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Discontinuity of the Magnetization in One-Dimensional $1/|x-y|^2$ Ising and Potts Models

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Results from percolation theory are used to study phase transitions in one-dimensional Ising and q-state Potts models with couplings of the asymptotic form $J_{x,y} \approx \text{const}/|x-y|^2$. For translation-invariant systems with well-defined $\lim_{x\to\infty} x^2 J_x = J^+$ (possibly 0 or ∞) we establish: (1) There is no long-range order at inverse temperatures β with $\beta J^+ \leqslant 1$. (2) If $\beta J^+ > q$, then by sufficiently increasing J_1 the spontaneous magnetization M is made positive. (3) In models with $0 < J^+ < \infty$ the magnetization is discontinuous at the transition point (as originally predicted by Thouless), and obeys $M(\beta_c) \geqslant 1/(\beta_c J^+)^{1/2}$. (4) For Ising ($q=2$) models with $J^+ < \infty$, it is noted that the correlation function decays as $\langle \sigma_x \sigma_y \rangle(\beta) \approx c(\beta)/|x-y|^2$ whenever $\beta < \beta_c$. Points 1–3 are deduced from previous percolation results by utilizing the Fortuin–Kasteleyn representation, which also yields other results of independent interest relating Potts models with different values of q.

KEY WORDS: $1/r^2$ interactions; one dimension; Fortuin–Kasteleyn representation; Ising model; Potts models; percolation; discontinuous transition; Thouless effect.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

The primary subject of this paper is a set of new rigorous results about phase transitions in one-dimensional Ising and Potts models with long-range interactions. In addition, we demonstrate the utility of some general comparison principles of Fortuin[22] in relating the phase structures of Ising, Potts, and percolation models. Since these two topics are of interest

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for separate reasons, our presentation in this introduction is split into two parts.

The results addressing specific properties of long-range models, which include the resolution of some much pondered issues, are based on the analogous results derived in Ref. 7 for a class of percolation models. The extension is enabled by the Fortuin–Kasteleyn representation[23,36] of Ising and Potts systems as dependent percolation models. Our main conclusions and some history of the subject are given in the first part of this introduction. The FK representation (discussed at length in Sections 2 and 4), the Fortuin[22] comparison inequalities, and our related results and applications are presented in Section 1.2.

1.1. Long-Range One-Dimensional Models

1.1.1. Ising Ferromagnets and Potts Models. It is generally recognized[40] that, in the context of equilibrium statistical mechanics, one-dimensional systems with rapidly decaying interactions are incapable of exhibiting long-range order at positive temperatures. A somewhat less trivial fact is that long-range order is nevertheless possible even in one dimension if the interactions decay sufficiently slowly. The dividing line between "rapidly decaying" and "long-range" interactions, as drawn by work starting with Refs. 19, 45, 20, 54, and 10, is in essence that of $1/|x-y|^2$ interactions.

The majority of the early work on long-range interactions focused on Ising ferromagnets: here, each site x of the lattice $\mathbb{Z}$ is assigned a spin variable σ_x, which can take on the values ± 1. The interaction between the spins is described by the Hamiltonian

$$\mathscr{H} = -\tfrac{1}{2} \sum_{\{x, y\}} J_{x,y}(\sigma_x \sigma_y - 1) \tag{1.1}$$

in which the $\{J_{x,y}\}$ are nonnegative (ferromagnetic), and are often taken to be translation-invariant, in which case we denote $J_{x,y} = J_{x-y}$. [The sum over $\{x, y\}$ counts each pair only once.] The order parameter for such models is the spontaneous magnetization, which can be expressed in two equivalent ways:

$$\begin{aligned} M(\beta) &= \beta^{-1} \frac{\partial f(\beta, h=0+)}{\partial h} \\ &= \langle \sigma_0 \rangle_+(\beta) \end{aligned} \tag{1.2}$$

Here $\langle \cdots \rangle_+(\beta)$ denotes expectation in the infinite-volume Gibbs state at inverse temperature β, with plus boundary conditions, and $f(\beta, h)$ is the free energy per site for such a system with $\mathscr{H}$ modified by the addition of the term $-h\sum\sigma_x$. More precisely, the plus state is obtained as a limit of finite-volume distributions (proportional to $e^{-\beta\mathscr{H}}$) with σ_x set equal to $+1$ on all the spins outside the finite region.

Among the results presented here are improved conditions for the non-vanishing of the order parameter in one-dimensional models, as well as a proof of the discontinuity of the order parameter in the special, borderline case of $1/|x-y|^2$ interactions. We find it beneficial for the study of these issues to consider them within a larger class of models.

Potts spin systems are a generalization of Ising models where each spin variable can take on one of q distinct values. The basic feature of the interaction is that the energy between any fixed pair of (interacting) spins depends only on whether or not the spin values agree. When the interaction always favors agreement, the model is said to be ferromagnetic.

