L^\varepsilon$. Then, either sum may be bounded above by

$$\begin{aligned}\sum_{\substack{N,\mathbf{a}\\ L\geqq N\geqq L^\varepsilon}} d_{0,(N,\mathbf{a})}\,\tau_{(N,\mathbf{a}),(L,0)} &\leqq(\text{const.})\sum_{\substack{N,\mathbf{a}\\ L\geqq N\geqq L^\varepsilon}}\tau_{(N,\mathbf{a}),(L,0)}\,e^{-N/\xi_c}\\ &=(\text{const.})\sum_{L\geqq N\geqq L^\varepsilon}\mathbb{G}_{L-N}\,e^{-N/\xi_c}\\ &\leqq(\text{const.})\,Le^{-L/\xi}e^{-(1/\xi-1/\xi_c)L^\varepsilon}.\end{aligned} \tag{6.12}$$

By symmetry, we obtain an analogous estimate if $L-R\geqq L^\varepsilon$. When $R-M\geqq L^\varepsilon$, it is seen that the contribution to the sum in (6.11) may be bounded by

$$\begin{aligned}\sum_{\substack{\mathbf{b},\mathbf{c}\\ M,R:\,R-M\geqq L^\varepsilon}}\tau_{0,(M,\mathbf{b})}\,\tau_{(R,\mathbf{c}),(L,0)}\,c_{(M,\mathbf{b}),(R,\mathbf{c})} &\leqq e^{-M/\xi}\sum_{\substack{\mathbf{b},\mathbf{c}\\ M,R:\,R-M\geqq L^\varepsilon}}\tau_{(R,\mathbf{c}),(L,0)}\,c_{(M,\mathbf{b}),(R,\mathbf{c})}\\ &\leqq(\text{const.})\,Le^{-L/\xi}e^{-(1/\xi-1/\xi_c)L^\varepsilon},\end{aligned} \tag{6.13}$$

with a similar estimate for (6.8). Thus, to within the stated error, we need only sum in the range given in (i).

Violation of the conditions in (ii) and (iii) leads to estimates of the same form. Indeed, it is easy to see that a "lateral over-extension" of $d_{0,(N,\mathbf{a})}(\leqq\tau_{0,(N,\mathbf{a})})$ when $N<L^\varepsilon$ and $|\mathbf{a}|\geqq 2L^\varepsilon$ produces a bound of the form (6.12), as is also the case, by symmetry, when $|\mathbf{d}|\geqq 2L^\varepsilon$. Given that $(Q-M)$ and $(P-Q)<L^\varepsilon$ (which we may assume by (1)), a displacement of $|\mathbf{b}-\mathbf{v}|$ or $|\mathbf{c}-\mathbf{v}|$ exceeding $3L^\varepsilon$ would be similarly costly.

Finally, observe that if any of the points **b**, **c**, **v** exceed $L^{1/2+\varepsilon}$ in magnitude, there must be some point in $C(0)$ with a huge lateral fluctuation – say, for example, $(Q', \mathbf{v}')\in C(0)$ with $|\mathbf{v}'|>L^{1/2+\varepsilon}$ and Q' of order L. The total contribution of such configurations may be bounded by

$$(6.14)\quad \sum_{\substack{|\mathbf{v}'|\geq L^{1/2+\varepsilon}\\ Q'}} \tau_{0,(Q',\mathbf{v})}\,\tau_{(Q',\mathbf{v}'),(L,0)} \leqq (\text{const.})\, L^{(d-1)/2} \sum_{Q'} \mathbb{G}_{Q'}\, e^{-(L-Q')/\xi} e^{-L^{\varepsilon}/\alpha}$$
$$\leqq (\text{const.})\, L^{(d+1)/2} e^{-L/\xi} e^{-L^{\varepsilon}/\alpha},$$

where we have used property (II iii) of Theorem 6.2 to control the tail of the sum. Thus we can restrict in the range given in (iv) at the stated cost.

Now let us apply the asymptotic formula for $h_{0,(L,\mathbf{a})}$ (Theorem 4.4 II) under the restrictions (i)–(iv). Noting that the error terms in this formula are uniform in the transverse coordinate (Theorem 4.4 (II ii)), and that, by our restrictions (ii)–(iv), we need only consider $|\mathbf{a}-\mathbf{b}|$, $|\mathbf{c}-\mathbf{d}|<L^{1/2+\varepsilon}$, the contribution to $P(\Phi_{\mathbf{v}}^{Q}\cap \mathscr{g}_{0,(L,0)})$ from case (Eq. (6.8)) is equal to

(6.15)
$$f^5(p) \sum_{\substack{N,M,R,P\\ \mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}} d_{0,(N,\mathbf{a})}\, e^{+N/\xi}\, c_{(M,\mathbf{b}),(Q,\mathbf{v})}\, e^{+(Q-M)/\xi}\, c_{(Q,\mathbf{v}),(R,\mathbf{c})}\, e^{+(R-Q)/\xi}\, d_{(P,\mathbf{d}),(L,0)}\, e^{+(L-P)/\xi}$$
$$\times e^{-L/\xi} \frac{\tilde{K}_2(p)}{(\alpha\pi(M-N))^{(d-1)/2}} e^{-(\mathbf{a}-\mathbf{b})^2/\alpha(M-N)} \frac{\tilde{K}_2(p)}{(\alpha\pi(P-R))^{(d-1)/2}} e^{-(\mathbf{c}-\mathbf{d})^2/\alpha(P-R)}$$
$$\times [1+O(L^{-(1-4\varepsilon)})],$$

while case (2) (Eq. (6.11)) yields a contribution of

(6.16)
$$f^4(p) \sum_{\substack{N,M,R,P\\ \mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}} d_{0,(N,\mathbf{a})}\, e^{+N/\xi}\, c^{Q|\mathbf{v}}_{(M,\mathbf{b}),(R,\mathbf{c})}\, e^{+(R-M)/\xi}\, d_{(P,\mathbf{d}),(L,0)}\, e^{+(L-P)/\xi}$$
$$\times e^{-L/\xi} \frac{\tilde{K}_2(p)}{(\alpha\pi(M-N))^{(d-1)/2}} e^{-(\mathbf{a}-\mathbf{b})^2/\alpha(M-N)} \frac{\tilde{K}_2(p)}{(\alpha\pi(P-R))^{(d-1)/2}} e^{-(\mathbf{c}-\mathbf{d})^2/\alpha(P-R)}$$
$$\times [1+O(L^{-(1-4\varepsilon)})].$$

Now by (i), we have restricted the sums to a range in which $(M-N)=Q[1+O(L^{\varepsilon})]$ and $(P-R)=(L-Q)[1+O(L^{\varepsilon})]$, so we may replace the factors of $(M-N)$ with Q and $(P-R)$ with $(L-Q)$ in the appropriate places in Eqs. (6.15) and (6.16) without altering the stated error. Next, using properties (ii)–(iv), it is seen that for all **a**, **b**, **c**, **d** under consideration, the factors of $(\mathbf{a}-\mathbf{b})^2$ and $(\mathbf{c}-\mathbf{d})^2$ are of order $\mathbf{v}^2+|\mathbf{v}|\,O(L^{\varepsilon})\leqq \mathbf{v}^2+O(L^{1/2+2\varepsilon})$; thus if we replace these factors by $\mathbf{v}^2$, our error term becomes $O(L^{-(1/2+2\varepsilon)})$. Collecting all relevant factors, we have

$$(6.17)\quad P(\Phi_{\mathbf{v}}^{Q}\cap \mathscr{g}_{0,(L,0)})=(S_1+S_2)\frac{\tilde{K}_2^2}{L^{(d-1)/2}}$$
$$\times \frac{e^{-L/\xi}}{(\alpha\pi L)^{(d-1)/2}} \frac{1}{(\alpha\pi(Q/L)(1-Q/L))^{(d-1)/2}} e^{-\mathbf{v}^2/\alpha L(1-Q/L)(Q/L)}$$
$$\times [1+O(L^{-(1/2+2\varepsilon)})]$$

where

$$S_1 = f^5(p) \sum_{\substack{N,M,R,P \\ \mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}} d_{0,(N,\mathbf{a})}\, e^{+N/\xi} c_{(M,\mathbf{b}),(Q,\mathbf{v})}\, e^{+(Q-M)/\xi} \times c_{(Q,\mathbf{v}),(R,\mathbf{c})}\, e^{+(R-Q)/\xi} d_{(P,\mathbf{d}),(L,0)}\, e^{+(L-P)/\xi} \tag{6.18}$$

and

$$S_2 = f^4(p) \sum_{\substack{N,M,R,P \\ \mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}} d_{0,(N,\mathbf{a})}\, e^{+N/\xi} c^{Q|\mathbf{v}}_{(M,\mathbf{b}),(R,\mathbf{c})}\, e^{+(R-M)/\xi} d_{(P,\mathbf{d}),(L,0)}\, e^{+(L-P)/\xi}. \tag{6.19}$$

Now it is clear that the error term will be unaffected if we sum S_1 and S_2 over unbounded ranges of indices – indeed, this will only incur another (relatively) exponentially small error. Thus, we may write

$$S_1 = f^5(p)\Big(\sum_N \mathbb{D}_N\, e^{+N/\xi}\Big)^2 \Big(\sum_N \mathbb{C}_N\, e^{+N/\xi}\Big)^2 \tag{6.20}$$

$$S_2 = f^4(p)\Big(\sum_N \mathbb{D}_N\, e^{+N/\xi}\Big)^2 \Big(\sum_{N,M>1} \mathbb{C}_{N+M}\, e^{+(N+M)/\xi}\Big) \tag{6.21}$$

instead of (6.18) and (6.19), where we have used

$$\mathbb{D}_N \equiv \sum_{\mathbf{b}\in\mathbb{Z}^{d-1}} d_{0,(N,\mathbf{b})} \tag{6.22}$$

and invoked the "sum rule"

$$\sum_{\mathbf{v}\in\mathbb{Z}^{d-1}} c^{Q|\mathbf{v}}_{(M,\mathbf{b}),(R,\mathbf{c})} = c_{(M,\mathbf{b}),(R,\mathbf{c})} \tag{6.23}$$

and translation invariance to obtain

$$\sum_{\mathbf{b},\mathbf{c}\in\mathbb{Z}^{d-1}} c^{Q|\mathbf{v}}_{(M,\mathbf{b}),(R,\mathbf{c})} = \mathbb{C}_{R-M}. \tag{6.24}$$

Finally, let us replace Q/L by φ and $\mathbf{v}^2/L$ by v^2 (a negligible error), and compare Eq. (6.17) to the statement of the theorem. Observe that one factor of $L^{-(d-1)/2}$ is cancelled by the definition of $\rho_\varphi(v)$. To obtain a conditional probability, we must place $\tau_{0,(L,0)}$ in the denominator of (6.17). Using the asymptotic formula

$$\tau_{0,(L,0)} \sim \frac{K_2(p)}{[\alpha(p)\,\pi L]^{(d-1)/2}}\, e^{-L/\xi(p)} \tag{6.25}$$

of Theorem 6.2, it is seen that the theorem is proved if we can establish $(\tilde{K}_2^2/K_2)(S_1+S_2)=1$.

To this end, we start by recalling (Lemma 4.3(i)) that $f(p)\sum_N \mathbb{C}_N\, e^{+N/\xi}=1$.

Now, according to the proof of Theorem 4.4 (I) (see, e.g., Eq. (4.17)), we may write

$$\hat{\mathbb{H}}(z) = [1-\hat{\mathbb{C}}(z)]^{-1} = F(z)[1-z e^{-1/\xi}]^{-1} \tag{6.26}$$

where $\tilde{K}_2$ is computed via

(6.27) $$f(p)\tilde{K}_2 = F(e^{1/\xi}).$$

Writing $F(z) = [1 - ze^{-1/\xi}][1 - \hat{\mathbb{C}}(z)]^{-1}$, we may expand $\hat{\mathbb{C}}(z)$ about $z = e^{1/\xi}$ with the result

(6.28) $$f(p)\tilde{K}_2 = e^{-1/\xi}[\hat{\mathbb{C}}'(e^{1/\xi})]^{-1},$$

where $\hat{\mathbb{C}}'$ is the derivative of $\hat{\mathbb{C}}$. Now

(6.29) $$e^{1/\xi}\hat{\mathbb{C}}'(e^{1/\xi}) = \sum_L L\mathbb{C}_L e^{+L/\xi},$$

so that

(6.30) $$\sum_L L\mathbb{C}_L e^{+L/\xi} = \frac{1}{f(p)\tilde{K}_2}.$$

However,

(6.31) $$f(p) \sum_{N,M>1} \mathbb{C}_{N+M} e^{+(N+M)/\xi} = f(p)\sum_L (L-1)\mathbb{C}_L e^{+L/\xi} = \sum_L L\mathbb{C}_L e^{+L/\xi} - 1;$$

whence

(6.32) $$(S_1 + S_2) = \frac{f^2(p)(\sum_N \mathbb{D}_N e^{+N/\xi})^2}{\tilde{K}_2}.$$

Finally, let us examine K_2. By definition, were we to write

(6.33 a) $$\mathbb{G}(z) = G(z)[1 - ze^{-1/\xi}]^{-1},$$

then

(6.33 b) $$f(p)K_2 = G(e^{1/\xi}).$$

Using the expression (6.6) at $\mathbf{k} = 0$ to compute $\hat{\mathbb{G}}(z)$, it is seen that

(6.34) $$G(z) = [1 - ze^{-1/\xi}][1 + \mathbb{K}(z)] + \hat{\mathbb{D}}^2(z)[1 - ze^{-1/\xi}][1 - \hat{\mathbb{C}}(z)]^{-1}$$

so that

(6.35) $$K_2 = \hat{\mathbb{D}}^2(z)\hat{K}_2$$

or

(6.36) $$K_2 = f^2(p)(\sum_N \mathbb{D}_N e^{+N/\xi})^2.$$

Combining (6.36) with the expression (6.32) for $(S_1 + S_2)$, the desired result is achieved. Note that we obtain the error estimate from (6.17) and the stipulation $\varepsilon \in (0, \frac{1}{4})$. □

6c. The absence of a roughening transition in two dimensions

The roughening problem in the context of percolation was introduced in [ACCFR]. Here we present a complete (but not altogether surprising) solution of the two-dimensional problem.

