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LETTER TO THE EDITOR

The density of Peierls contours in $d=2$ and the height of the Wedding Cake

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Abstract. For the two-dimensional nearest neighbour Ising magnet above the critical temperature, the density of Peierls contours that separate two distant points is shown to be strictly positive. As a direct consequence it is shown, in the Wedding Cake model, that in the corresponding regime, the height of the surface scales with the linear dimensions of the system.

This letter addresses a problem which might have been posed a long time ago but, for various reasons, has become a topic of current interest: suppose that σ is a 'typical' spin configuration of the two-dimensional (2D) Ising magnet† and let a and b denote two points in $\mathbb{Z}^2$. How many Peierls contours in σ separate a from b? Below the critical temperature, due to percolation of one or the other species (which is the characteristic of this phase [3]), it is not difficult to show that this number is typically of order unity no matter how far the points are from one another. However, inside the single phase regime, one is tempted to speculate that the number of separating contours scales with the distance between a and b. Such an assertion, stated somewhat more precisely, is the subject of theorem 1 below.

A closely related question concerns the Wedding Cake model of the three-dimensional wetting transition [1]. In this model, one considers a subclass of the one-step solid-on-solid surfaces: first, these surfaces are restricted to lie in the positive half-space and, if defined over some $\Lambda\subset\mathbb{Z}^2$, are required to vanish at the boundary $\partial\Lambda$. Second, whenever a contour separating regions of different heights is crossed, going from the outside to the inside, it is insisted that the height of the surface *increases.* Denoting by S_Λ the set of all surfaces which satisfy this criterion, the model is defined by assigning to each $s\in S_\Lambda$, a weight proportional to $\exp[-2\beta|s|]$ where $|s|$ is the surface area of s. The connection between this model and the 2D nearest neighbour Ising magnet in zero external field was uncovered in [2]. There it was shown that these surfaces—along with their weights—are in one-to-one correspondence with those of the Ising configurations on Λ for the model with unit strength coupling, $T=1/\beta$ and, plus (or minus) boundary conditions on $\partial\Lambda$. The principal result in this letter translates into the statement that, e.g. for the square $\Lambda_{N,N}\subset\mathbb{Z}^2$ of scale N, the height at the centre of the Wedding Cake is typically of order N provided that β is smaller than the critical value, β_c, of the Ising model.

† For reasons explained elsewhere [4] (and probably [6]), the model, as presently stated, is likely to be trivial in any dimension exceeding two. However, the analogue problem for the q-state Potts model is easily formulated and, for sufficiently large $q(d)$, non-trivial in $d>2$. These subjects are discussed in more detail in [6]; here, attention will be confined to the simplest cases.

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The infinite temperature ($\beta = 0$) version of this problem was investigated, independently, in the works [4] and [6]. In these papers (as will be the case in this letter) the problem is formulated in the language of a first passage time problem. Let $\sigma \in \Omega \equiv \{-1, +1\}^{\mathbb{Z}^2}$ denote an Ising configuration. To the bonds (ℓ) which connect pairs of neighbouring sites, let us assign the values ($V_\ell(\sigma)$) where $V_\ell = 1$ if the endpoints of ℓ are different in sign and 0 otherwise. Notice that the bonds *dual* to the bonds that carry a value of one are organized into circuits† which separate regions of opposite type.

The relevant first passage problem is defined by tallying up the number of circuits (or contours) which must be crossed in going from one point to another. Explicitly, if a and b are points in $\mathbb{Z}^2$, and $\mathcal{P} : a \to b$ is a path connecting these points, let us define

$$T_{\mathcal{P}} = \sum_{\ell \in \mathcal{P}} V_\ell \tag{1}$$

and

$$T_{a,b} = \inf_{\mathcal{P} : a \to b} T_{\mathcal{P}}. \tag{2}$$

When a is the origin and b is n units along the x_1 axis, this object will be denoted by T_n. The analogue of the first passage time is called the interfacial density, Θ and is defined by

$$\Theta = \lim_{n \to \infty} \frac{T_n}{n}. \tag{3}$$

(The existence of this limit is an application of the standard theorems on subadditive processes and will be discussed in proposition 1 below.) In [4] and [6] it was shown, among other things, that in the independent Bernoulli (percolation) site problem on the square lattice at density $\frac{1}{2}$, Θ is non-zero.