Two convenient representations for the spin variables are as taking values in the set $\{1,..., q\}$, or as unit vectors allowed to point only to the vertices of a fixed $(q-1)$-dimensional "tetrahedron." For the latter representation, we will use traditional vector notation $\boldsymbol{\sigma}_x \in \mathbb{R}^{q-1}$ to describe the state of a spin. The dot product of any two such vectors assumes only two values and satisfies

$$\boldsymbol{\sigma}_x\cdot\boldsymbol{\sigma}_y=(q\delta_{\sigma_x,\sigma_y}-1)/(q-1)$$

A Potts model is described therefore by a Hamiltonian

$$\mathscr{H}=-\sum_{\{x,y\}}J_{x,y}(\delta_{\sigma_x,\sigma_y}-1)=-\sum_{\{x,y\}}\mathscr{J}_{x,y}(\boldsymbol{\sigma}_x\cdot\boldsymbol{\sigma}_y-1) \tag{1.3}$$

with $\mathscr{J}_{xy}=[(q-1)/q]\,J_{x,y}$. The case $q=2$ coincides with the previously defined Ising model.

The order parameter for a general Potts model is

$$\begin{aligned} M_q(\beta) &= \beta^{-1}\frac{\partial f(\beta, h=0+)}{\partial h} \\ &= \langle\hat{\mathbf{e}}_1\cdot\boldsymbol{\sigma}_0\rangle_1(\beta)=\frac{q}{q-1}\left\langle\left(\delta_{\sigma_0,1}-\frac{1}{q}\right)\right\rangle_1(\beta) \end{aligned} \tag{1.4}$$

The free energy $f(\beta, h)$ is defined here by adding to the Hamiltonian the term $-h\sum_x\hat{\mathbf{e}}_1\cdot\boldsymbol{\sigma}_x$, in which $\hat{\mathbf{e}}_1$ is a unit vector in the direction of a fixed vertex of the tetrahedron. The symbol $\langle\cdots\rangle_1(\beta)$ represents the thermal average in the infinite-volume Gibbs measure constructed with "1 boundary conditions." The equality between the thermodynamic definition and the other two expressions is discussed in Section 2 (Theorem 2.4) and in the Appendix (Theorem A.1).

We shall generally assume here that even though the $J_{x,y}$ are long-range, the quantity $|J| = \sup_x \sum_y J_{x,y}$ is finite [since otherwise in the translation-invariant case $M_q(\beta) \equiv 1$ for all $\beta > 0$]. By a general result, $M = 0$ if $\beta\,|J| < 1$. Thus, the critical point β_c, defined as the largest β below which M vanishes, is strictly positive. Whether or not β_c is finite (i.e., whether or not there is a phase transition at nonzero temperature) depends on the asymptotic behavior of $J_{x,y}$, since in one dimension, finite-range interactions cannot produce symmetry-breaking.

1.1.2. Main Results for Long-Range Models. Following are the three main results of this paper for one-dimensional, long-range models. We assume translation invariance and also (for simplicity) that the following limit exists:

$$J^+ = \lim_{x \to \infty} x^2 J_x \tag{1.5}$$

(it may be 0 or $+\infty$). To extend some of the results beyond this assumption, use may be made of the monotonicity of M_q in β and in the J_x, which is well known for $q = 2$ and follows for all $q \geqslant 1$ from the FK representation presented in Section 2.

a. Presence of Long-Range Order. If $\beta J^+ > q$, then $M_q > 0$ is achieved by making any single J_x sufficiently large with β and all the other couplings held fixed.

b. Absence of Long-Range Order. If $\beta J^+ \leqslant 1$ (which depends only on the asymptotic behavior of βJ_x), then $M_q = 0$, regardless of individual J_x values.[4] In particular, if $x^2 J_x \to 0$, then $M_q(\beta) = 0$ for all $\beta < \infty$.

c. Discontinuity of Magnetization. If $0 < J^+ < \infty$, then at some β_c in $(1/J^+, \infty)$, $M_q(\beta)$ jumps from zero to at least $(J^+ \beta_c)^{-1/2}$.

The above discontinuity can be summarized by the dichotomy

$$\text{either} \quad M_q = 0 \qquad \text{or} \quad \beta J^+ M_q^2 \geqslant 1 \tag{1.6}$$

For a succinct statement of a and b, let β^* be defined as

$$\beta^*(q) = \inf\{\beta J^+ \mid M_q > 0\}$$

where the inf is over all choices of $\{\beta J_x\}$. Then

$$1 \leqslant \beta^*(q) \leqslant q \tag{1.7}$$

[4] Berbee[11] recently obtained a result of this sort for Ising models by an independent argument which shows that $M_2(\beta) = 0$ whenever $\beta J^+ < 1/2$.

It may be interesting to note that (1.6) can be viewed as a renormalized version of the lower bound in (1.7). Tracing back the proof, one finds (as explained in Ref. 7) that a natural way of thinking about it is to first understand why $\beta J^+ < 1$ implies $M = 0$, and then to notice that in a natural sense βJ^+ may be replaced in that argument by its renormalized value $\beta J^+ M^2$. A similar renormalization (carried out, however, in a very different way) first appeared in the work of Anderson, Yuval, and Hamann [AYH].[9,10] The dichotomy (1.6) also reminds one of a dichotomy proposed by Thouless, who was the first to argue for the existence of a discontinuity. We discuss the relation of these dichotomies below.

A fourth result, established only for the case of Ising (and the analogous independent percolation) models, is that for $0 < J^+ < \infty$, the two-point correlation function $\langle \sigma_0 \sigma_x \rangle(\beta)$ decays as $c(\beta)/x^2$ for any $\beta < \beta_c$. The constant $c(\beta)$ must diverge as $\beta \nearrow \beta_c$, since for any Ising ferromagnet the susceptibility $\chi(\beta) \equiv \sum_x \langle \sigma_0 \sigma_x \rangle(\beta)$ diverges as $\beta \nearrow \beta_c$.[50,6] In view of this, the phase transition is not really of first order despite the discontinuity in the order parameter.