The set up is as follows: We start with a box of scale L

(6.37a) $$\Lambda_L = \{\mathbf{x} \in \mathbb{Z}^d \,|\, |\mathbf{x}| \leqq L\},$$

with upper boundary

(6.37b) $$\partial\Lambda_L^+ = \{\mathbf{x} \in \Lambda_L \,|\, |\mathbf{x}| = L, x_d > 0\}$$

and lower boundary

(6.37c) $$\partial\Lambda_L^- = \{\mathbf{x} \in \Lambda_L \,|\, |\mathbf{x}| = L, x_d < 0\}.$$

Notive that $\partial\Lambda_L^+ \cup \partial\Lambda_L^-$ does not include the points where $x_1 = 0$. We will be concerned with the behavior of configurations inside Λ_L. In particular, let us define the event

(6.38) $\mathscr{I}_L = \{\omega \,|\, \text{there is no path of occupied bonds between } \partial\Lambda_L^+ \text{ and } \partial\Lambda_L^-\}$.

The event $\mathscr{I}_L$ has the (dual) interpretation of the presence of an *interface* separating $\partial\Lambda_L^+$ from $\partial\Lambda_L^-$. The idea is to study the density $p > p_c$ Bernoulli measures conditioned on the event $\mathscr{I}_L$. In the limit $L \to \infty$, there are two essentially different possible outcomes:

(1) The interface remains localized.
(2) The interface fluctuates outside of any finite region.

Obviously (1) and (2) must be quantified. However, given any acceptable definition, a transition from behavior (1) to behavior (2) which occurs at a value of p distinct from p_c isa called a *roughening transition.* It is believed but it is not proven that a roughening transition occurs in three or more dimensions.

There are two reasonable criteria to distinguish cases (1) and (2). First, we can extract some limiting measure and determine whether or not it is Bernoulli. Second, we can examine the so-called roughening order parameter [ACCFR]:

Let $\mathbf{h} \in \mathbb{Z}^d$ be the point $(0, \ldots, h)$ which is h units above the midplane. We define

(6.39a) $$T_L^+(h) = \{\omega \,|\, C(\mathbf{h}) \cap \partial\Lambda_L^+ \neq \emptyset\}$$

and

(6.39b) $$T_L^-(h) = \{\omega \,|\, C(\mathbf{h}) \cap \partial\Lambda_L^- \neq \emptyset\}.$$

Clearly, when $\omega \in \mathscr{I}_L$, it cannot be the case that both $T_L^+(h)$ and $T_L^-(h)$ happen. We denote the difference between the conditional probabilities by $\theta_L(h)$:

(6.40a) $$\theta_L(h) = P(T_L^+(h) | \mathscr{I}_L) - P(T_L^-(h) | \mathscr{I}_L).$$

Presumably, for finite L and $h>0$, $\theta_L(h)$ is positive. One is interested in the limit

(6.40b)
$$\theta(h)=\lim_{L\to\infty}\theta_L(h)$$

should such a limit exist. If $\theta(h)$ is zero for every h, we are in the rough phase; otherwise, the interface is said to be rigid. Of secondary importance, one could consider the $h\uparrow\infty$ limit of $\theta(h)$. It is natural to assume that if the interface is rigid, $\theta(h)$ will tend to the percolation density.

We will examine the question of roughening in two dimensions from both perspectives. In particular, we will show:

Theorem 6.4a. *For the two-dimensional bond percolation problem, whenever $p>p_c$, the vague $L\to\infty$ limits of the Λ_L density-p Bernoulli measures conditioned on the event $\mathscr{I}_L$ are the ordinary density-p Bernoulli measures.*

and

Theorem 6.4b. *Under the conditions of Theorem* 6.4a $\theta(h)=0$ *for all h.*

The study of these interfacial problems is, for the most part, the analysis of the dual model. For the bond percolation model on $\mathbb{Z}^d$ (or indeed for any two-dimensional percolation model), this amounts to the study of a two-dimensional *sub*critical system which contains a long path – a familiar topic in this paper.

Indeed, when a bond of $\mathbb{Z}^d$ is vacant, it is customary to represent this by the event that the corresponding bond of $(\mathbb{Z}+\frac{1}{2})^2$ is occupied. On the square lattice, the event that the sites dual to $\mathbf{x}$ and $\mathbf{y}$, $\mathbf{x}^*$ and $\mathbf{y}^*$, are connected by a path of dual bonds will be denoted by $\mathscr{g}^*_{\mathbf{x}^*,\mathbf{y}^*}$. When boundary conditions are modified, we will denote the corresponding events by $\mathscr{k}^*_{\mathbf{x}^*,\mathbf{y}^*}$, etc. Also, we will use $C^*(\mathbf{x}^*)$ to denote the dual connected cluster of the point $\mathbf{x}^*$.

Observe that $\tau^*_{\mathbf{x}^*,\mathbf{y}^*}(p)\equiv P(\mathscr{g}^*_{\mathbf{x}^*,\mathbf{y}^*})$ is precisely the ordinary connectivity, $\tau_{\mathbf{x},\mathbf{y}}$, at density $(1-p)$. This exact (self) duality, although somewhat helpful, is not essential to our arguments. It is worth noting that in $d=2$, $p>p_c$ implies that the dual correlations decay exponentially [K]. Here we will use the symbol $\xi\equiv\xi(1-p)$ to denote the rate of decay of the dual connectivities when the (direct) bond density is p. In addition, we will use various other symbols whose arguments, strictly speaking, should be $(1-p)$.

Defining

(6.41)
$$\Lambda_L^*=\{\mathbf{x}^*\in(\mathbb{Z}+\tfrac{1}{2})^2\,|\,\exists\,\mathbf{y}\in\Lambda_L \text{ such that } |\mathbf{x}^*-\mathbf{y}|=\tfrac{1}{2}\},$$

it is seen that the event $\mathscr{I}_L$ is precisely the event that one of the (dual) connected clusters of $(-(L+\frac{1}{2}), +\frac{1}{2})$ or $(-(L+\frac{1}{2}), -\frac{1}{2})$ (restricted to Λ_L^*) contains one of the points $(+(L+\frac{1}{2}), +\frac{1}{2})$ or $(+(L+\frac{1}{2}), -\frac{1}{2})$. In particular, this is essentially the event $\tilde{\mathscr{k}}_{(-L,0),(+L,0)}$, about which we now have some detailed information.

In order to avoid further typographical and linguistical complications, we will "identify" the two nearby points $((L+\frac{1}{2}), \pm\frac{1}{2})$ and similarly identify the pair $(-(L+\frac{1}{2}), \pm\frac{1}{2})$. Thus, for example, we will use $C^*(((L+\frac{1}{2}), \pm\frac{1}{2}))$ as notation for $C^*(((L+\frac{1}{2}), +\frac{1}{2})) \cup C^*(((L+\frac{1}{2}), -\frac{1}{2}))$, and $\mathscr{G}^*_{((L+\frac{1}{2}), \pm\frac{1}{2}), \mathbf{x}^*}$ as notation for the union of the events $\mathscr{G}^*_{((L+\frac{1}{2}), +\frac{1}{2}), \mathbf{x}^*}$ and $\mathscr{G}^*_{((L+\frac{1}{2}), -\frac{1}{2}), \mathbf{x}^*}$.

To prove Theorems 6.4 a, b we will need the following:

Lemma 6.5. *Let B, $L \in \mathbb{Z}^+$, $B \ll L$, and consider the density-p Bernoulli measure on the configurations in Λ_L. Denote by $\mathscr{E}_B$ the event that the dual cluster $C^*(((-(L+\frac{1}{2}), \pm\frac{1}{2}))\|_{\Lambda_L^*}$ enters the box Λ_B^*:*

$$\mathscr{E}_B = \{\omega \mid C^*(((-(L+\tfrac{1}{2}), \pm\tfrac{1}{2}))\|_{\Lambda_L^*} \cap \Lambda_B^* \neq \emptyset\}.$$

Then $\forall p > p_c$, $\exists c(p) < \infty$ such that as $L \to \infty$

$$P(\mathscr{E}_B | \mathscr{I}_L) \leq \frac{c(p)\, B}{\sqrt{L}}.$$

Proof. Assume $B \ll L$ and suppose $\omega \in \mathscr{E}_B \cap \mathscr{I}_L$. Then there is a dual cluster spanning Λ_L^*, and this cluster intersects Λ_B^*. It must be the case that either:

(a) there is a point $\mathbf{y}^* \in \partial \Lambda_{3B}^*$ such that the events

$$\mathscr{G}^*_{((-L-\frac{1}{2}), \pm\frac{1}{2}), \mathbf{y}^*} \quad \text{and} \quad \mathscr{G}^*_{((L+\frac{1}{2}), \pm\frac{1}{2}), \mathbf{y}^*}$$

occur disjointly (and within Λ_L^*); *or*

(b) there are points $\mathbf{x}^* \in \Lambda_L^* \setminus \Lambda_{3B}^*$ and $\mathbf{y}^* \in \Lambda_B^*$ such that the events

$$\mathscr{G}^*_{((-L-\frac{1}{2}), \pm\frac{1}{2}), \mathbf{x}^*}, \quad \mathscr{G}^*_{((L+\frac{1}{2}), \pm\frac{1}{2}), \mathbf{x}^*} \quad \text{and} \quad \mathscr{G}^*_{\mathbf{x}^*, \mathbf{y}^*}$$

occur disjointly (and, of course, within Λ_L^*).

Note that the two cases above are not disjoint, but they do exhaust $\mathscr{E}_B \cap \mathscr{I}_L$. Roughly speaking, the first case accounts for those ω in which the full interface intersects $\partial \Lambda_{3B}^*$, while the second accounts for those ω in which the interface only sends out a branch which intersects $\partial \Lambda_{3B}^*$, as well as some ω in which the full interface actually intersects $\partial \Lambda_{3B}^*$.

Let us estimate the probability of the first case. Using subadditivity, and the van den Berg-Kesten inequality (2.12), the probability of case (a) is bounded above by

$$\sum_{\mathbf{y}^* \in \partial \Lambda_{3B}^*} \tau^*_{((-L-\frac{1}{2}), \pm\frac{1}{2}), \mathbf{y}^*} \, \tau^*_{\mathbf{y}^*, ((+L+\frac{1}{2}), \pm\frac{1}{2})}. \tag{6.42}$$

We will now estimate (6.42) by means of the asymptotic formulas of Theorem 6.2. Indeed, letting $\mathbf{y}^* = (Y, y)$, the two correlation functions in (6.42) may be bounded above by a constant times

$$e^{-2L/\xi} e^{-y^2/\alpha(L-Y)} e^{-y^2/\alpha(L+Y)} \frac{1}{[L-Y]^{\frac{1}{2}}} \frac{1}{[L+Y]^{\frac{1}{2}}}. \tag{6.43}$$

Since Y and $|y|(=O(B))$ are small compared with L, we may replace the denominators in ((6.43) by (const.) L and discard the Gaussian factors altogether. Then, using Theorem 6.2 to estimate $P(\mathscr{I}_L)$, the conditional probability can be bounded above by $c_a/\sqrt{L}$ for each value of y. Since there are only of the order of B terms, case (a) has an upper bound of the stated form.

Next we consider case (b), in which $C^*(((L+\frac{1}{2}), \pm\frac{1}{2}))$ only visits Λ_B^* as a subsidiary operation. In these instances, the probability can be bounded above by

$$\sum_{\substack{\mathbf{x}^*\in\Lambda_L^*\setminus\Lambda_{3B}^* \\ \mathbf{y}^*\in\partial\Lambda_B^*}} \tau^*_{((-L-\frac{1}{2}),\pm\frac{1}{2}),\mathbf{x}^*}\,\tau^*_{\mathbf{x}^*,((+L+\frac{1}{2}),\pm\frac{1}{2})}\,\tau^*_{\mathbf{x}^*,\mathbf{y}^*}. \tag{6.44}$$

We may of course bound $\tau^*_{\mathbf{x}^*,\mathbf{y}^*}$ above by $e^{-|\mathbf{x}^*-\mathbf{y}^*|/\xi}$. However, here, it is worth noting that for each $\mathbf{y}^*\in\Lambda_B^*$, $|y_i^*|\leqq 2|x_i^*|$, so we may degrade the inequality further:

$$\tau^*_{\mathbf{x}^*,\mathbf{y}^*}\leqq e^{-|\mathbf{x}^*|/2\xi}. \tag{6.45}$$

Clearly, for $|\mathbf{x}^*|$ large (say bigger than $\sqrt{L}$), this extra decay renders the contribution of such configurations negligible. For smaller values of $|\mathbf{x}^*|$, we may use the asymptotic formulas of Theorem 6.2 to obtain, for L sufficiently large,

$$\begin{aligned}&\tau^*_{((-L-\frac{1}{2}),\pm\frac{1}{2}),\mathbf{x}^*}\,\tau^*_{\mathbf{x}^*,((+L+\frac{1}{2}),\pm\frac{1}{2})}\,\tau^*_{\mathbf{x}^*,\mathbf{y}^*}\\ &\leqq(\text{const})\frac{e^{-x^2/\alpha(L+X)}}{[L+X]^{\frac{1}{2}}}\frac{e^{-x^2/\alpha(L-X)}}{[L-X]^{\frac{1}{2}}}\,e^{-2L/\xi}e^{-|\mathbf{x}^*|/2\xi}\end{aligned} \tag{6.46}$$

where we have used the notation $\mathbf{x}^*=(X,x)$. Taking X out of the various denominators on the grounds that we are in the "small $\mathbf{x}^*$ region," the resulting expression may be freely summed over all $\mathbf{x}^*$ starting at $|\mathbf{x}^*|=2B$. Doing this for each $\mathbf{y}^*$ in Λ_B^*, our upper bound on (6.44) is

$$\frac{c_b\,e^{-2L/\xi}}{L}\,B e^{-B/\xi} \tag{6.47}$$

for some constant c_b. Dividing (6.47) by the known large-L behavior of $P(\mathscr{I}_L)$, we get a contribution which (for B large) is considerably smaller than that of case (a). □