The key issue, at least in [4], is the simultaneous star-percolation of both the plus and minus sites‡ which is known to occur in the density $\frac{1}{2}$ site-percolation problem on $\mathbb{Z}^2$ [7]. In a recent paper by Higuchi, a similar result is established for the entire region $\beta < \beta_c$ of the 2D Ising magnet. The contents of this letter should be thought of as a minor corollary to the work [8].

Very recently, the author has learned that the results presented here can also be obtained by the methods of [6], and, in fact, they will appear in the revised version [14]. However, the derivation in this letter, while probably less generalizable than the techniques of [6], has simplified this problem to the point where a complete proof will be written in the few paragraphs following the preliminary business section. Furthermore, in accordance with the opening remarks, the proof here uses elementary techniques which date at least as far back as [15].

In what follows, let us denote the rectangles $\{x \in \mathbb{Z}^2 \,|\, |x_1| \leq M, |x_2| \leq N\}$ by $\Lambda_{N,M}$. The finite volume conditional measures that will be of use in this letter are just the ones in which all the spins outside of $\Lambda_{N,M}$ have been set to plus or minus. These measures will be denoted by $\langle \cdot \rangle^{+}_{\beta;(N,M)}$, etc. (In this letter, angle brackets will be used both for expectations and probabilities.) At inverse temperature β (always assumed

† It should cautiously be observed that in this letter, the separating curves are indeed finite circuits—as opposed to objects that are infinite in extent—because for the problems of interest, all connected clusters of the same sign are of finite extent with probability one.

‡ The reader will recall that two sites are deemed to be star-connected if none of their coordinates differ by more than unity. The plus sites are said to star-percolate when there is an infinite star-connected plus cluster and similarly for the minuses.

smaller than β_c), the unique infinite volume Gibbs measure will be denoted by angle brackets with a subscripted β: $\langle\cdot\rangle_\beta$.

The starting point will be to establish the existence of a contour density in the regime of interest.

Proposition 1. For all $\beta < \beta_c$, the limit

$$\Theta = \lim_{n\to\infty} \frac{T_n}{n}$$

exists with probability one (wp1) and in L^1 and is, wp1, a constant.

Proof. A proof of the wp1 and L^1 convergence of T_n/n may be taken directly from the proof for ordinary first passage percolation found, e.g., in [11]. Many of the arguments date back to [9].

Denote by $T_{n,m}$, $m \geqslant n$, the objects $T_{(n,0),(m,0)}$. It is observed that the $T_{n,m}$ are subadditive

$$T_{n,m+r} \leqslant T_{n,m} + T_{m,m+r} \tag{4}$$

bounded (below by zero and above by $m-n$) and $\forall n, m$, $T_{n,m}$ is equal in distribution to $T_{0,m-n} \equiv (T_{m-n})$. It follows from Kingman's subadditive ergodic theorem [12] that

$$\lim_{n\to\infty} \frac{T_n}{n} \equiv \Theta \tag{5}$$

exists wp1 and in L^1. Now observe that if $\boldsymbol{k}$ is any vector in $\mathbb{Z}^2$ and $\mathcal{T}_k(\sigma)$ is the translation of the configuration σ by $\boldsymbol{k}$ units then $\Theta(\mathcal{T}_k(\sigma)) = \Theta(\sigma)$. Since $\langle\cdot\rangle_\beta$ is *the* unique (invariant) Gibbs measure when $\beta \leqslant \beta_c$ (see, e.g., [13, ch IV, corollary 1.29]) translational averages are, wp1, equal to thermal expectations. From this it follows that Θ is, wp1, a constant. For more details on these sorts of arguments see, e.g., [10, ch IV].

We now consider reduction to a problem at low density. The bond events of interest will be the simultaneous occurrence of star-connected plus and star-connected minus crossings of large blocks as defined below.

Lemma 1. Let $L \gg 1$ and consider the 4×1 block of sites that occupy the region $\Lambda_{4L,L}$. Let $\mathcal{B}_L^+$ denote the event that:

(i) In the left quarter of $\Lambda_{4L,L}$, there is a star-connected top bottom crossing by $+$ spins.

(ii) In the right quarter of $\Lambda_{4L,L}$, there is a star-connected top bottom crossing by $+$ spins.

(iii) There is a star-connected left right crossing of the entire block by $+$ spins.