1.1.3. Relations to Previous Work. In the area of long-range one-dimensional systems, the literature on Potts and on percolation models is of fairly recent origin.[15,49] On other hand, there is a rich history concerning one-dimensional gases and Ising models. Much of it focuses on two related issues: (1) sharp conditions for the existence of long-range order in one-dimensional models, and (2) the nature of the phase transition in the borderline models, with $J_x \approx 1/x^2$. These questions were of interest for a number of different reasons. For the general theory of statistical mechanics they are of obvious interest, in particular since the $1/x^2$ one-dimensional models offer examples exhibiting rather unusual behavior. Furthermore, such systems arise in the study of time evolution phenomena, such as the Kondo problem. Finally, the study of long-range models has involved a number of important methods (such as energy–entropy arguments, renormalization methods, and multiscale contour analysis) whose applicability extends to a variety of problems.

The criticality of the falloff rate $J_x \approx 1/x^2$ follows from the early results of Dobrushin,[19] Ruelle[45] and Dyson.[20] The former works established that there is no long-range order if $J_x \approx 1/x^s$ with $s > 2$, and the latter showed (using hierarchical models) that for any $s < 2$, long-range order is possible, and occurs at sufficiently high values of β. Thus $s = 2$ is the critical power dividing short- and long-range forces in one dimension. Somewhat sharper results were in fact obtained. Dyson's result is that if eventually $J_x \geqslant x^{-2} \log \log x$, then there is spontaneous magnetization at large enough values of β. Later results of Rogers and Thompson[47] amount to the statement that if for every ε eventually $J_x \leqslant \varepsilon x^{-2} \log^{-1/2} x$, then there is no long-range order at any temperature.

Interest in the case $s=2$ was greatly stimulated by the discovery of Anderson and Yuval[8] and Hamann[30] of a close connection between such Ising models and the Kondo effect in metals.[5] Furthermore, attention was drawn to this case by an argument of Thouless[54] that for Ising models with $J_x \approx 1/x^2$, the order parameter is discontinuous at the critical point (the Thouless effect). Thouless studied the $1/x^2$ case by means of a beautiful energy–entropy analysis rooted in a classic argument of Landau and Lifshitz.[40] While it is now understood that, for reasons mentioned below, Thouless' argument does not really imply a discontinuity, the notion that there is one received strong support from the renormalization group analysis of AYH[9,10] and from Dyson's[20] proof that there is a discontinuity in a related hierarchical model.

The energy–entropy argument made by Thouless (see also Ref. 51) led to the dichotomy that for the $1/x^2$ Ising model, at any β either $M(\beta)=0$ or else

$$\beta J^+ M^2(\beta) \geqslant \alpha \tag{1.8}$$

where $\alpha=\alpha(\beta)$ is determined by the behavior of the truncated two-point function $\langle \sigma_x, \sigma_y \rangle_+^T = \langle \sigma_x \sigma_y \rangle_+ - M^2$ for $\beta \geqslant \beta_c$, according to

$$\frac{1}{L^2} \sum_{|x|,|y| \leqslant L} \langle \sigma_x, \sigma_y \rangle_+^T \sim L^{-2\alpha} \qquad \text{as} \quad L \to \infty \tag{1.9}$$

Assuming that

$$\langle \sigma_x, \sigma_y \rangle_+^T \sim 1/|x-y|^{\vartheta} \qquad \text{as} \quad |x-y| \to \infty \tag{1.10}$$

(ϑ would be denoted by $-1+\eta$ in "conventional" critical exponent notation), one has for the quantity in (1.8)

$$\alpha = \tfrac{1}{2} \min\{\vartheta, 1\} \tag{1.11}$$

Thouless concluded from (1.8), under the two assumptions that $\beta_c < \infty$ and that $\vartheta(\beta)$ does not tend to zero as $\beta \searrow \beta_c$, that $M(\beta)$ must have a discontinuity at β_c. Note that even had the assumption on ϑ been satisfied in the strongest way, α would be $\frac{1}{2}$, with which (1.8) falls short of (1.6).

Surprisingly, some further renormalization group analysis (which came much later and was based on Ref. 10) and numerical work suggested that $\vartheta(\beta)$ actually tends to *zero* as $\beta \searrow \beta_c$.[13] Moreover, some rigorous

[5] More recently, an analogy has been drawn between these problems and macroscopic quantum tunneling.[16,18]

results now show[55] that in the class of $1/x^2$ models the quantity $\vartheta(\beta_c+0)$ and hence also α take arbitrarily small values. It seems, therefore, on the basis of what is now known, that the mechanism considered by Thouless does not account for the discontinuity phenomenon which he first predicted. Nevertheless, he was quite right in that there is a discontinuity, and that it can be derived by means of a dichotomy.