As an immediate corollary to the above Lemma, we have:

Proof of Theorem 6.4a. To estblish the vague convergence of $P(-|\mathscr{I}_L)$ to Bernoulli measure at the appropriate density, it is sufficient to demonstrate that the probability of any local event (e.g., a cylinder event) converges to its Bernoulli value. To this end, let $\mathscr{A}$ be any event which is determined by the configurations in Λ_B. Obviously,

$$P(\mathscr{A}|[\mathscr{E}_B]^c\cap\mathscr{I}_L)=P(\mathscr{A}) \tag{6.48}$$

where $[-]^c$ denotes complementation. But then

$$\begin{aligned}|P(\mathscr{A}|\mathscr{I}_L)-P(\mathscr{A})|&=|(P(\mathscr{A}|[\mathscr{E}_B]^c\cap\mathscr{I}_L)-P(\mathscr{A}))\,P([\mathscr{E}_B]^c|\mathscr{I}_L)+P(\mathscr{A}|[\mathscr{E}_B]\cap\mathscr{I}_L)\\ &\quad-P(\mathscr{A}))\,P([\mathscr{E}_B]|\mathscr{I}_L)|=|(P(\mathscr{A}|[\mathscr{E}_B]\cap\mathscr{I}_L)-P(\mathscr{A}))\,P([\mathscr{E}_B]|\mathscr{I}_L)|\leqq P([\mathscr{E}_B]|\mathscr{I}_L)\\ &\leqq\frac{cB}{\sqrt{L}}\to 0\end{aligned} \tag{6.49}$$

and the desired result is established. □

Proof of Theorem 6.4b. For $\mathbf{x}\in\mathbb{Z}^2$, $|\mathbf{x}|<L$, denote by $T_L^+(\mathbf{x})$ and $T_L^-(\mathbf{x})$ the events described in (6.39a) and (6.39b) with $\mathbf{h}$ replaced by $\mathbf{x}$. We will show that $\forall\mathbf{x}$,

$$\lim_{L\to\infty} P(T_L^+(\mathbf{x})|\mathscr{I}_L)=\lim_{L\to\infty} P(T_L^-(\mathbf{x})|\mathscr{I}_L)=\tfrac{1}{2}P_\infty(p), \tag{6.50}$$

from which one obtains $\theta\equiv 0$.

Let $\mathbf{x}\in\mathbb{Z}^2$, $|\mathbf{x}|<L$. For L very large, $P(T_L^+(\mathbf{x})\cup T_L^-(\mathbf{x}))$ is essentially equal to $P_\infty(p)$. We denote the difference, $P(T_L^+(\mathbf{x})\cup T_L^-(\mathbf{x}))-P_\infty(p)$, by $\varepsilon_L(p;x)$ and note that for $p>p_c$, ε_L is exponentially small in L. By the Harris-FKG inequality (2.10),

$$P(T_L^+(\mathbf{x})\cup T_L^-(\mathbf{x})|\mathscr{I}_L)\leqq P_\infty(p)+\varepsilon_L(p;x). \tag{6.51}$$

However, with the conditioning, the two events are disjoint, and thus, in the $L\uparrow\infty$ limit, they cannot both have probability exceeding $(\frac{1}{2})P_\infty(p)$. Therefore, to prove (6.50), it is sufficient to establish that for our (arbitrary) $\mathbf{x}$,

$$\lim_{L\to\infty} P(T_L^+(\mathbf{x})|\mathscr{I}_L)\geqq\tfrac{1}{2}P_\infty(p). \tag{6.52}$$

To prove (6.52), let $B\gg|\mathbf{x}|$ be a fixed but large integer, and assume that already $L\gg B$. We first assert that in $\mathscr{I}_L$, the event

$$\Gamma_L=\{\omega\,|\,C^*((-L-\tfrac{1}{2},\pm\tfrac{1}{2}))\text{ is contained in the region }|x_2|<L^{2/3}\} \tag{6.53}$$

almost always occurs. Indeed, by bound of Theorem 6.2 (II iii), we have

$$P(\Gamma_L^c|\mathscr{I}_L)\leqq e^{-O(L^{1/3})}. \tag{6.54}$$

Next, note that given an $\omega\in\mathscr{I}_L\cap[\mathscr{E}_B]^c$, there are (topologically speaking) only two possibilities: either the connected component of occupied bonds from $(-\mathrm{L}-\frac{1}{2},\pm\frac{1}{2})$ to $(L+\frac{1}{2},\pm\frac{1}{2})$ lies above or it lies below Λ_B^*. We will denote these two possibilities be $[\mathscr{E}_B]^{c+}$ and $[\mathscr{E}_B]^{c-}$, respectively. Observe that $[\mathscr{E}_B]^{c+}\cap[\mathscr{E}_B]^{c-}\neq\emptyset$ (since both events contain configurations in which the interface surrounds Λ_B^*), although, as will emerge later, this intersection is of small conditional probability. By symmetry,

$$P([\mathscr{E}_B]^{c+}|\mathscr{I}_L)=P([\mathscr{E}_B]^{c-}|\mathscr{I}_L), \tag{6.55}$$

and by Lemma 6.5, these quantities tend to (at least) $\frac{1}{2}$ as $L\to\infty$.

Since neither the event Γ_L, nor the event $[\mathscr{E}_B]^c$ significantly restricts the configurations in $\mathscr{I}_L$ as $L\to\infty$, to prove (6.52) it suffices to establish that e.g. in the configurations which lie in $\mathscr{I}_L\cap[\mathscr{E}_B]^{c-}\cap\Gamma_L$, the point $\mathbf{x}$ is connected to $\partial\Lambda_L^+$ with conditional probability approximately equal to $P_\infty(p)$. Such a proof would also establish that $P([\mathscr{E}_B]^{c+}\cap[\mathscr{E}_B]^{c-}|\mathscr{I}_L)$ is negligible.

To this end, let $\Xi_{T,\lambda}^*$ denote the box which is the translate of Λ_T^* by $(1+\lambda)T$ units in the x_2-direction. We take $T\geqq 2B$ and $\lambda\geqq\frac{1}{4}$. Although L is large, we may take T as large as $2L^{2/3}$. Consider the event that the interface visits $\Xi_{T,\lambda}^*$:

$$\mathscr{E}_{\Xi_{T,\lambda}}=\{\omega\,|\,C^*((-L-\tfrac{1}{2},\pm\tfrac{1}{2}))\|_{\Lambda_L^*}\cap\Xi_{T,\lambda}^*\neq\emptyset\}. \tag{6.56}$$

We will demonstrate that for B (and T) large, there is a constant $c<\infty$ such that

$$P(\mathscr{E}_{\Xi_{T,\lambda}} \mid \mathscr{I}_L \cap [\mathscr{E}_B]^{c-}) \leqq c T^3 e^{-T^{1/3}/\alpha} \tag{6.57}$$

as $L \to \infty$, where $\alpha = \alpha(p)$ is the constant appearing in Theorem 6.2.

Let $\omega \in \mathscr{I}_L \cap [\mathscr{E}_B]^{c-}$. Then there is point $\mathbf{y}^* = (0^*, y)$ with $y \leqq -B$, and two disjoint (dual) paths:

$$\mathscr{P}_1: \text{ from } (-L-\tfrac{1}{2}, \pm\tfrac{1}{2}) \text{ to } \mathbf{y}^*$$
$$\mathscr{P}_2: \text{ from } \mathbf{y}^* \text{ to } (L+\tfrac{1}{2}, \pm\tfrac{1}{2}).$$

Denote by $\mathscr{D}\Xi^*_{T,\lambda}$ the set of dual sites a distance no more than $(\frac{1}{8})\,T$ from $\Xi^*_{T,\lambda}$. In order for the event $\mathscr{E}_{\Xi_{T,\lambda}}$ to occur, it is clear that either:

(a) at least one of the paths, $\mathscr{P}_1$ or $\mathscr{P}_2$, enters the region $\mathscr{D}\Xi^*_{T,\lambda}$; or
(b) some point in $\Xi^*_{T,\lambda}$ is connected to $\partial\mathscr{D}\Xi^*_{T,\lambda}$ by a path which (necessarily) takes place in the complement of $\mathscr{P}_1 \cup \mathscr{P}_2$.

These cases are not disjoint, but do exhaust $\mathscr{I}_L \cap [\mathscr{E}_B]^c$. As in the proof of Lemma 6.5, case (a) roughly accounts for those configurations in which the "full" interface enters $\mathscr{D}\Xi^*_{T,\lambda}$, while case (b) also takes into account those configurations in which the interface only sends out a branch which intersects $\mathscr{D}\Xi^*_{T,\lambda}$.

Case (b) has probability smaller than (const.) $T^4 e^{-T/\xi} P(\mathscr{I}_L)$, and hence, in the large-L limit, has conditional probability generously bounded by the estimate in (6.57) for T sufficiently large.

Let us now bound the probability of case (a). We assume, for the sake of argument, that it is $\mathscr{P}_2$ which visits $\mathscr{D}\Xi^*_{T,\lambda}$, and we denote by $\mathbf{v}^* = (V, v)$ a generic point in $\mathscr{P}_2 \cap \mathscr{D}\Xi^*_{T,\lambda}$. We may, in fact, assume that the path $\mathscr{P}_2$ starts at $\mathbf{y}^*$, then goes to $\mathbf{v}^*$ and then on to $(L+\frac{1}{2}, \pm\frac{1}{2})$ (otherwise, by relabelling, we could identify the path which visits $\mathscr{D}\Xi^*_{T,\lambda}$ as $\mathscr{P}_1$). For each $\mathbf{y}^*$ and $\mathbf{v}^*$, we may bound the contribution by

$$\tau^*_{((-L-\frac{1}{2}), \pm\frac{1}{2}), \mathbf{y}^*}\, \tau_{\mathbf{y}^*, \mathbf{v}^*}\, \tau^*_{\mathbf{v}^*, ((+L+\frac{1}{2}), \pm\frac{1}{2})}. \tag{6.58}$$

To obtain our estimate, we will sum (6.58) over all $y<0$ and all $\mathbf{v}^* \in \mathscr{D}\Xi^*_{T,\lambda}$. First observe that the contribution from term with $|y|>2T$ is of the same form as the contribution in case (b). Indeed, if $|y|>2T$, we can use the crude bounds $\tau^*_{\mathbf{v}^*, ((+L+\frac{1}{2}), \pm\frac{1}{2})} \leqq 2e^{-(L-2T)/\xi}$, $\tau_{\mathbf{y}^*, \mathbf{v}^*} \leqq e^{-[(1/8)T+|y|]/\xi}$ and then sum (6.58) freely over $|y| \geqq 2T$ (which, of course, "pins" the factor $\tau^*_{((-L-\frac{1}{2}), \pm\frac{1}{2}), \mathbf{y}^*}$ near its value when $|y|=2T$). This yields a contribution which is exponentially small in T, and thus clearly smaller than the bound in (6.57).

Evidently, the principal contribution comes from case (a) and $|y|<2T$. In this case, we use the bounds

$$\tau^*_{((-L-\frac{1}{2}), \pm\frac{1}{2}), \mathbf{y}^*} \leqq \frac{\text{const}}{\sqrt{L}}\, e^{-L/\xi} \tag{6.59}$$

from Theorem 6.2 (II ii), and

$$\tau^*_{\mathbf{v}^*, ((+L+\frac{1}{2}), \pm\frac{1}{2})} \leqq e^{-(L-V)/\xi}. \tag{6.60}$$

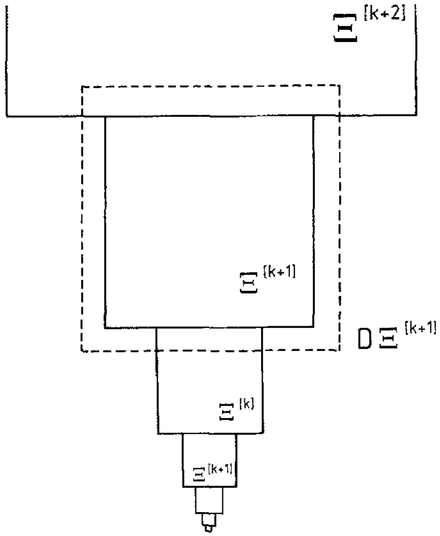

Fig. 7. Arrangement of the boxes $\Xi^{[k]}$

In (6.60), we have neglected some possible additional aid from power law corrections. Now when $V<(1/16)T$, we may use the estimate $\tau_{\mathbf{y}^*,\mathbf{v}^*}\leqq e^{-(1/8)T/\xi}$, and the conditional result is again exponentially small in T. Finally, assuming that $V\geqq(1/16)\,T$ (and assuming that T is large enough to satisfy $T^{2/3}\leqq(1/16)\,T$), we may use property (II iii) of Theorem 6.2 to bound $\tau_{\mathbf{y}^*,\mathbf{v}^*}$. Multiplying the result by the number of such terms, we obtain the stated estimate (6.57).