(Cf figure 1 below for a visualization of this event along with the described boundary condition.) Then, for $\beta < \beta_c$, there are finite, positive constants α, C and κ such that

$$\langle \mathcal{B}_L^+ \rangle^-_{\beta;(8L,5L)} \geqslant 1 - CL^\alpha \exp[-\kappa L].$$

Remarks. (*a*) Notice that, by the FKG [5] property, in as far as the region $\Lambda_{8L,5L}$ is concerned, the above is the worst case conditional probability for the event $\mathcal{B}_L^+$. (*b*) The use of 4×1 blocks—as opposed to, say, 3×1 blocks—is for the visual convenience of figure 2 and is otherwise of no particular significance.

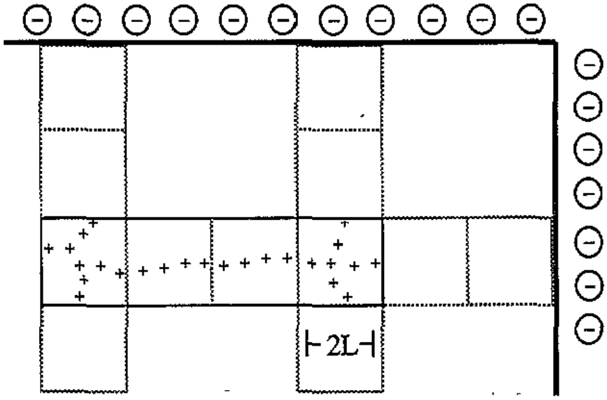

Figure 1.

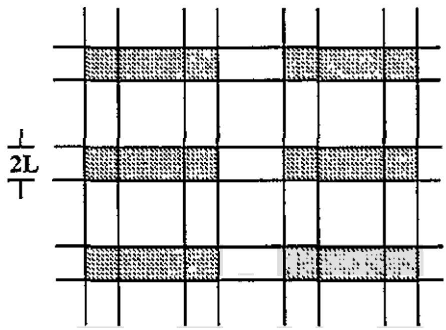

Figure 2.

Proof. An abbreviated statement of [8] theorem 2 is as follows: let Λ, $V \subset \mathbb{Z}^2$ with $\Lambda \subset V$ and let $\mathscr{A}$ denote any event defined on the configurations in Λ. Then, for any $\beta < \beta_c$, the difference between the conditional probability of $\mathscr{A}$ given any configuration outside V and the infinite volume probability of $\mathscr{A}$ decays, 'exponentially', with the distance between Λ and V. As a direct consequence, one has the estimate

$$\langle \mathscr{B}_L^+ \rangle_\beta - \langle \mathscr{B}_L^+ \rangle^-_{\beta;(8L,5L)} = |\langle \mathscr{B}_L^+ \rangle^-_{\beta;(8L,5L)} - \langle \mathscr{B}_L^+ \rangle_\beta| \leq C_1 L^{\alpha_1} \exp[-\kappa_1 L] \tag{6}$$

where κ_1 is a positive, finite constant.

Now to prevent any of the events (i)–(iii) from occurring, there must be connected paths of minus spins in $\Lambda_{4L,L}$ that are of linear extent at least as large as L. Theorem 3 of [8] may be paraphrased as: let x and y be points in $\mathbb{Z}^2$ and let $K_{x,y}$ denote the event that x and y belong to the same connected cluster of plus (or minus) spins. Then, if $\beta < \beta_c$,

$$\langle K_{x,y} \rangle_\beta \leq C_2 \exp[-\kappa_2 |x-y|] \tag{7}$$

where C_2 and κ_2 are positive constants and $|x-y|$ is the Euclidian (or any other convenient, equivalent) distance between x and y. Evidently $\langle (\mathscr{B}_L^+)^c \rangle_\beta$ (the probability that $\mathscr{B}_L^+$ does not occur) enjoys an upper bound similar to the right-hand side of (6). From these two estimates, the desired result follows immediately. $\square$

Let us now implement the block bond construction: consider the bond lattice $\mathbb{B}_L$ which consists of overlapping translates of $\Lambda_{4L,L}$ and $\Lambda_{L,4L}$ as depicted in figure 2. The bond events will be appropriate translates of the event $\mathscr{B}_L^+$ together with its spin reversal:

$$\mathscr{B}_L = \mathscr{B}_L^+ \cap \mathscr{B}_L^-. \tag{8}$$

It is noted that, for large L, the individual bond events occur with high probability.