An apparently correct explanation of the phenomena discussed above can be found in the renormalization group analysis of AYH[9,10] which led to a discontinuity without any assumption about ϑ. On the rigorous side, the mere fact that in the critical, and hence delicate, case $s=2$ there is a phase transition was finally given a rigorous proof by Fröhlich and Spencer.[25] While that work proved that $\beta^* < \infty$, it did not present any sensible upper bound on β^*, for which one should distinguish between the roles of the short-range and the long-range couplings J_x, as emphasized in Refs. 9 and 10. Later, studying long-range independent bond percolation models, Newman and Schulman[44] proved by an independent argument (which involves rigorous renormalization methods) the existence of percolation in $1/x^2$ systems. The results of Ref. 44 yield $\beta^* \leqslant 1$ for independent percolation, which corresponds to the $q \to 1$ limit of Potts models. That $q=1$ result is in fact optimal, since the results of Aizenman and Newman[7] include the opposite inequality $\beta^* \geqslant 1$. In fact, the comparison inequalities discussed in Section 1.2, which show that β^* is a nondecreasing function of q (for $q \geqslant 1$), while β^*/q is nonincreasing, establish links between the Ising and the percolation problems studied in Refs. 25 and 44.

The results of Aizenman and Newman[7] form a percolation version of the dichotomy (1.6). That work concerned independent as well as certain dependent percolations, and in part was based on a rigorous renormalization-type argument. While the argument is quite different from that of AYH[9,10] (concerning the Ising model), the structure it presents resembles the "flow diagram" of Ref. 10. In this paper we show how to make the results of Ref. 7 applicable to Ising (and more generally Potts) models.

It may also be mentioned here that by the use of rigorous low-temperature expansions, Imbrie[34] proved that $\vartheta=2$ for Ising models with very large β. Evidently, the work of Ref. 13 mentioned above concerning the behavior of ϑ as $\beta \searrow \beta_c$ refers to a truly intermediate phase.

Let us end this part of the introduction by mentioning the following open problem. Show that for all the systems of the kind considered here, with interactions decaying asymptotically as $1/|x-y|^2$, the inequality $\beta M^2 \geqslant 1$ in (1.6) is actually saturated at β_c. Such a result, which is suggested by the methods of Refs. 7, 10, and 44, would have important applications in the study of the intermediate phase mentioned above.

1.2. Relations among the Order Parameters in Different Models

1.2.1. Random Cluster Models. The percolation results mentioned above apply to systems of which the simplest is the independent bond percolation model, in which the bonds (corresponding to all the pairs $b=\{x, y\}$) are independently occupied with probabilities

$$K_{x-y}=1-\exp(-\beta J_{x-y})\approx \beta J_{x-y} \tag{1.12}$$

The occupied bonds are regarded as connecting, and the order parameter $M(\{\beta J\})$ is the percolation probability, which is often denoted by P_∞ ($\{\beta J\}$ denotes here the set of all the couplings).

However, the results of Ref. 7 were formulated so that they apply also to a class of dependent percolation models. It turns out that this class includes the random cluster models, with parameter values $q \geqslant 1$, which describe the statistics of a diagrammatic expansion of the q-state Potts models (for which q is integer). As Fortuin and Kasteleyn[23,36] discovered, the random cluster models allow a natural interpolation to noninteger q, with the limit $q \to 1$ corresponding to independent percolation. For our discussion of Potts models it is of central importance to identify the geometric content of their physical quantities. That is presented in Section 2. Here we just introduce the direct definition of the random cluster models.

A general bond percolation model on the one-dimensional lattice $\mathbb{Z}$ is described in terms of bond occupation variables $n=\{n_b\}$, which take the value 1, meaning the bond $b=\{x, y\}$ is occupied, or 0, meaning that the bond is vacant. The model is defined by a probability distribution on the occupation configurations of all bonds, which we will typically assume to be translation-invariant. In independent percolation the occupation variables are mutually independent, and the probability of a configuration n for any finite collection of bonds $\tilde{\Lambda}$ is the product

$$\mathscr{B}_\beta(n)=\prod_{\{x, y\}:n_{x,y}=1}(1-e^{-\beta J_{x,y}})\prod_{\{x, y\}:n_{x,y}=0} e^{-\beta J_{x,y}} \tag{1.13}$$

Note that in places where the translation invariance plays no role, we refer to the couplings as $\beta J_{x,y}$ instead of βJ_{x-y}. In the random cluster models (where the bond variables are dependent), we construct the infinite-volume measure by suitable limits of ensembles defined with a finite length L. Putting aside the question of boundary conditions, the finite-volume probability measures of the random cluster model parametrized by $q>0$ are defined by the weights[6]

$$\mathscr{W}_{q,\beta}(n)=\mathscr{B}_\beta(n)\, q^{c(n)}/Z_{q,\beta}(\tilde{\Lambda}) \tag{1.14}$$

[6] The case $q=0$ can also be defined (by a limiting procedure) and is of interest, but it will not be discussed in this work.

where $c(n)$ is the total number of connected clusters in $[-L, L]$ with respect to the occupied bonds of n, and $Z_{q,\beta}(\tilde{\Lambda})$ is the normalization constant, which makes $\mathscr{W}$ a probability distribution.