We will now use the estimate (6.57) to show that, with probability close to $\frac{1}{2}$, there is a large region which the interface does not enter. To this end, consider the sequence of boxes, $(\Xi^{[k]})$, which are translated of Λ_{2^kB} stacked one on top of the next as shown in Fig. 7. Each such box (or, more precisely, the * of each such box) satisfies the criterion to be a $\Xi^*_{T,\lambda}$. Thus, using the bound (6.57), the conditional probability of the interface "entering" $\Xi^{[k]}$ – an event denoted by $\mathscr{E}_{\Xi^{[k]}}$ – may be bounded above by

$$P(\mathscr{E}_{\Xi^{[k]}}\,|\,\mathscr{I}_L\cap[\mathscr{E}_B]^{c-})\leqq(\mathrm{const})(2^kB)^3\exp(-\alpha^{-1}[2^kB]^{1/3}). \tag{6.61}$$

We may estimate the conditional probability of $\bigcup_{k=1}^{N}\mathscr{E}_{\Xi^{[k]}}$, where N is the smallest integer satisfying $2^NB>L^{2/3}$, by summing (6.61) over k. The result is exponentially small in a power of B. Beyond $\Xi^{[N]}$, we are in a region of safety: here we may use the event Γ_L to ensure that the interface does not interfere with any of the larger $\Xi^{[k]}$'s (in Λ_L). Thus, given the event $\mathscr{I}_L$, with probability larger than

$$\frac{1}{2}\left(1-\frac{c'B}{\sqrt{L}}-c''(B)\right) \tag{6.62}$$

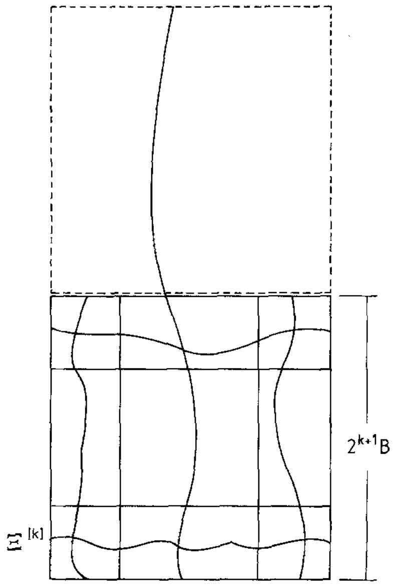

Fig. 8. The event W_k

with $c''(B)$ tending rapidly to zero with B, the interface fails to enter any of the regions $\Xi^{[k]}$, $k=0, \ldots, M$, where M is the smallest integer such that $2^M B \geqq L$ and

$$\Xi^{[0]} \equiv \Lambda_B. \tag{6.63}$$

Thus we have shown that a component of the distribution of configurations inside the region $\bigcup_{k=0}^{M} \Xi^{[k]}$ is the usual Bernoulli measure; furthermore the weight of this component is very close to $\frac{1}{2}$.

Our final job is to show that, given this lack of interference, the point $\mathbf{x}$ is connected to $\partial\Lambda_L^+$ $\left(\text{by a path inside } \bigcup_{k=0}^{M} \Xi^{[k]}\right)$ with probability close to $P_\infty(p)$. In fact, such arguments are by now quite standard (see, e.g., [C; CC3; CC4; CCD]). First, note that $\mathbf{x}$ is connected to $\partial\Xi^{[0]}$ by a path of occupied bonds with probability exceeding $P_\infty(p)$. Next, consider the event W_k that (see Fig. 8):

(1) the four $B2^{k+1} \times B2^{k-1}$ rectangles inside the boundary of $\Xi^{[k]}$ are crossed – the long way – by paths of occupied bonds, and
(2) there is a path of occupied bonds crossing (upward) the $B2^{k+2} \times B2^{k+1}$ rectangle whose bottom half is $\Xi^{[k]}$.

It is not hard to show that

$$P(W_k) \geqq 1 - (\text{const.})\, 2^k B^2 e^{-2^k B/\xi}. \tag{6.64}$$

Thus, by the Harris-FKG inequality (or subadditivity)

(6.65) $$P\left(\bigcap_{k=0}^{M} W_k\right) \geqq 1 - e^{-O(B)}.$$

It is clear that (once B is large enough), if $\mathbf{x}$ is connected to $\partial \Xi^{[0]}$, and if all the events W_k occur, then $\mathbf{x}$ is connected to $\partial \Lambda_L^+$.

Thus, given $[\mathscr{E}_B]^{c-}$, we have demonstrated that $P(T_L^+(h))$ can be made arbitrarily close to $P_\infty(p)$ by taking B (and hence L) large. It then follows from Eqs. (6.62), (6.65) and (6.49) that $\forall h$, $\theta(h) = 0$. □

Appendix

Here we prove Proposition 3.4, restated below for convenience.

Proposition 3.4. *Let* $\mathbb{C}_L(p)$ *and* $\mathbb{K}_L(p)$ *be defined as in* Eqs. (3.25) *and* (3.38), *and let* $\xi(p)$ *and* $\xi_c(p)$ *be the decay rates of* $\tau_{0,L}(p)$ *and* $\mathbb{C}_L(p)$ *as given by* Eq. (1.2) *and Proposition* 3.2. *Then for every* $p \in (0, p_c)$, *either* $\xi_c(p) = \xi_k(p) = \frac{1}{2}\xi(p)$ *or* $\exists D(p) < \infty$ *such that uniformly in* L,

$$\mathbb{K}_L(p) \leqq D(p)\, \mathbb{C}_L(p).$$

As a first step in establishing Proposition 3.4, we will prove the following weaker statement (which is given as the corollary to the Proposition 3.4 in Sect. 3):

Proposition A 1. *Let* $\xi_c(p)$ *and* $\xi_k(p)$ *be defined as in Propositions* 3.2 *and* 3.3. *Then for every* $p \in (0, p_c)$, $\xi_k(p) = \xi_c(p)$.

Proof. Consider the "tunnel" region

(A.1) $$\mathbf{T} = \mathbb{Z} \times [-\mathrm{T}, -\mathrm{T}+1, \ldots, +\mathrm{T}]^{d-1},$$

and its intersection with the slab $\mathbf{S}(L)$ (cf. Eq. (2.3)):

(A. 2) $$\mathbf{T}(L) = \mathbf{S}(L) \cap \mathbf{T}.$$

Let us define tunnel direct connectivity events, $\mathscr{c}_{\mathbf{x},\mathbf{y}}^T$ and $\mathscr{k}_{\mathbf{x},\mathbf{y}}^T$, in which the required connections occur via paths of occupied bonds with both endpoints in $\mathbf{T}$. For $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$, $\mathbf{x} \in \mathbf{P}(0) \cap \mathbf{T}$, $\mathbf{y} \in \mathbf{P}(L) \cap \mathbf{T}$, $L \geqq 1$, we have

(A. 3) $$\mathscr{c}_{\mathbf{x},\mathbf{y}}^T = \{\omega \mid \mathbf{y} \in C(\mathbf{x})\|_{\mathbf{T}(L)}, C(\mathbf{x})\|_{\mathbf{T}(L)} \cap \mathbf{P}(0) = \{\mathbf{x}\}, \\ C(\mathbf{x})\|_{\mathbf{T}(L)} \cap \mathbf{P}(L) = \{\mathbf{y}\}, |C(\mathbf{x})\|_{\mathbf{T}(L)} \cap \mathbf{P}(j)| \geqq 2 \,\forall\, 1 \leqq j \leqq L-1\}$$

(A. 4) $$\mathscr{k}_{\mathbf{x},\mathbf{y}}^T = \{\omega \mid \mathbf{y} \in C(\mathbf{x})\|_{\mathbf{T}}, |C(\mathbf{x})\|_{\mathbf{T}} \cap \mathbf{P}(j)| \geqq 2 \,\forall\, 1 \leqq j \leqq L-1\},$$

and the corresponding connectivity functions are

(A. 5) $$c_{\mathbf{x},\mathbf{y}}^T(p) = P(\mathscr{c}_{\mathbf{x},\mathbf{y}}^T)$$

(A. 6) $$k_{\mathbf{x},\mathbf{y}}^T(p) = P(\mathscr{k}_{\mathbf{x},\mathbf{y}}^T).$$

As usual, we will denote the on-axis connectivity functions between the origin and $(L, 0, \ldots, 0)$ by $c_{0,L}^T$ and $k_{0,L}^T$.

It is straightforward to show that $\xi_c^T(p)$ and $\xi_k^T(p)$, defined by the limits of $L^{-1} \log c_{0,L}^T$ and $L^{-1} \log k_{0,L}^T$, exist (see, e.g., the proof of Proposition 3.3).

Furthermore, the functions $k_{0,L}^T$ are monotone increasing in T, from which it easily follows that, as $T\to\infty$,

(A. 7) $$\xi_k^T(p)\uparrow \xi_k(p),$$

where $\xi_k(p)$ is the correlation length of $\mathbb{K}_L$ as defined in Proposition 3.3. Unfortunately, the quantities $c_{0,L}^T$ do not necessarily have this monotone property since some of the contributing events are negative – specifically, paths which occur outside the tunnel could connect other points of $\mathbf{P}(0)$ or $\mathbf{P}(L)$ to $C(0)\|_{\mathbf{S}(L)}$.

We claim that $\forall T$

(A. 8) $$\xi_k^T(p)=\xi_c^T(p).$$

First, we clearly have $\mathscr{k}_{0,L}^T\supset \mathscr{c}_{0,L}^T$, which implies $\xi_k^T(p)\geqq\xi_c^T(p)$. To obtain the opposite inequality, let $\mathbf{A}(j)$ denote the set of all "horizontal" bonds between the points with $x_1=j$ and points with $x_1=j+1$, and let $\mathbf{B}(j)$ denote the set of all "vertical" bonds in the plane $\mathbf{P}(j)$. Then by vacating all bonds in $(\mathbf{A}(0)\cup \mathbf{B}(0))\cap\mathbf{T}$ and $(\mathbf{A}(L-1)\cup\mathbf{B}(L))\cap\mathbf{T}$, and occupying the all bonds in $\mathbf{B}(1)\cap\mathbf{T}$ and $\mathbf{B}(L-1)\cap\mathbf{T}$, in addition to occupying the two (possibly just vacated) bonds between the points $x_1=0$ and $x_1=(1,0,\ldots,0)$, and $x_1=(L-1,0,\ldots,0)$ and $x_1=(L,0,\ldots,0)$, we can "transform" an $\omega\in\mathscr{k}_{0,L}^T$ into an $\omega\in\mathscr{c}_{0,L}^T$. Although this is expensive, the price is uniform in L, i.e.

(A. 9) $$c_{0,L}^T\geqq\Phi(T)\,k_{0,L}^T$$

which $\Phi(T)=e^{-O(T^{d-1})}$. This establishes (A. 8).

Next, consider the event $\mathscr{f}_{\mathbf{P}(j),\mathbf{T}}$ defined by

(A.10) $$\mathscr{f}_{\mathbf{P}(j),\mathbf{T}}=\{\omega\,|\,\text{the region } \mathbf{P}(j)\setminus\mathbf{T} \text{ is } \textit{not} \text{ connected to the region } \mathbf{T} \text{ by a path of occupied bonds}\}.$$

We claim that whenever $p<p_c$, $f(T)\equiv P(\mathscr{f}_{\mathbf{P}(j),\mathbf{T}})>0$. Indeed, by translation invariance

(A.11) $$\chi=\sum_{\substack{\mathbf{x}\in\mathbf{P}(0)\\ \mathbf{y}:y_2,\ldots,y_d=0}}\tau_{\mathbf{x},\mathbf{y}}.$$

Thus the expected number of connections between $\mathbf{P}(j)\setminus\mathbf{T}$ and $\mathbf{T}$ is given by

(A.12) $$\sum_{\substack{\mathbf{x}\in\mathbf{P}(0)\\ \mathbf{y}\in\mathbf{T}}}\tau_{\mathbf{x},\mathbf{y}}\leqq(2T)^{d-2}\chi<\infty.$$

By the Bortel-Cantelli lemma, with probability one, only a finite number of the $\mathscr{g}_{\mathbf{x},\mathbf{y}}$ events contributing to the sum in (A.12) occur. Using FKG properties of the Bernoulli measure, it is readily established that, with positive probability, *none* of these events occur, i.e. $f(T)>0$.

Finally, it should be observed that the intersection of $\mathscr{f}_{\mathbf{P}(0),\mathbf{T}}$, $\mathscr{f}_{\mathbf{P}(L),\mathbf{T}}$ and $\mathscr{c}_{0,L}^T$ is contained in the event $\mathscr{c}_{0,L}$. Thus

(A.13) $$\begin{aligned}c_{0,L}&\geqq P(\mathscr{f}_{\mathbf{P}(0),\mathbf{T}}\cap\mathscr{f}_{\mathbf{P}(L),\mathbf{T}}\cap\mathscr{c}_{0,L}^T)=P(\mathscr{f}_{\mathbf{P}(0),\mathbf{T}}\cap\mathscr{f}_{\mathbf{P}(L),\mathbf{T}})\,c_{0,L}^T\\&\geqq f^2(T)\,c_{0,L}^T,\end{aligned}$$

where in the last step we have used the Harris-FKG inequality (2.10). Taking logs, dividing by L and letting $L\to\infty$, (A.8), (A.12) and (A.13) imply $\xi_c\geqq\xi_k^T\,\forall T$. However $\xi_c\leqq\xi_k=\lim_{T\uparrow\infty}\xi_k^T$, so the correlation lengths agree. □

We divide the remainder of the proof into two stages, first imposing cylinder and then strict cylinder conditions on the k-type (free) direct correlation functions. For this, we will have to introduce two other direct correlation functions, both of which are defined in the cylinder.