Although bond events corresponding to rectangles which share a corner are strongly correlated, separated bonds interact 'weakly' as will be exemplified in lemma 2 below. Indeed, observe that $\mathbb{B}_L$ can be partitioned into four disjoint sublattices, $\mathbb{B}_L^{[1]}$-$\mathbb{B}_L^{[4]}$, each of which consist of non-overlapping 4×1 blocks. One of these sublattices has been highlighted in figure 2. If S is a bond on one of these lattices, let us define $B_L^{[S]}(\sigma)$ to be 1 if the appropriate translate and/or rotation of the event $\mathscr{B}_L$ occurs in the configuration σ, and 0 otherwise. The relevant probabilities of (bond) events which take place within any one of these sublattices can be estimated, for large L, by product measure at high density.

Lemma 2. Let $\varepsilon > 0$ and, for $\beta < \beta_c$ suppose that L has been chosen large enough to ensure that $\langle \mathscr{B}_L^+ \rangle^-_{\beta;(8L,5L)} > 1 - \varepsilon/2$. Let $(B_L^{[S]} | S \in \mathbb{B}_L^*)$, $* = [1], [2], [3]$ or $[4]$ denote the collection of bond events which occur on the lattice $\mathbb{B}_L^*$. Then, if F is any function of the collection $(B_L^{[S]})$, which is increasing (in the sense of [5]), $\langle F \rangle_\beta$ is at least as large as the average of F is an independent ensemble of bonds at density $1-\varepsilon$.

Proof. This is a direct transcription of the following (easily proved) statement: let $X_1, \ldots, X_m, \ldots$ denote a collection of $\{0, 1\}$ valued random variables and suppose that for any j, and any configuration $(X_1, \ldots, X_j)$,

$$\mathrm{Prob}(X_{j+1} = 1 | (X_1, \ldots, X_j)) \geqslant s. \tag{9}$$

Then, the average of any FKG increasing function of the (X_k) is at least as large as the corresponding average when the (X_k) are independent Bernoulli variables with $\mathrm{Prob}(X_j = 1) = s$. A proof of this statement, which follows directly using induction, can also be extracted from the proof of the first lemma in [16].

For the case at hand, let us denote by $\mathscr{B}_L^{+[S]}$ and $\mathscr{B}_L^{-[S]}$ the positive and negative pieces out of which the event $B_L^{[S]} = 1$ is composed. If $\omega \subset \Omega$ is any configuration of spins inside any collection of blocks belonging to some bonds $B_L^{[T]}$, $T \neq S$, by the FKG property of the nearest neighbour Ising system,

$$\langle \mathscr{B}_L^{+[S]} | \omega \rangle_\beta \geqslant \langle \mathscr{B}_L^+ \rangle^-_{\beta;(8L,5L)} \tag{10a}$$

and

$$\langle \mathscr{B}_L^{-[S]} | \omega \rangle_\beta \geqslant \langle \mathscr{B}_L^+ \rangle^+_{\beta;(8L,5L)}. \tag{10b}$$

The desired result is now straightforward. □

It is now permissible to act as though the bonds within any sublattice are independent and at high density. In the forthcoming low density first passage argument, the events $B_L^{[S]} = 1$ should be thought of as barriers or 'blocked passages' on the dual (bond) lattice—where all the action will take place—while the events $B_L^{[S]} = 0$ are represented as open passages on the dual lattice.

Lemma 3. Let $\varepsilon > 0$ be a sufficiently small number, $\beta < \beta_c$ and let L be chosen large enough so that

$$\langle \mathscr{B}_L^+ \rangle^-_{\beta;(8L,5L)} \geqslant 1 - \varepsilon/2.$$

Let $N \gg L$ and consider the collection of bond events $B_L^{[S]}$ which are determined by *the spins inside* $\Lambda_{N,N}$. *Then there are finite constants* γ, K, s *and* δ *such that either* in the finite volume measure given by $\langle\cdot\rangle^+_{\beta;(2N,2N)}$ or in the infinite volume state, there are more than γN disjoint circuits of barrier bond events inside $\Lambda_{N,N}$ with probability exceeding $1 - KN^s\,\mathrm{e}^{-\delta N}$.