One natural construction of the infinite-volume measure is by starting from the measure (1.14) for the variables associated with the set of bonds $\tilde{\Lambda}_L^{\mathrm{f}} = \{b = \{x, y\} \mid |x|, |y| \leqslant L\}$ and letting $L \to \infty$. As we shall see, that limit (yielding a probability measure on the set of bonds of the infinite lattice) exists, and describes—when properly interpreted—the "free boundary condition" state of the q-state Potts model. However, the order parameter M is detected better in the state constructed with the nonsymmetric plus, or $\sigma = 1$, boundary conditions. The corresponding construction for the random cluster model is to take

$$\tilde{\Lambda}_L^{\mathrm{w}} = \{b = \{x, y\} \mid |x| \leqslant L, y \in \mathbb{Z}\}$$

and regard all the sites of $[-L, L]$ that are connected by the occupied bonds of $\tilde{\Lambda}_L^{\mathrm{w}}$ to the complement $[-L, L]^c$ as belonging to the same cluster, or equivalently treat all the bonds with both ends in $[-L, L]^c$ as occupied. With the above interpretation of $c(n)$, the measures defined by (1.14) also converge (Section 2), and yield what we call the "wired state." With $* = \mathrm{w}$ (wired) or f (free), we denote by $\mu_{q,\beta}^*(dn)$ or $\mathrm{Prob}_{q,\beta}^*(\cdots)$ these two states of the random cluster models. (These models have other interesting boundary conditions, involving interfaces, which we shall not discuss here.)

The close relation between the Potts and the random cluster models, which we discuss more completely in Section 2, leads to the following expression for the order parameter defined for Potts models by (1.2) and (1.4):

$$M_q(\{\beta J\}) = \mathrm{Prob}_{q,\beta}^{\mathrm{w}}(\text{the origin belongs to an infinite cluster}) \tag{1.15}$$

The relation (1.15) allows us now, following Refs. 23 and 36, to extend the definition of the order parameter to all real values of $q \geqslant 1$.[7]

1.2.2. Comparison Inequalities. As a preliminary fact, let us point out that the random cluster models with $q \geqslant 1$ (and $J \geqslant 0$) satisfy the FKG inequality (Ref. 22, Lemma 2) (see Section 2). Hence, some of the most basic, and quite useful, inequalities of Ising systems extend to Potts and random cluster models. In particular:

(o) For each $q \geqslant 1$ the order parameter $M_q(\{\beta J\})$ is monotone-increasing (by which we always mean nondecreasing) in β and in the $J_{x,y}$.

More striking are comparison inequalities relating models with different values of q. The basic technical results are contained in the original

[7] The restriction to $q \geqslant 1$ is for technical convenience (see Theorem 2.2).

paper of Fortuin (Ref. 22, Lemma 3). What we want to emphasize here is that when these general inequalities are applied to the order parameter and to other physically natural quantities, they yield a very versatile tool. (In particular, they can be used for a simple derivation of some of the other results of Ref. 22 and of related works.) The basic statements concern the monotonicity of $M_q(\{\beta J\})$ as q and β vary, with $q \geqslant 1$. To state them simply, at any fixed $\{J_{x,y}\}$:

$$\text{(i)} \quad M_q(\{\beta J\}) \text{ decreases as } q \text{ increases with } \beta \text{ fixed (or decreasing)} \tag{1.16}$$

and

$$\text{(ii)} \quad M_q(\{\beta J\}) \text{ increases as } q \text{ increases with fixed or increasing } \beta/q \tag{1.17}$$

A slightly stronger, though more cumbersome, version of (ii) is

(ii′) If $q' \geqslant q$ ($\geqslant 1$), but for each $\{x, y\}$

$$[\exp(\beta' J'_{x,y}) - 1]/q' \geqslant [\exp(\beta J_{x,y}) - 1]/q \tag{1.18}$$

then

$$M_q(\{\beta J\}) \leqslant M_{q'}(\{\beta' J'\})$$

Some comparisons with $q < 1$ are also attainable. The above relations actually extend beyond the order parameter, to expectation values of all "monotone" functions of the bond occupation variables; see Section 4.

1.2.3. Applications of the Comparison Inequalities. Let $\beta_c(q)$ denote the critical point where $M_q(\{\beta J\})$ first becomes positive in a random cluster model with fixed $J_{x,y}$. As immediate consequences of the above inequalities,

$$\beta_c(q') \geqslant \beta_c(q) \geqslant \frac{q}{q'} \beta_c(q'), \qquad \text{for all} \quad q' \geqslant q \geqslant 1 \tag{1.19}$$

A striking consequence is that if there is a phase transition for *some* value q_0 in $[1, \infty)$, i.e., if $0 < \beta_c(q_0) < \infty$, then there is a phase transition for *every* $q \geqslant 1$! In particular, an *independent* percolation transition implies an Ising transition, and vice versa.

In the context of $1/x^2$ models, the first rigorous proof of the existence of a phase transition was that of Fröhlich and Spencer[25] for Ising models. By another method, such a result was subsequently derived for independent percolation by Newman and Schulman,[44] whose upper bound $\beta^*(1) \leqslant 1$ [for the quantity $\beta^*(q)$ discussed in Sections 1.1.2 and 1.1.3] turned out to be optimal.[7] We now see, by (1.19), that these problems are related. In one direction, the mere existence of a percolation transition can be deduced from the results of Ref. 25, and in the other the results of Ref. 44 yield the much improved bound $\beta^*(2) \leqslant 2$. Similarly, the result of Aizenman and Newman[7] that $\beta^*(1) \geqslant 1$ now easily extends to all $q \geqslant 1$. Thus, we get here $1 \leqslant \beta^*(q) \leqslant q$. The gap that is left is caused by the structure of (1.19). For completeness let us remark that this gap is closed in a forthcoming paper of Imbrie and Newman,[55] and that $\beta^*(q) = 1$ for all real $q \geqslant 1$. That improvement is quite relevant for the study of the intermediate phase of Refs. 13, 55.