Definition A1. For $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$, $\mathbf{x} \in \mathbf{P}(0)$, $\mathbf{y} \in \mathbf{P}(L)$, $L \geqq 1$, the *non-strict direct connectivity functions* are defined by

$$\text{(A.14a)} \qquad \mathscr{C}^{**}_{\mathbf{x},\mathbf{y}} = \{\omega \mid \mathbf{y} \in C(\mathbf{x})\|_{\mathbf{S}(L)}, |C(\mathbf{x})\|_{\mathbf{S}(L)} \cap \mathbf{P}(j)| \geqq 2 \ \forall 1 \leqq j \leqq L-1\},$$

$$\text{(A.14b)} \qquad c^{**}_{\mathbf{x},\mathbf{y}}(p) = P(\mathscr{C}^{**}_{\mathbf{x},\mathbf{y}}),$$

with the convention $c^{**}_{\mathbf{x},\mathbf{y}}(p) = 0$ for $L=0$; and

$$\text{(A.15)} \qquad \mathbb{C}^{**}_L(p) = \sum_{\mathbf{y} \in \mathbf{P}(L)} c^{**}_{\mathbf{x},\mathbf{y}}(p).$$

The connections contributing to $\mathbb{C}^{**}_L$ are required to take place in the cylinder, but not to respect the strict cylinder condition; thus the relationship between $\mathbb{C}^{**}_L$ and $\mathbb{C}_L$ is analogous to that between $\tilde{\mathbb{H}}_L$ and $\mathbb{H}_L$ (cf. Eq. (3.10)). Halfway between $\mathbb{C}_L$ and $\mathbb{C}^{**}_L$, we have:

Definition A 2. For $\mathbf{x}$, $\mathbf{y} \in \mathbb{Z}^d$, $\mathbf{x} \in \mathbf{P}(0)$, $\mathbf{y} \in \mathbf{P}(L)$, $L \geqq 1$, the *half-strict direct connetivity functions* are defined by

$$\text{(A.16a)} \qquad \mathscr{C}^{*}_{\mathbf{x},\mathbf{y}} = \{\omega \mid \mathbf{y} \in C(\mathbf{x})\|_{\mathbf{S}(L)}, C(\mathbf{x})\|_{\mathbf{S}(L)} \cap \mathbf{P}(0) = \{\mathbf{x}\}, \\ |C(\mathbf{x})\|_{\mathbf{S}(L)} \cap \mathbf{P}(j)| \geqq 2 \ \forall 1 \leqq j \leqq L-1\},$$

$$\text{(A.16b)} \qquad c^{*}_{\mathbf{x},\mathbf{y}}(p) = P(\mathscr{C}^{*}_{\mathbf{x},\mathbf{y}}),$$

again with the convention $c^{*}_{\mathbf{x},\mathbf{y}}(p) = 0$ for $L=0$; and

$$\text{(A.17)} \qquad \mathbb{C}^{*}_L(p) = \sum_{\mathbf{y} \in \mathbf{P}(L)} c^{*}_{\mathbf{x},\mathbf{y}}(p).$$

Remark. Since

$$\text{(A.18)} \qquad \mathbb{K}_L \geqq \mathbb{C}^{**}_L \geqq \mathbb{C}^{*}_L \geqq \mathbb{C}_L,$$

if follows from Proposition A1 that sharp decay rates exist for both $\mathbb{C}^{**}_L(p)$ and $\mathbb{C}^{*}_L(p)$, and that these decay rates are equal to $\xi_c(p)$.

Finally, we will also need:

Definition A 3. For $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in \mathbb{Z}^d$, the *three-point connectivity event* is

$$\text{(A.19)} \qquad \mathscr{G}_{\mathbf{x},\mathbf{y},\mathbf{z}} = \{\omega \mid \mathbf{y}, \mathbf{z} \in C(\mathbf{x})\}.$$

In much of what follows, we will make extensive use of the tree-diagram decomposition techniques introduced in [AN]. The general strategy will be to pick an ω purported to satisfy certain conditions, and then to demonstrate that ω satisfies various *other* conditions. This is often exhibited by implementing "local growing rules," by which we mean the following: The full configuration, ω, will be regarded as a fixed, deterministic object. Then various subsets of ω, e.g., $C(0, \omega)$, may be grown in a sequence of time steps: One may start at $t=0$ with $\{0\}$. By checking, according to some set of local rules, whether or not some bond on the boundary of the current cluster is actually occupied or vacant, one obtains the cluster at time $t=n+1$ from the cluster at time $t=n$. The only potential hold-up in such a procedure – which in any case can

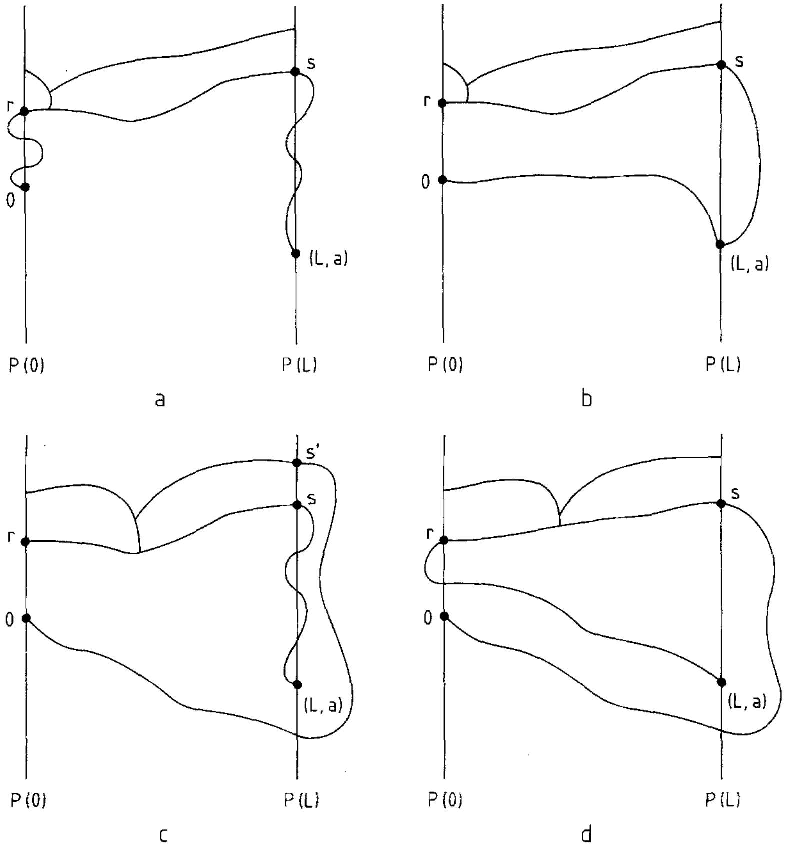

Fig. A 1. Configurations in $\mathscr{k}^{\textcircled{1}}_{0,(L,\mathbf{a})}$ corresponding to cases (**a**), (**b**), (**c**) and (**d**)

be circumvented – is the possibility that various stages in a proof could (deterministically) take an infinite amount of time. Such situations will have Bernoulli probability zero (since we are always below threshold), and we will be content with w.p.1 statements. For more details on these methods, see [AN].

In the following lemma, we impose non-strict cylinder conditions on the free direct connectivity.

Lemma A 2. *Let* $L>1$. *Then* $\forall p\in(0,p_c)$, $\exists D_1(p)$, $D_2(p)<\infty$ *such that*

$$\mathbb{K}_L \leqq D_1\, \mathbb{C}^{**}_L + D_2 \sum_{0\leqq N<P\leqq L} \mathbb{C}^{*}_{P-N}\, e^{-2[L-(P-N)]/\xi}.$$

Proof. Let $\omega\in\mathscr{k}_{0,(L,\mathbf{a})}$. We consider two situations which we denote by case ① and case ②, respectively. Case ① consists of those configurations in $\mathscr{k}_{0,(L,\mathbf{a})}$ for which, in $C(0)$, the $\mathscr{c}^{**}$ event happens somewhere:

(A.20) $\qquad \mathscr{k}^{\textcircled{1}}_{0,(L,\mathbf{a})} = \{\omega\in\mathscr{k}_{0,(L,\mathbf{a})} \mid \exists\, \mathbf{y}\in C(0)\cap\mathbf{P}(0),\ \mathbf{u}\in\mathbf{P}(L) \text{ such that } \omega\in\mathscr{c}^{**}_{\mathbf{y},\mathbf{u}}\}.$

Case ② is the event that no such pair $\{\mathbf{y},\mathbf{u}\}$ exists.

We claim that under the circumstances which define case ①, the following stronger statement is true: There are points, $\mathbf{r}\in\mathbf{P}(0)$ and $\mathbf{s}\in\mathbf{P}(L)$, both in $C(0)$, such that the event $\mathscr{c}^{**}_{\mathbf{r},\mathbf{s}}$ occurs; and (disjointly from $\mathscr{c}^{**}_{\mathbf{r},\mathbf{s}}$ unless otherwise noted) one of the following events occurs (see Fig. A.1):

(a) there are two disjoint paths, one from 0 to $\mathbf{r}$ and one from $(L, \mathbf{a})$ to $\mathbf{s}$: explicitly, $\omega \in c^{**}_{\mathbf{r},\mathbf{s}} \circ \mathscr{g}_{0,\mathbf{r}} \circ \mathscr{g}_{(L,\mathbf{a}),\mathbf{s}}$;

(b) 0, $(L, \mathbf{a})$ and $\mathbf{s}$ belong to the same connected cluster: explicitly, $\omega \in c^{**}_{\mathbf{r},\mathbf{s}} \circ \mathscr{g}_{0,(L,\mathbf{a}),\mathbf{s}}$;

(b) 0, $(L, \mathbf{a})$ and $\mathbf{r}$ belong to the same connected cluster: explicitly, $\omega \in c^{**}_{\mathbf{r},\mathbf{s}} \circ \mathscr{g}_{0,(L,\mathbf{a}),\mathbf{r}}$;

(c) 0 is connected to $\mathbf{s}$, and there is some $\mathbf{s}' \neq \mathbf{s}$, $\mathbf{s}' \in \mathbf{P}(L)$ such that (not necessarily disjointly from $c^{**}_{\mathbf{r},\mathbf{s}}$) $\omega \in c^{**}_{\mathbf{r},\mathbf{s}'}$, and $(L, \mathbf{a})$ is disjointly connected to $\mathbf{s}'$: explicitly, $\exists \mathbf{s}' \in \mathbf{P}(L)$ such that $\omega \in (c^{**}_{\mathbf{r},\mathbf{s}'} \cap c^{**}_{\mathbf{r},\mathbf{s}}) \circ \mathscr{g}_{0,\mathbf{s}} \circ \mathscr{g}_{(L,\mathbf{a}),\mathbf{s}'}$;

(c′) same as (c) but with sides reversed: explicitly, $\exists \mathbf{r}' \in \mathbf{P}(0)$ such that $\omega \in (c^{**}_{\mathbf{r}',\mathbf{s}} \cap c^{**}_{\mathbf{r},\mathbf{s}}) \circ \mathscr{g}_{(L,\mathbf{a}),\mathbf{r}} \circ \mathscr{g}_{0,\mathbf{r}'}$;

(d) there are two disjoint paths, one from 0 to $\mathbf{s}$ and one from $(L, \mathbf{a})$ to $\mathbf{r}$; explicitly, $\omega \in c^{**}_{\mathbf{r},\mathbf{s}} \circ \mathscr{g}_{0,\mathbf{s}} \circ \mathscr{g}_{(L,\mathbf{a}),\mathbf{r}}$.

It should be remarked that the above events are not necessarily disjoint; however, anything except (a) is absurdly improbable – these events have been listed, roughly, in decreasing order of likelihood.

To show that (a)–(d) exhaust all possibilities, suppose that $\mathbf{y} \in C(0)$ is in the plane $\mathbf{P}(0)$, $\mathbf{u} \in C(0)$ is in the plane $\mathbf{P}(L)$, and the configuration ω satisfies $\omega \in c^{**}_{\mathbf{y},\mathbf{u}}$. With suitable interpretations of various sentences, there is no need to assume that $\mathbf{y} \neq 0$ and $\mathbf{u} \neq (L, \mathbf{a})$; however, in what follows, it pays to think of these points as distinct – and to refer to Fig. A1. Denote by $\mathbf{y}_1, \mathbf{y}_2, \ldots$, the points of $\mathbf{P}(0)$ which belong to $C(\mathbf{y})\|_{\mathbf{S}(L)}$ and by $\mathbf{u}_1, \mathbf{u}_2, \ldots$, the points of $\mathbf{P}(L)$ which belong to $C(\mathbf{u})\|_{\mathbf{S}(L)}$. It is worth observing that $C(\mathbf{y})\|_{\mathbf{S}(L)} = C(\mathbf{u})\|_{\mathbf{S}(L)}$, and that in fact $\forall i, j$

$$C(\mathbf{y}_i)\|_{\mathbf{S}(L)} = C(\mathbf{u}_j)\|_{\mathbf{S}(L)} \tag{A.21}$$

and

$$\omega \in c^{**}_{\mathbf{y}_i,\mathbf{u}_j}. \tag{A.22}$$

Now if $\mathbf{x} \in C(0)$, then either $\mathbf{x}$ belongs to $C(\mathbf{y})\|_{\mathbf{S}(L)}$ or there must be a path between $\mathbf{x}$ and one of the $\mathbf{y}_i, \mathbf{u}_j$ which takes place in the complement of $C(\mathbf{y})\|_{\mathbf{S}(L)}$; indeed, if we grow the connected cluster of any $\mathbf{x} \in C(\mathbf{y}) \setminus C(\mathbf{y})\|_{\mathbf{S}(L)}$, the only mechanism that the growing cluster has to reach $C(\mathbf{y})\|_{\mathbf{S}(L)}$ is through one of the points $\mathbf{y}_i, \mathbf{u}_j$.