Proof. In what is to follow, only the infinite volume statement will receive an explicit proof; the finite volume result can be recovered by an easy application of [8] theorem 2. Let W denote a generic self-avoiding path on the dual of $\mathbb{B}_L$ that starts at the 'origin' and ends up at the boundary of the collection of bonds under consideration. Let $\mathscr{W}$ be the collection of all walks of this type. For $W \in \mathscr{W}$, let $|W|$ be its length (number of bonds of $\mathbb{B}_L$ crossed). It is claimed that with very high probability, any such walk will enjoy fewer than $\frac{1}{2}|W|$ open bonds. Indeed, supposing that W consists of $n_{[1]}(W), \ldots, n_{[4]}(W)$ steps on (the dual of) lattices $\mathbb{B}_L^{[1]}, \ldots, \mathbb{B}_L^{[4]}$. The probability that W enjoys $g_{[1]}, \ldots, g_{[1]}$ open bonds on these respective lattices is bounded above by

$$\min\left\{\binom{n_{[1]}}{g_{[1]}}\varepsilon^{g_{[1]}}, \ldots, \binom{n_{[4]}}{g_{[4]}}\varepsilon^{g_{[4]}}\right\}.$$

It is noted that each of combinatoric factors is bounded above by $2^{|W|}$ *and that at least* one of the gs is as big as $\frac{1}{8}|W|$. The number of ways that $g_{[1]} + g_{[2]} + g_{[3]} + g_{[4]}$ can add up to any K with $|W| \geqslant K \geqslant \frac{1}{2}|W|$ is bounded by a power of $|W|$. Using W^* as notation *for the event that in a spin configuration* σ, *the walk* W *does experience as many as* $\frac{1}{2}|W|$ open bonds, the above tells us that

$$\langle W^* \rangle_\beta \leqslant c|W|^A 2^{|W|} \varepsilon^{|W|/8} \tag{11}$$

for some finite, positive constants c and A.

Now the number of walks of length $|W|$ starting at the origin is (eventually) less than $3^{|W|}$. Summing over all possible lengths of walks—observing that the shortest possible walks contain at least the order of N/L steps—one has that the probability of observing any walk event is bounded above via

$$\left\langle \bigcup_{W \in \mathscr{W}} W^* \right\rangle_\beta \leqslant KN^s\,\mathrm{e}^{-\delta N}. \tag{12}$$

As a consequence of the standard discrete geometric lemmas (here what is needed can be easily derived from [17, theorem 4]), the fact that none of the W^* events occur implies that the number of disjoint circuits of barrier bond events is the order of (half) the minimal path length. □

Remark. The above mentioned circuits are, of course, *bond* disjoint and this leads to the possibility that (at most) two of the circuits could collide at any given 'site'. Although this would not create an insurmountable difficulty in what is to follow, for the purposes of keeping this letter simple, let us replace, in the statement of lemma 3, the word 'disjoint' with the phrase 'bond and site disjoint' at the expense of losing half the estimated number of circuits. Further, note that the choice of $\Lambda_{2N,2N}$ was, of course, for typographical convenience and $\Lambda_{fN,fN}$ with any $f > 1$ would have worked equally well.

As an immediate corollary to lemma 3, one has:

Theorem 1. Whenever $\beta < \beta_c$, $\Theta > 0$. Furthermore, there are finite, positive constants c_1 and c_2 such that for large M, the height of the Wedding Cake defined over $\Lambda_{M,M}$ at the origin is larger than a $c_1 M$, with probability exceeding $1 - \exp[-c_2 M]$.

Proof. Observe that the connectivity properties of the bond events are inherited by the underlying (star-connected) paths out of which these events are comprised. Lemma 3 then implies that, with high probability (either in $\Lambda_{M,M}$ with + boundary conditions or in the infinite volume problem) there are of the order M *pairs* of star-connected circuits, one of each sign, surrounding the origin and lying inside $\Lambda_{M,M}$. Note that these circuit-pairs are not only disjoint from one another, they are actually separated by a distance of order L. It is easily shown that between each successive pair of circuit-pairs (excuse the language), there must be a Peierls contour: for example, take the minus ring from the inside pair and the plus ring from the outside pair. Between this inner minus ring and outer plus ring there has to be a boundary circuit separating the interior of the inner and the exterior of the outer. The presence, with the above stated probability (in either measure) of the order of M Peierls contours inside $\Lambda_{M,M}$ that surround the origin, clearly implies the desired results. □

Concluding note: Higuchi has, in fact proved his principal results (theorems 2 and 3) in the presence of non-zero magnetic field $0 \leqslant |h| < h_c(\beta)$. All the results derived in this letter go through in these cases as well—at the expense of clouding the statements of lemmas 1-3 and theorem 1. However (in my opinion) the presence of a non-vanishing magnetic field yields an unphysical random surface problem.

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