The comparison inequalities between different q's have a number of other applications beyond the context of one-dimensional models. Two of these are discussed in Section 4. One concerns dilute (in the "bond" sense) Ising, or Potts, ferromagnets. Here we find it convenient to compare the diluted $q > 1$ system to the corresponding diluted $q = 1$ model, since the effect of dilution on independent percolation is to reproduce an ordinary percolation model with some new occupation density. This line of reasoning leads to a technology for handling dilute ferromagnets which is simultaneously rigorous, intuitive, and simple. More details may be found in Ref. 5.

The other application given in Section 4 concerns the phase transitions in logarithmic wedges of $\mathbb{Z}^d$ (which for $d = 2$ have the shape of a "stingy slice of pie"). Using the comparison inequalities, we extend results obtained previously concerning the existence of an *intermediate phase* for percolation and Ising systems in wedges, to Potts wedges with any $q \geqslant 1$.

1.3. Organization of the Paper

The remainder of this paper is organized as follows. Section 2 contains the FK representation and some general properties of the finite- and infinite-volume states of Potts and random cluster models, with some of the proofs relegated to the Appendix. In Section 3 we use this representation for a derivation of the dichotomy (1.6) from the percolation results of Ref. 7. We include there also a discussion of the correlation function decay for long-range Ising models with $\beta < \beta_c$. Section 4 is devoted to the comparison principles and the applications mentioned above.

2. FORTUIN–KASTELEYN REPRESENTATION OF POTTS SPIN SYSTEMS

In this section, we review the geometric representation of the order parameter and the correlation functions of Potts models by means of the random cluster representation due to Fortuin and Kasteleyn.[23,36] We also derive the FKG property of the associated equilibrium states and use it to establish some preliminary results on the states obtained in the infinite-volume limit. Much of this already appears in Ref. 22. The results of this section apply in a quite general setting.

2.1. Derivation of the Random Cluster Representation

We start with a finite $\Lambda \subset \mathbb{Z}^d$ and consider the q-state Hamiltonian $\mathscr{H}$ in the first of the two forms in (1.3). The partition function for such a system with free boundary conditions is given by

$$\mathscr{Z}_\Lambda^{\mathrm{f}} = \mathrm{Tr}_\Lambda e^{-\beta \mathscr{H}(\sigma_\Lambda)} \tag{2.1}$$

where σ_Λ is notation for a generic configuration of spins in Λ and Tr means sum over all such configurations.

Denoting by $\bar{\Lambda}^{\mathrm{f}} = \{\{x, y\} \mid x, y \in \Lambda\}$ the set of all the "bonds" in Λ, which is consistent with the notation used for the random cluster models in Section 1.2.1, we have

$$\exp[-\beta \mathscr{H}(\sigma_\Lambda)] = \prod_{\{x, y\} \in \bar{\Lambda}^{\mathrm{f}}} \exp[\beta J_{x,y}(\delta_{\sigma_x, \sigma_y} - 1)] \tag{2.2}$$

The basic idea is now to expand such products, wherever they occur, by means of the identity

$$\exp[\beta J_{x,y}(\delta_{\sigma_x, \sigma_y} - 1)] = (1 - \lambda_{x,y}) + \lambda_{x,y} \delta_{\sigma_x, \sigma_y} \tag{2.3}$$

with $\lambda_{x,y} \equiv 1 - \exp(-\beta J_{x,y})$. The terms in the expansion of the product (2.2) are in one-to-one correspondence with a bond function $n: \bar{\Lambda}^{\mathrm{f}} \to \{0, 1\}$. Explicitly, for each term in the expansion we set $n_b = 1$ on those bonds $b = \{x, y\}$ for which the corresponding term in the product is $\lambda_{x,y} \delta_{\sigma_x, \sigma_y}$, and $n_b = 0$ if the corresponding term is $(1 - \lambda_{x,y})$. Of course, it is worthwhile to think of the configurations geometrically. Thus, we will refer to n as a bond configuration, the bonds being "occupied," meaning "connecting," if $n_b = 1$, and "vacant" if $n_b = 0$.

Writing (2.2) in the expanded form, we have

$$\mathscr{Z}_\Lambda^{\mathrm{f}} = \sum_{n = \{n_b\}} \prod_{b \in \bar{\Lambda}^{\mathrm{f}}: n_b = 1} \lambda_b \prod_{b \in \bar{\Lambda}^{\mathrm{f}}: n_b = 0} (1 - \lambda_b) \left(\mathrm{Tr} \prod_{b: n_b = 1} \delta_{\sigma_x, \sigma_y} \right) \tag{2.4}$$

We will now evaluate the trace, which depends on the configuration n. For each configuration, the set Λ divides into connected components (or clusters). In the corresponding trace in (2.4), the delta functions require that the spins take a constant value within each such component (with respect to n); otherwise, the trace is unconstrained. Hence, we pick up a factor of q for each connected component of n (where we regard also a single disconnected site as a cluster). Denoting by $c(n)$ their number, we have

$$\mathscr{Z}^{\mathrm{f}}_{\Lambda} = \sum_{n} \mathscr{B}_{\beta}(n)\, q^{c(n)} \tag{2.5}$$

where $\mathscr{B}_{\beta}(n) \equiv \mathscr{B}_{\{\beta J\}}(n)$ is the weight we encountered in (1.13), giving there the probability of n in a Bernoulli (independent) percolation model with the bond occupation probabilities

$$K_{x,y} = \lambda_{x,y} \quad [=1-\exp(-\beta J_{x,y})]$$

A quick comparison of (2.5) with (1.14) shows that the partition function $\mathscr{Z}^{\mathrm{f}}_{\Lambda}$ is exactly the normalization constant (or the generating function) $Z_{q,\beta}(\tilde{\Lambda}^{\mathrm{f}})$ of the random cluster model. To complete the correspondence, we shall discuss the relation of the states (in particular, the order parameters and the correlation functions) and other than free boundary conditions. Let us start with the former.