Let us start growing the clusters of 0 and $(L, \mathbf{a})$ – in the complement of $C(\mathbf{y})\|_{\mathbf{S}(L)}$. The most reasonable possibility is that the growing cluster of the origin hits one of the $\mathbf{y}_i (\equiv \mathbf{r})$, while the cluster of $(L, \mathbf{a})$ finds one of the $\mathbf{u}_j (\equiv \mathbf{s})$. This implies possibility (a). On the other hand, before both of these events occur, it could be the case that the growing clusters meet. This would (sooner or later) imply possibility (b) or (b′). Finally, and least plausible of all, is the situation in which the two "easy way" events do not both happen and the growing clusters never meet. This means that one or both of the growing clusters joins with $C(\mathbf{y})\|_{\mathbf{S}(L)}$ via the wrong entrance. If one of the "easy way" events occurs, we get (c) or (c′), while if neither occurs, we get (d).

Using subadditivity and the van den Berg-Kesten inequality (2.12), the total contribution from these configurations is bounded by

$$\sum_{\substack{\mathbf{r} \in \mathbf{P}(0) \\ \mathbf{s} \in \mathbf{P}(L)}} \tau_{0,\mathbf{r}} \, \tau_{(L,\mathbf{a}),\mathbf{s}} \, c^{**}_{\mathbf{r},\mathbf{s}} \tag{A.23a}$$

from situation (a),

(A.23 b)
$$\sum_{\substack{\mathbf{r}\in\mathbf{P}(0)\\ \mathbf{s}\in\mathbf{P}(L)\\ \mathbf{x}\in\mathbb{Z}^d}} \tau_{0,\mathbf{x}}\,\tau_{(L,\mathbf{a}),\mathbf{x}}\,\tau_{\mathbf{x},\mathbf{r}}\,c^{**}_{\mathbf{r},\mathbf{s}}$$

from situation (b), with something similar from (b′),

(A.23 c)
$$\sum_{\substack{\mathbf{r}\in\mathbf{P}(0)\\ \mathbf{s}\in\mathbf{P}(L),\mathbf{s}'\in\mathbf{P}(L)}} \tau_{0,\mathbf{s}}\,\tau_{(L,\mathbf{a}),\mathbf{s}'}\,P(c^{**}_{\mathbf{r},\mathbf{s}'}\cap c^{**}_{\mathbf{r},\mathbf{s}})$$

from situation (c), with something similar from (c′), and

(A.23 d)
$$\sum_{\substack{\mathbf{r}\in\mathbf{P}(0)\\ \mathbf{s}\in\mathbf{P}(L)}} \tau_{0,\mathbf{s}}\,\tau_{(L,\mathbf{a}),\mathbf{r}}\,c^{**}_{\mathbf{r},\mathbf{s}}$$

from situation (d).

Summing (A.23 a) first over **a**, then **s** and **r**, yields a net contribution of (const) $\mathbb{C}^{**}_L$. The later (A.23) equations, under similar summation procedures, all produce $\mathbb{C}^{**}_L e^{-O(L)}$, and thus may be dismissed entirely. Equation (A.23 c) must be broken into $|\mathbf{s}-\mathbf{s}'|<(\text{const})\,L$ and $|\mathbf{s}-\mathbf{s}'|>(\text{const})\,L$. We have thus derived

(A.24)
$$\mathbb{K}^{①}_L \leqq D_1\,\mathbb{C}^{**}_L,$$

where we have used the notation

(A.25)
$$\mathbb{K}^{①}_L \equiv \sum_{\mathbf{a}\in\mathbb{Z}^d} k^{①}_{0,(L,\mathbf{a})}.$$

We now turn to the analysis of case ②. Thus take $\omega\in k^{②}_{0,(L,\mathbf{a})}$. Pick any **y** in $\mathbf{P}(0)$ that is in the connected cluster of the origin, and such that $C(\mathbf{y})\|_{\mathbf{S}(L)}$ contains an occupied path to $\mathbf{P}(L)$. Let $\mathbf{u}\in\mathbf{P}(L)$ denote a point in $C(\mathbf{y})\|_{\mathbf{S}(L)}$. Denote by $\mathbf{z}_1, \mathbf{z}_2, \ldots$, those points in $\mathbf{P}(0)$ that are attached to the origin but do not belong to $C(\mathbf{y})\|_{\mathbf{S}(L)}$, and by $\mathbf{v}_1, \mathbf{v}_2, \ldots$, those points in $\mathbf{P}(L)\cap C(0)\backslash C(\mathbf{y})\|_{\mathbf{S}(L)}$. The hypothesis of case ② demands that $C(\mathbf{y})\|_{\mathbf{S}(L)}$ does not produce the event $c^{**}_{\mathbf{y},\mathbf{u}}$, evidently,

(i) $\exists M$, $1\leqq M\leqq L-1$, such that $C(\mathbf{y})\|_{\mathbf{S}(L)}\cap\mathbf{P}(M)$ contains only a single point. To compensate for this deficiency, it must also be the case that:

(ii) the set of **z**'s and/or the set of **v**'s is not empty.

We denote by $N_1\leqq N_2\leqq\ldots\leqq N_k$ the distances satisfying (i), that is

(A.26)
$$|C(\mathbf{y})\|_{\mathbf{S}(L)}\cap\mathbf{P}(N_j)|=1 \quad \text{with } 1\leqq N_j\leqq L-1,$$

and by the vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots \mathbf{b}_k\in\mathbb{Z}^{d-1}$ the other coordinates of the intersection of $C(\mathbf{y})\|_{\mathbf{S}(L)}$ with these planes. Observe that in the region $\mathbf{S}(N_1)$, $C(\mathbf{y})\|_{\mathbf{S}(L)}$ contains a realization of the event $c^{*}_{\mathbf{y},(N_1,\mathbf{b}_1)}$,[3] while similarly, in the region $N_k\leqq x_1\leqq L$, $C(\mathbf{y})\|_{\mathbf{S}(L)}$ contains a realization of $c^{*}_{(N_k,\mathbf{b}_k),\mathbf{u}}$. In general, when $k\geqq 2$, we have a realization of the event $c^{*}_{(N_j,\mathbf{b}_j),(N_{j+1},\mathbf{b}_{j+1})}$ for each $j<k$.

[3] More precisely, in the region $\mathbf{S}(N_1)$, we have a realization of the mirror image of the event $c^{*}_{(N_1,\mathbf{b}_1),\mathbf{y}}$. Of course, by symmetry, these have the same probability, which is the only issue of importance here

Let $\mathfrak{R}(z)$ denote the maximum extent, in the x_1-direction, of the z-clusters:

(A.27a) $$\mathfrak{R}(z)=\max\{x_1 \mid \mathbf{x}\in C(\mathbf{z}_i)\|_{\mathbf{S}(L)}, i=1,2,\ldots\}.$$

Similarly, define

(A.27b) $$\mathfrak{R}(v)=\min\{x_1 \mid \mathbf{x}\in C(\mathbf{v}_i)\|_{\mathbf{S}(L)}, i=1,2,\ldots\}.$$

In cases when $\mathfrak{R}(z)<\mathfrak{R}(v)$, the various pieces of $C(0)$ are forced to cooperate in order to produce the event $\mathscr{k}_{0,(L,\mathbf{a})}$. In particular, let N be defined by

(A.28a) $$N=\max\{N_0, N_1, N_2, \ldots, N_{k+1} \mid N_j \leqq \mathfrak{R}(z)\}$$

and

(A.28b) $$P=\min\{N_0, N_1, \ldots, N_{k+1} \mid N_j \geqq \mathfrak{R}(v)\},$$

where $N_0\equiv 0$ and $N_{k+1}\equiv L$. It is seen that if $N=N_i$, then P must equal N_{i+1}. On the other hand, if it happens that $\mathfrak{R}(z)\geqq\mathfrak{R}(v)$, we may either deal with the case separately (which in light of what follows is inadvisable), or pick, by fiat, any N and P, $N<P$, satisfying $N\leqq\mathfrak{R}(z)$ and $P\geqq\mathfrak{R}(v)$.
Let us summarize the situation as it now stands:

In the region to the left of $\mathbf{P}(N)$,
(1) $\exists\mathbf{y}\in\mathbf{P}(0)$ and $(N,\mathbf{b})\in\mathbf{P}(N)$ such that $C(\mathbf{y})\|_{\mathbf{S}(L)}$ contains $(N,\mathbf{b})$
(2) $\exists\mathbf{z}\in\mathbf{P}(0)$ and $(N,\mathbf{b}')\in\mathbf{P}(N)$ such that $C(\mathbf{z})\|_{\mathbf{S}(L)}$ contains $(N,\mathbf{b}')$
(3) $C(\mathbf{y})\|_{\mathbf{S}(L)}\cap C(\mathbf{z})\|_{\mathbf{S}(L)}=\emptyset$.

In the region to the right of $\mathbf{P}(P)$, we have the analogs of (1)–(3):

(1′) $\exists\mathbf{u}\in\mathbf{P}(L)$ and $(N,\mathbf{c})\in\mathbf{P}(N)$ such that $C(\mathbf{u})\|_{\mathbf{S}(L)}$ contains $(N,\mathbf{c})$
(2′) $\exists\mathbf{v}\in\mathbf{P}(L)$ and $(N,\mathbf{c}')\in\mathbf{P}(N)$ such that $C(\mathbf{v})\|_{\mathbf{S}(L)}$ contains $(N,\mathbf{c}')$
(3′) $C(\mathbf{u})\|_{\mathbf{S}(L)}\cap C(\mathbf{v})\|_{\mathbf{S}(L)}=\emptyset$.

In between the planes, we have a realization of the event $c^*_{(N,\mathbf{b}),(P,\mathbf{c})}$.
We claim that, in fact, a slightly stronger statement is true:
(4) $\mathbf{y}$ and $\mathbf{z}$ as described in (1)–(3) can be chosen in such a way that

(A.29a) $$C(\mathbf{y})\|_{\mathbf{S}(N)}\subset C(0)\|_{\mathbf{H}(N)}$$

and

(A.29b) $$C(\mathbf{z})\|_{\mathbf{S}(N)}\subset C(0)\|_{\mathbf{H}(N)},$$

with an analogous (4′) statement.

Equation (A.29a) is obvious: Indeed, no point in $\mathbf{P}(0)$, including the origin, can give rise to an occupied path intersection $C(\mathbf{y})\|_{\mathbf{S}(L)}$ in the region $x_1\geqq N$ without passing through the point $(N,\mathbf{b})$. Thus $C(\mathbf{y})\|_{\mathbf{S}(N)}\subset C(0)\|_{\mathbf{H}(N)}$. Equation (A.29b) may not necessarily hold for all of the $\mathbf{z}_i$. However, were it the case that for some chosen $\mathbf{z}$, all paths from 0 to $C(\mathbf{z})\|_{\mathbf{S}(N)}$ entered the region $x_1\geqq N$ before intersecting $C(\mathbf{z})\|_{\mathbf{S}(N)}$, then along any of these paths there must be another $\mathbf{z}_i\neq\mathbf{z}$ satisfying (2)–(4).

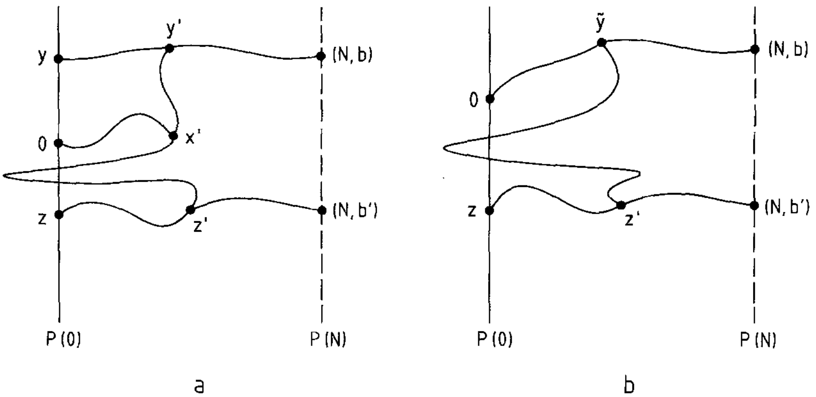

Fig. A2. Pieces of configurations in $\mathcal{k}^{②}_{0,(L,\mathbf{a})}$ corresponding to cases **a** (α) and **b** (β)

We claim that (1)–(4) imply only a few essentially equivalent possibilities (see Fig. A2):

(α) there is some point $\mathbf{x}^* \in \mathbf{H}(N)$, and points $\mathbf{y}^*$, $\mathbf{z}^* \in \mathbf{S}(N)$ such that the following collection of mutually disjoint paths occur in $\mathbf{H}(N)$:
(a) 0 to $\mathbf{x}^*$, (b) $\mathbf{x}^*$ to $\mathbf{y}^*$, (c) $\mathbf{x}^*$ to $\mathbf{z}^*$, (d) $(N, \mathbf{b}')$ to $\mathbf{z}^*$,
(e) $\mathbf{z}^*$ to $\mathbf{z}$, (f) $\mathbf{y}^*$ to $\mathbf{y}$, (g) $(N, \mathbf{b})$ to $\mathbf{y}^*$;

(β') there are points $\tilde{\mathbf{y}}$, $\mathbf{z}^* \in \mathbf{S}(N)$ such that the following collection of mutually disjoint paths occur in $\mathbf{H}(N)$:
(a) 0 to $\tilde{\mathbf{y}}$, (b) $\tilde{\mathbf{y}}$ to $\mathbf{z}^*$, (c) $\tilde{\mathbf{y}}$ to $(N, \mathbf{b})$,
(d) $\mathbf{z}^*$ to $\mathbf{z}$, (e) $(N, \mathbf{b}')$ to $\mathbf{z}^*$;

(β'') the same as (β') but with the letters $\mathbf{y}$ and $\mathbf{z}$ exchanged and the letters $\mathbf{b}$ and $\mathbf{b}'$ exchanged.