The equilibrium (Gibbs) state of a finite system in Λ with free boundary conditions is described by a probability measure on the space of spin configurations for which the expectation values of observables are given by

$$\langle g(\sigma) \rangle^{\Lambda}_{\mathrm{f}} = \mathrm{Tr}_{\Lambda}\, e^{-\beta \mathscr{H}(\sigma_{\Lambda})} g(\sigma_{\Lambda}) / \mathscr{Z}^{\mathrm{f}}_{\Lambda} \tag{2.6}$$

By substituting (2.3) in (2.6) and expanding as before, one obtains the following expression for the state $\langle \cdots \rangle^{\Lambda}_{\mathrm{f}}$:

$$\langle g(\sigma) \rangle^{\Lambda}_{\mathrm{f}} = \sum_{n} \mathscr{W}_{q,\beta}(n)\, E^{\mathrm{f}}_{n}(g(\sigma)) \tag{2.7}$$

Here, for each n, $E^{\mathrm{f}}_{n}(\cdot)$ is a very simple average over the spins σ, the spins being just constrained to be constant on each connected cluster with the values for different clusters being independent symmetric variables, and $\mathscr{W}_{q,\beta}(n)$ is the probability distribution for the bond configuration n given by the random cluster formula (1.14).

For other boundary conditions—say, a fixed spin configuration, $\sigma_{\Lambda^c} \equiv \{\eta_y\}$ on $\Lambda^c \equiv \mathbb{Z}^d \setminus \Lambda$, with the couplings between Λ and Λ^c determined by the $J_{x,y}$—similar representations may be obtained. Indeed, it is seen that only a few adjustments have to be made: First, there will be additional bonds between Λ and Λ^c, corresponding to the interaction with the fixed spins in Λ^c. Hence, the relevant set of bonds is not $\tilde{\Lambda}^{\mathrm{f}}$, but $\tilde{\Lambda}^{+} = \{\{x, y\} \mid$

$x \in \Lambda$, $y \in \mathbb{Z}^d\}$. A new feature is that, unless the boundary spins are all in agreement, some simple consistency constraints will be placed on the collection of allowable bond configurations. Furthermore, for notational purpose, we find it convenient to modify the notion of what constitutes "connected components."

Following *mutatis mutandis* the discussion of the free boundary condition, one finds that the partition functions $\mathscr{Z}^{\eta}_{\Lambda}$ and the states $\langle \cdots \rangle^{\Lambda}_{\eta}$ are still given by Eqs. (2.5) and (2.7), with, however, *the following modifications*:

1. $n = \{n_b\}$ is defined on the bonds of $\tilde{\Lambda}^+$ (rather than $\tilde{\Lambda}^{\mathrm{f}}$). However, only those configurations n contribute that do not include an occupied path connecting two sites in Λ^c with different values of η.

2. We modify the definition of $c(n)$ to count only those distinct connected clusters that are not connected to any boundary site (where the spins are fixed). This definition applies in the formulas (1.14) and (2.5) for the weights $\mathscr{W}_{q,\beta}(n)$, which are still normalized to form a probability distribution and for $\mathscr{Z}^{\eta}_{\Lambda}$ [which again equals the normalizing factor in (1.14)].

3. In the expectations $E^{\eta}_{n}(\cdot)$, which enter in the extended version of (2.7), spins in clusters connected to the boundary assume only the value of the boundary site(s) to which they are connected (the above constraint on n assures that no contradiction is created), while the other spins are distributed as in (2.7), i.e., subject only to the constraints to be constant on each connected component.

For the simplicity of some expressions given below, let us add the following new convention:

4. In defining the connected clusters, we regard all the boundary sites (in Λ^c) with the *same value of* η, as well as those sites of Λ connected to them, as connected. This can be alternatively stated by saying that we treat all the bonds having both ends in Λ^c and the same value of η as occupied.

The most relevant for this work are the $\hat{\mathbf{e}}_1$ boundary conditions appearing in (1.4), which generalize the plus boundary conditions of the Ising model. In that case, condition 1 is satisfied for all n, and the modifications introduced in 3 and 4 are also particularly simple. The corresponding probability measure $\mathscr{W}(n)$ is exactly the wired state $\mu^{\mathrm{w}}_{q,\beta}(dn_{\Lambda})$ of the random cluster model discussed in Section 1.2.1.

Of particular interest are the following applications of the formula(s) (2.7).

Lemma 2.1. (a) For any Potts model in a finite volume Λ, the magnetization with $\hat{\mathbf{e}}_1$ boundary condition [see (1.4)] satisfies

$$\langle \hat{\mathbf{e}}_1 \cdot \boldsymbol{\sigma}_x \rangle_1^A \equiv \frac{1}{q-1} \langle (q\delta_{\sigma_x,1} - 1) \rangle_1^A$$
$$= \mu_A^{\mathrm{w}}(x \leftrightarrow A^c) \equiv M_q^A(\{\beta J\}) \tag{2.8}$$

where $x \leftrightarrow A$ means, for x a site and A either a site or a set of sites, that x is connected to A (in the appropriate sense as discussed above) by a path of occupied bonds, and $\mu_A^{\mathrm{w}}(\cdot)$ is the probability measure of the wired state with the given values of q and $\{\beta J\}$.