To see this, let $\mathscr{P}_{[1]}$ be an occupied self-avoiding path from $\mathbf{y}$ to $(N, \mathbf{b})$ in $\mathbf{S}(N)$ and $\mathscr{P}_{[2]}$ a similar type of path from $\mathbf{z}$ to $(N, \mathbf{b}')$. Obviously, we have $\mathscr{P}_{[1]} \cap \mathscr{P}_{[2]} = \emptyset$.

Now, start to grow $C(0)$ via local rules. Eventually, the growing cluster must meet $\mathscr{P}_{[1]}$ or $\mathscr{P}_{[2]}$. Let us say, for the sake of argument, that it meets $\mathscr{P}_{[1]}$ first, which could of course happen "immediately," e.g., if $y = 0$. (If the growing cluster hits $\mathscr{P}_{[2]}$ before $\mathscr{P}_{[1]}$, in what follows we will have to make the exchange $\mathbf{y} \leftrightarrow \mathbf{z}$, which may lead to the β'' case.) We denote this first point of contact by $\mathbf{y}^*$. Continue now to locally grow $C(0)$ in the complement of the bonds of $\mathscr{P}_{[1]}$. There are two possibilities:

(α) The growing $C(0)$ finds $\mathscr{P}_{[2]}$.
(β) The process runs out of occupied bonds before intersecting $\mathscr{P}_{[2]}$.

In the former set of circumstances, denote by $\mathbf{z}^*$ the first point of intersection of $C(0)$ with $\mathscr{P}_{[2]}$, and the stated result is evident. As for the β incident, when $C(0)$ ceases to grow, denote the "stopped" cluster by $\underline{C}(0)$. We must now give the (untouched sites of) $\mathscr{P}_{[1]}$ a chance to find $\mathscr{P}_{[2]}$. (Since $\mathbf{z} \in C(0)\|_{\mathbf{H}(L)}$, this must happen eventually.) Thus let us denote by $\mathbf{y}^{**}$ any point of $\mathscr{P}_{[1]}$ whose cluster – grown in the complement of the bonds of $\mathscr{P}_{[1]}$ and, by necessity, in the complement of $\underline{C}(0)$ – reaches $\mathscr{P}_{[2]}$. Denote by $\mathbf{z}^*$ the first point on $\mathscr{P}_{[2]}$ which is reached by the cluster of $\mathbf{y}^{**}$. If $\mathscr{P}_{[1]}$ passes through $\mathbf{y}^*$ Before $\mathbf{y}^{**}$, we may identify $\tilde{\mathbf{y}}=\mathbf{y}^{**}$, while if $\mathscr{P}_{[1]}$ passes through $\mathbf{y}^{**}$ before $\mathbf{y}^*$, we may identify $\tilde{\mathbf{y}}=\mathbf{y}^*$. In either case, statement (β') (and similarly (β'')) is verified.

Obviously, to the right of $\mathbf{P}(P)$, the story is the same. Let us denote the probabilities of the left and right α-type events by $A_N(\mathbf{b}, \mathbf{b}', \mathbf{y}, \mathbf{y}^*, \mathbf{z}, \mathbf{z}^*, \mathbf{x}^*, 0)$ and $A_P^\dagger(\mathbf{c}, \mathbf{c}', \mathbf{u}, \mathbf{u}^*, \mathbf{v}, \mathbf{v}^*, \mathbf{w}^*, (L, \mathbf{a}))$, and similarly for the β-type events. We must consider all nine pairs of left and right possibilities, but each of these terms has a bound of the same order of magnitude. We will illustrate with only one example. Let us consider, then,

$$\text{(A.30)} \quad \sum A_N(\mathbf{b}, \mathbf{b}', \mathbf{y}, \mathbf{y}^*, \mathbf{z}, \mathbf{z}^*, \mathbf{x}^*, 0)\, c^*_{(N,\mathbf{b}),(P,\mathbf{c})}\, A_P^\dagger(\mathbf{c}, \mathbf{c}', \mathbf{u}, \mathbf{u}^*, \mathbf{v}, \mathbf{v}^*, \mathbf{w}^*, (L, \mathbf{a})).$$

We start by bounding the A correlations via the van den Berg-Kesten inequality. For example,

$$\text{(A.31)} \quad A_N(\mathbf{b}, \mathbf{b}', \mathbf{y}, \mathbf{y}^*, \mathbf{z}, \mathbf{z}^*, \mathbf{x}^*, 0) \leqq \tau_{0,\mathbf{x}^*}\, \tau_{\mathbf{x}^*,\mathbf{y}^*}\, \tau_{\mathbf{x}^*,\mathbf{z}^*}\, \tau_{\mathbf{z}^*,(N,\mathbf{b}')} \times \tau_{\mathbf{z}^*,\mathbf{z}}\, \tau_{\mathbf{y}^*,\mathbf{y}}\, \tau_{(N,\mathbf{b}),\mathbf{y}^*},$$

and similarly for $A_P^\dagger$. As it turns out, in order to perform the summation, the procedure must begin at $(L, \mathbf{a})$, since otherwise there are not enough free indices available. Thus, at fixed N and P, let us consider the partial sum

$$\text{(A.32)} \quad \sum_{\substack{\mathbf{w}^*: W \geqq P \\ \mathbf{u}^*, \mathbf{v}^*: L \geqq U, V \geqq P \\ \mathbf{a}, \mathbf{c}, \mathbf{c}' \in \mathbb{Z}^{d-1} \\ \mathbf{v}, \mathbf{u} \in \mathbf{P}(L)}} \tau_{\mathbf{w}^*,(L,\mathbf{a})}\, \tau_{\mathbf{w}^*,\mathbf{u}^*}\, \tau_{\mathbf{w}^*,\mathbf{v}^*} \tau_{\mathbf{v}^*,(P,\mathbf{c}')}\, \tau_{\mathbf{v}^*,\mathbf{v}}\, \tau_{\mathbf{u}^*,\mathbf{u}}\, \tau_{(P,\mathbf{c}),\mathbf{u}^*}\, c^*_{(N,\mathbf{b}),(P,\mathbf{c})},$$

where W, U, and V are the x_1-coordinates of $\mathbf{w}^*$, $\mathbf{u}^*$, and $\mathbf{v}^*$. Summing (A.32) over $\mathbf{a}$ and $\mathbf{c}'$, then $\mathbf{v}$ and $\mathbf{u}$, gives the upper bound

$$\text{(A.33)} \quad \sum_{\substack{\mathbf{w}^*, \mathbf{u}^*, \mathbf{v}^* \\ W \geqq P \\ L \geqq U, V \geqq P}} \mathbb{G}_{|L-W|}\, \mathbb{G}_{V-P}\, \mathbb{G}_{L-V}\, \mathbb{G}_{L-U}\, \tau_{\mathbf{w}^*,\mathbf{u}^*}\, \tau_{\mathbf{w}^*,\mathbf{v}^*}\, \tau_{(P,\mathbf{c}),\mathbf{u}^*}\, c^*_{(N,\mathbf{b}),(P,\mathbf{c})}.$$

Next, we sum over the transverse coordinates of $\mathbf{v}^*$, then $\mathbf{w}^*$, then $\mathbf{u}^*$, which gives us

$$\text{(A.34)} \quad \sum_{\substack{L \geqq U, V \geqq P \\ W \geqq P}} \mathbb{G}_{|L-W|}\, \mathbb{G}_{V-P}\, \mathbb{G}_{L-V}\, \mathbb{G}_{L-U}\, \mathbb{G}_{|W-U|}\, \mathbb{G}_{|W-V|}\, \mathbb{G}_{U-P}\, c^*_{(N,\mathbf{b}),(P,\mathbf{c})}.$$

Observe now that if we sum over the indices W, Y, U, and V, we get an upper bound of a constant times $e^{-2(L-P)/\xi}$ without having touched our $c^*_{(N,\mathbf{b}),(P,\mathbf{c})}$ term. Finally, we may freely sum over $\mathbf{c}$ to get the desired $\mathbb{C}^*_{P-N}$ term, which frees up the index $\mathbf{b}$ for future ease of summation. Performing the entire procedure again on the other side of $\mathbf{P}(N)$, we get a total upper bound of

$$\text{(A.35)} \quad (\text{const}) \sum_{0 \leqq N < P \leqq L} \mathbb{C}^*_{P-N}\, e^{-2(L-(P-N))/\xi}$$

for the $\alpha-\alpha$ term. A similar analysis on the β and mixed terms yields a bound of the same order of magnitude as (A.35). This provides a bound on the $\mathbb{K}_L^{②}$ term, which, together with (A.24), is the desired result. □

Corollary. *Let $p\in(0, p_c)$. Then either $\xi_c(p)=\frac{1}{2}\xi(p)$ or $\exists D_3(p)<\infty$ such that uniformly in L,*

$$\mathbb{K}_L(p)\leqq D_3(p)\,\mathbb{C}_L^{**}(p).$$

Proof. First, let us define η via

$$\frac{1}{\xi_k}=\frac{2}{\xi}-6\eta. \tag{A.36}$$

It should be observed that for any integers Q, M, $0<Q<M$, and for any $\mathbf{s}$, $\mathbf{t}\in\mathbb{Z}^{d-1}$, we have the obvious bound

$$c_{0,(M,\mathbf{s})}^{**}\geqq a(p)\,c_{0,(Q,\mathbf{t})}^{*}\,c_{(Q,\mathbf{t}),(M,\mathbf{s})}^{**} \tag{A.37}$$

with $a(p)>0$ a harmless patching factor. However, since the (mirror images of the) events $\mathscr{c}_{0,(Q,\mathbf{t})}^{*}$ are disjoint, the stronger statement

$$c_{0,(M,\mathbf{s})}^{**}\geqq a(p)\sum_{\mathbf{t}\in\mathbb{Z}^{d-1}} c_{0,(Q,\mathbf{t})}^{*}\,c_{(Q,\mathbf{t}),(M,\mathbf{s})}^{**} \tag{A.38}$$

is also true. Summing (A.38) over $\mathbf{s}$, one obtains the bound

$$\mathbb{C}_M^{**}\geqq a(p)\,\mathbb{C}_Q^{*}\,\mathbb{C}_{M-Q}^{**}. \tag{A.39}$$

Using the assumption that $2/\xi>1/\xi_c=2/\xi-6\eta$, it is clear that, uniformly in S, we can find an $\Omega(p)>0$ such that

$$\mathbb{C}_S^{**}\geqq\Omega e^{-(2/\xi-5\eta)S}. \tag{A.40}$$

Whence, after a little $\mathbb{C}^*$ algebra, the statement of Lemma A.2 becomes

$$\mathbb{K}_L\leqq\mathbb{C}_L^{**}\left[D_1+D_2\frac{1}{\Omega a}\sum_{0\leqq N<P\leqq L} e^{-4\eta(L-(P-N))}\right]. \tag{A.41}$$

The sum in (A.41) is bounded above by a finite constant uniformly in L. □

Finally, we relate the direct connectivities with strict and non-strict cylinder boundary conditions:

Proof of Proposition 3.4. For $\mathbf{x}\in\mathbf{P}(0)$, define

$$\mathscr{c}_{0,(L,\mathbf{a});\mathbf{x}}^{**}=\{\omega\in\mathscr{c}_{0,(L,\mathbf{a})}^{**}\,|\,\mathbf{x}\in C(0)\|_{\mathbf{S}(L)}\} \tag{A.42a}$$

and

$$c_{0,(L,\mathbf{a});\mathbf{x}}^{**}=P(\mathscr{c}_{0,(L,\mathbf{a});\mathbf{x}}^{**}). \tag{A.42b}$$

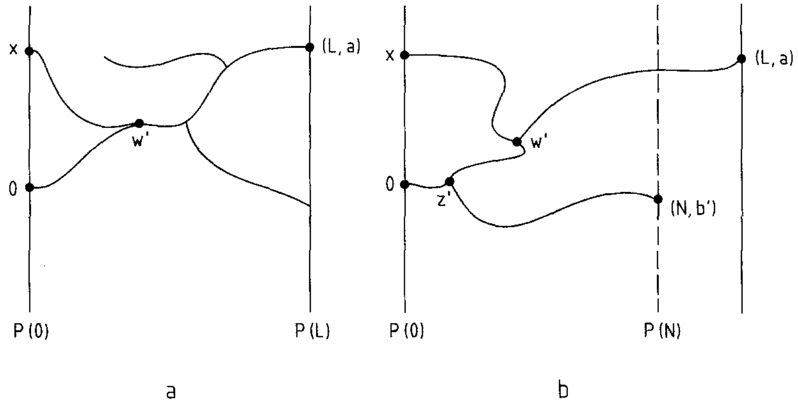

Fig. A3. Configurations in $c^{**}_{0,(L,\mathbf{a});\mathbf{x}}$ corresponding to cases (**a**) and (**b**)

We claim that $\exists D_4(p) < \infty$ such that

(A.43) $$\sum_{\mathbf{a}\in\mathbb{Z}^{d-1}} c^{**}_{0,(L,\mathbf{a});\mathbf{x}} \leqq D_4\, e^{-\eta|\mathbf{x}|}\, \mathbb{C}^{**}_L$$

where η has been defined in (A.36).

If $\eta = 0$, Eq. (A.43) is obviously true. Otherwise, let us take an $\omega \in c^{**}_{0,(L,\mathbf{a});\mathbf{x}}$. It is seen that inside $\mathbf{S}(L)$, there must be a point $\mathbf{w}^*$ and three disjoint occupied paths:

(A.44a) $\mathscr{P}_{[1]}$: from $\mathbf{w}^*$ to $(L, \mathbf{a})$,

(A.44b) $\mathscr{P}_{[2]}$: from 0 to $\mathbf{w}^*$,

(A.44c) $\mathscr{P}_{[3]}$: from $\mathbf{x}$ to $\mathbf{w}^*$.

Now grow the cluster of $\mathscr{P}_{[1]}$ in the complement of $\mathscr{P}_{[2]}$ and $\mathscr{P}_{[3]}$. Denote by $\underline{C}(\mathscr{P}_{[1]} \perp (\mathscr{P}_{[2]} \cup \mathscr{P}_{[3]}))$ the fully grown cluster generated by this procedure. It is not unreasonable to hope that (see Fig. A3):

(a) $\underline{C}(\mathscr{P}_{[1]} \perp (\mathscr{P}_{[2]} \cup \mathscr{P}_{[3]}))$ contains a realization of the event $k_{\mathbf{w}^*,(L,\mathbf{a})}$.

However, should this not come to pass, there is a rightmost plane, $\mathbf{P}(N)$, $N \leqq L-1$, and a $\mathbf{b} \in \mathbb{Z}^{d-1}$ such that

(A.45) $$\{(N, \mathbf{b})\} = \underline{C}(\mathscr{P}_{[1]} \perp (\mathscr{P}_{[2]} \cup \mathscr{P}_{[3]})) \cap \mathbf{P}(N).$$

Then, it has to be the case that either

(b) $\mathscr{P}_{[2]}$ is connected to $\mathbf{P}(N)$ in the complement of $\underline{C}(\mathscr{P}_{[1]} \perp (\mathscr{P}_{[2]} \cup \mathscr{P}_{[3]}))$, or
(b′) $\mathscr{P}_{[3]}$ is connected to $\mathbf{P}(N)$ in the complement of $\underline{C}(\mathscr{P}_{[1]} \perp (\mathscr{P}_{[2]} \cup \mathscr{P}_{[3]}))$.

Case (a) implies $\exists\, \mathbf{w}^* \in \mathbf{S}(L)$ such that

(A.46a) $$\omega \in g_{0,\mathbf{w}^*} \circ g_{\mathbf{x},\mathbf{w}^*} \circ k_{\mathbf{w}^*,(L,\mathbf{a})},$$

and the probability of this is bounded above by

(A.47a) $$\sum_{\mathbf{w}^*\in\mathbb{Z}^d} \tau_{0,\mathbf{w}^*}\, \tau_{\mathbf{x},\mathbf{w}^*}\, k_{\mathbf{w}^*,(L,\mathbf{a})}.$$

Case (b) may be described by the fact that $\exists \mathbf{w}^* \in \mathbf{S}(L)$, $w_1^* \equiv W$, and N with $W<N<L$, $\mathbf{b}, \mathbf{b}' \in \mathbb{Z}^{d-1}$, and $\mathbf{z}^* \in \mathbf{S}(N)$ such that

(A.46b) $$\omega \in \mathcal{G}_{0,\mathbf{z}^*} \circ \mathcal{G}_{\mathbf{z}^*,\mathbf{w}^*} \circ \mathcal{G}_{\mathbf{x},\mathbf{w}^*} \circ \mathcal{G}_{\mathbf{w}^*,(N,\mathbf{b})} \circ \mathcal{G}_{\mathbf{z}^*,(N,\mathbf{b}')} \circ \mathcal{C}^{**}_{(N,\mathbf{b}),(L,\mathbf{a})},$$

similarly for case (b′) with 0 replaced by $\mathbf{x}$. The event described in (A.46b) has probability bounded above by

(A.47b) $$\sum_{\mathbf{w}^*,\mathbf{z}^*,N,\mathbf{b},\mathbf{b}'} \tau_{0,\mathbf{z}^*}\, \tau_{\mathbf{z}^*,\mathbf{w}^*}\, \tau_{\mathbf{x},\mathbf{w}^*}\, \tau_{\mathbf{w}^*,(N,\mathbf{b})}\, \tau_{\mathbf{z}^*,(N,\mathbf{b}')}\, C^{**}_{(N,\mathbf{b}),(L,\mathbf{a})},$$

with a corresponding equation describing situation (b′).
Summing (A.47a) over $\mathbf{a} \in \mathbb{Z}^{d-1}$, we obtain

(A.48) $$\sum_W \mathbb{K}_{L-W} \sum_{\mathbf{w}} \tau_{0,(W,\mathbf{w})}\, \tau_{\mathbf{x},(W,\mathbf{w})},$$

where we have used the notation $\mathbf{w}^* = (W, \mathbf{w})$. Now, we use the a priori bound (2.7) and the obvious bounds $|\mathbf{w}^*| > W$ and $|\mathbf{w}^* - \mathbf{x}| > W$ to write

(A.49) $$\tau_{0,(W,\mathbf{w})}\, \tau_{\mathbf{x},(W,\mathbf{w})} \leqq e^{-(2/\xi)(|\mathbf{w}^*| + |\mathbf{w}^* - \mathbf{x}|)} \leqq e^{-2(1/\xi - 2\eta)W} e^{-2\eta}.$$

The bound in (A.49) may be further degraded by noting that

(A.50) $$2(|\mathbf{w}^*| + |\mathbf{w}^* - \mathbf{x}|) \geqq |\mathbf{x}| + |\mathbf{w}| + W,$$

so that (A.48) can be bounded above by

(A.51) $$e^{-\eta|\mathbf{x}|} \sum_{W,\mathbf{w}} \mathbb{K}_{L-W}\, e^{-(2/\xi - 4\eta)W} e^{-\eta|\mathbf{w}|} e^{-\eta W}.$$

We now use the corollary to Lemma A.2 to bound $\mathbb{K}_{L-W}$ above by a constant times $\mathbb{C}^{**}_{L-W}$. Next, since the decay rate of $\mathbb{C}^*_L$ is ξ_c (see Proposition A1 and the Remark following Definition A2), it follows from the definition of η (cf. Eq. (A.36)), that for W large enough, $e^{-(2/\xi - 4\eta)W}$ may be bounded above by (another constant times) $\mathbb{C}^*_W$. Then, according to Eq. (A.39), the two $\mathbb{C}$'s may be combined to obtain a total upper bound of

(A.52) $$D_{4'}\, e^{-\eta|\mathbf{x}|} \mathbb{C}^{**}_L \sum_{\mathbf{w}^*} e^{-\eta|\mathbf{w}^*|}$$

on case (a). Performing the sum over $\mathbf{w}^*$ just modifies the constant $D_{4'}$.

The (b) (and (b′)) cases follow from a similar, but somewhat more laborious analysis. Summing (A.47b) over $\mathbf{a}$, $\mathbf{b}'$, and then $\mathbf{b}$, we get

(A.53) $$\sum_{\substack{0 \leqq W \leqq N \\ 0 \leqq Z \leqq N \\ \mathbf{w},\mathbf{z},N}} \mathbb{G}_{N-Z}\, \mathbb{G}_{N-W}\, \mathbb{C}^{**}_{L-N}\, \tau_{0,\mathbf{z}^*}\, \tau_{\mathbf{z}^*,\mathbf{w}^*}\, \tau_{\mathbf{x},\mathbf{w}^*},$$

where we have again used $\mathbf{w}^* = (W, \mathbf{w})$, $\mathbf{z}^* = (Z, \mathbf{z})$. Next, we note that the uniform bounds of Proposition 3.1 imply $\mathbb{G}_{N-W}\, \mathbb{G}_{N-Z} \leqq \beta^{-2} e^{-(2/\xi)N} e^{+(1/\xi)(Z+W)}$. Using this and the a priori upper bound $\tau_{0,\mathbf{z}^*} \leqq e^{-|\mathbf{z}^*|/\xi}$, the argument of the sum in (A.53) is bounded above by a constant times

$$\text{(A.54)} \qquad \mathbb{C}^{**}_{L-N}\, e^{-(2/\xi - 5\eta)N} e^{-5N\eta} e^{+(1/\xi)(Z+W)} e^{-|\mathbf{z}^*|/\xi} e^{-|\mathbf{w}^*-\mathbf{z}^*|/\xi} e^{-|\mathbf{w}^*-\mathbf{x}|/\xi}.$$

Now, we use the inequalities

$$\text{(A.55a)} \qquad \frac{1}{\xi}|\mathbf{z}^*| \geqq \left(\frac{1}{\xi} - 2\eta\right)|Z| + 2\eta|\mathbf{z}^*|$$

$$\text{(A.55b)} \qquad \frac{1}{\xi}|\mathbf{w}^* - \mathbf{x}| \geqq \left(\frac{1}{\xi} - 2\eta\right)|W| + 2\eta|\mathbf{w}^* - \mathbf{x}|$$

$$\text{(A.55c)} \qquad \frac{1}{\xi}|\mathbf{w}^* - \mathbf{z}^*| \geqq 5\eta|\mathbf{w}^* - \mathbf{z}^*|,$$

together with the bound $\mathbb{C}^{**}_{L-N} e^{-(2/\xi - 5\eta)N} \geqq (\text{const})\, \mathbb{C}^{**}_L$, valid for N sufficiently large, to estimate the quantity in (A.54) by

$$\text{(A.56)} \qquad (\text{const})\, \mathbb{C}^{**}_L\, e^{-5\eta N} e^{+2\eta(Z+W)} e^{-2\eta|\mathbf{z}^*|} e^{-2\eta|\mathbf{w}^*-\mathbf{x}|} e^{-5\eta|\mathbf{w}^*-\mathbf{z}^*|}.$$

Recalling that $W, Z \leqq N$, the above is pointwise bounded by

$$\text{(A.57)} \qquad (\text{const})\, \mathbb{C}^{**}_L\, e^{-\eta N} e^{-2\eta|\mathbf{z}^*|} e^{-2\eta|\mathbf{w}^*-\mathbf{x}|} e^{-5\eta|\mathbf{w}^*-\mathbf{z}^*|}.$$

Finally, observing that

$$\text{(A.58)} \qquad 2\eta|\mathbf{z}^*| + 2\eta|\mathbf{w}^* - \mathbf{x}| + 2\eta|\mathbf{w}^* - z^*| \geqq \eta|\mathbf{x}| + \eta|\mathbf{z}^*|,$$

so that

$$\text{(A.59)} \qquad 2\eta|\mathbf{z}^*| + 2\eta|\mathbf{w}^* - \mathbf{x}| + 5\eta|\mathbf{w}^* - \mathbf{z}^*| \geqq \eta|\mathbf{x}| + \tfrac{1}{2}(\eta|\mathbf{z}^*| + \eta|\mathbf{w}^*|),$$

we are left with

$$\text{(A.60)} \qquad (\text{const.})\, e^{-\eta|\mathbf{x}|} \mathbb{C}^{**}_L \sum_{\substack{\mathbf{w}^*: W \leqq N \\ \mathbf{z}^*: Z \leqq N \\ N}} e^{-\eta N} e^{-(\eta/2)(|\mathbf{z}^*| + |\mathbf{w}^*|)} \equiv D_{4''}\, e^{-\eta|\mathbf{x}|} \mathbb{C}^{**}_L$$

for the b (and (b′)) estimates. Together with (A.52), this establishes Eq. (A.43).

It is now straightforward to show that (if $\eta > 0$) $\mathbb{C}^{**}_L$ and $\mathbb{C}^*_L$ are bounded by multiples of one another. Indeed, decomposing $\mathbb{C}^{**}_L$ according to the number of intersections of $C(0)\|_{\mathbf{S}(L)}$ with the plane $\mathbf{P}(0)$ (as was done for the $\mathbb{H}$'s in Eqs. (3.13)–(3.14)):

$$\text{(A.61)} \qquad \mathbb{C}^{**}_{L,k} = \sum_{\mathbf{a} \in \mathbb{Z}^{d-1}} P(\{\omega \in c^{**}_{0,(L,\mathbf{a})} \| |C(0)\|_{\mathbf{S}(L)} \cap \mathbf{P}(0)| = k\}),$$

we have the identity

(A.62) $$\mathbb{C}^{*}_{L+1}=\sum_{k}\mathbb{C}^{**}_{L,k}(1-p)^{2(d-1)}p(1-p)^{k-1}.$$

Thus, by the Jensen inequality,

(A.63) $$\mathbb{C}^{*}_{L+1}\geqq\mathbb{C}^{**}_{L}(1-p)^{2(d-1)}p(1-p)^{\bar{K}_L-1},$$

where

(A.64) $$\bar{K}_L\equiv\sum_{k}k\frac{\mathbb{C}^{**}_{L,k}}{\mathbb{C}^{**}_{L}}.$$

Now, by (A.43),

(A.65) $$\mathbb{C}^{**}_{L,k}\leqq\sum_{\substack{|\mathbf{x}|\geqq k^{1/(d-1)}\\ \mathbf{a}\in\mathbb{Z}^{d-1}}}c^{**}_{0,(L,\mathbf{a});\mathbf{x}}\leqq D_5\,e^{-\eta k^{1/(d-1)}}\mathbb{C}^{**}_{L}$$

with $D_5(p)<\infty$. Hence $\bar{K}_L(p)<\infty$, so that by (A.63)

(A.66) $$\mathbb{C}^{*}_{L}\leqq D_6\,\mathbb{C}^{**}_{L}$$

with $D_6(p)<\infty$ for $p<p_c$, and we are halfway there.

If the steps of this proof (i.e., Eqs. (A.42)–(A.66)) are carefully examined, it is not terribly difficult to see that the derivation proceeds unhindered if, in each $\mathbb{C}$-term, a $*$ is removed. The result is

(A.67) $$\mathbb{C}_{L}\leqq D_7\,\mathbb{C}^{*}_{L}$$

with $D_7(p)<\infty$ for $p<p_c$. Combining the corollary to Lemma A.2 with the bounds (A.66) and (A.67), we have the desired result. □

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