(b) The two-point function with free boundary conditions satisfies

$$\langle \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_y \rangle_f^A \equiv \left\langle \frac{1}{q-1} (q\delta_{\sigma_x,\sigma_y} - 1) \right\rangle_{\mathrm{f}}^A$$
$$= \mu_A^{\mathrm{f}}(x \leftrightarrow y) \equiv \tau_A^{\mathrm{f}}(x, y) \tag{2.9}$$

with a similar formula holding for the $\hat{\mathbf{e}}_1$ boundary condition, which corresponds to the wired state $\mu_A^{\mathrm{w}}(\cdot)$ (with the corresponding interpretation of $x \leftrightarrow y$).

Since the proof is by a completely elementary application of (2.7), let us skip it here, except for commenting on the peculiar function $[1/(q-1)](q\delta_{\sigma_x,\sigma_y} - 1)$ appearing in the second form of each of the above quantities. The expectation value of that function is a natural measure of the bias of the spin σ_x toward the value σ_y. When the conditional distribution of σ_x is totally independent of σ_y the expectation is 0, and when they are perfectly correlated the (conditional) expectation is 1. A remarkable feature of the formula (2.7) is that it represents the physical correlations by means of averages over these two "totally polarized" cases.

Following are some additional comments on the measures induced on the bond configurations n by the states of the Potts model.

Remarks. 1. Unlike the free or the fixed $\hat{\mathbf{e}}_1$ (or $\hat{\mathbf{e}}_k$) boundary conditions, the general *fixed spin* boundary conditions introduce somewhat disruptive constraints, requiring the presence of interfaces. Such general boundary conditions are discussed in the Appendix.

2. Although the random cluster measures may appear somewhat forbidding due to the nonlocal nature of the q-dependent factor, these measures are nevertheless quite manageable. In the free and the wired states, for each bond b the conditional probability of the event $n_b = 1$, conditioned on the occupation status of all the other bonds, is covered by only two alternatives:

$$\mathrm{Prob}^*(n_b = 1 \mid \{n_{b'}\}_{b' \neq b}) = \begin{cases} \lambda_b & \text{if the ends of } b \text{ are in the same *cluster regardless of } n_b \\ \dfrac{\lambda_b}{\lambda_b + q(1-\lambda_b)} & \text{otherwise} \end{cases} \tag{2.10}$$

where $* = \mathrm{f}$ (free b.c.) or w (wired b.c.). It may be noted that for an infinite system the right side of (2.10) is not an everywhere continuous function of $\{n\}$. On the other hand, that function has certain monotonicity properties, about which we say more in the next subsection.

3. An interesting application of ideas related to the FK representation has recently been made by Swendsen and Wang,[53] who present a fast Monte Carlo algorithm, which proceeds by alternate updates of spin and bond variables. A point of view emphasized by Sokal[52] is that there exists a natural joint probability distribution of bond and spin variables [whose weights are implicit in (2.4)], with the following properties:

(a) The marginal distribution of the spin variables is the Potts model (2.2).
(b) The marginal distribution of the bond occupation variables is the random cluster model (1.14).
(c) The conditional distributions of each set of variables, given the other, have a particularly simple form. [One of these conditional distributions is the $E_n^{\mathrm{f}}(\cdot)$ of (2.7).]

The Swendsen–Wang algorithm consists of alternate applications of the two conditional distributions, and hence the joint distribution is a very natural framework for it. For us, however, it suffices to focus primarily on the random cluster measures, which we use as an extremely convenient tool for the study of the infinite-volume states of the Potts models. We also find them of interest as dependent percolation models (in which q becomes a continuous parameter).

2.2. Harris–FKG Inequalities

In 1960, Harris[32] proved a correlation inequality for the Bernoulli percolation problem, which was extended in Ref. 22 to the random cluster measures. The analogue of this inequality for the case of ferromagnetic Ising spin measures, and in a more general context, was the subject of Ref. 24. Such inequalities have also been studied in the context of reliability theory.[21,48] The general FKG inequality is stated in the setup of measures on a partially ordered set Ω. In the application discussed here, Ω is the set of bond configurations.

To fix the notation, we let $\tilde{\Lambda}$ be the collection of bonds of a set Λ, and Ω the collection of bond configurations, i.e., functions $n: \tilde{\Lambda} \to \{0, 1\}$. To state the FKG inequality, we first need the notion of increasing events, which is induced by the natural partial order on Ω:

Definition 1. For two configurations $n, n' \in \Omega$ the relation

$$n' \succ n \tag{2.11}$$

(n dominates n') means that for all $b \in \tilde{\Lambda}$, $n'_b \geqslant n_b$. In other words, the set of occupied bonds of n' includes that of n.

2. A function is said to be *increasing* if it is nondecreasing with respect to this partial order. An event is increasing if its indicator is an increasing function. *Decreasing* functions and events are the negatives and complements of increasing functions and events.

3. For a pair of probability measures on Ω, we say that μ *dominates* ν in the FKG sense, denoted $\mu \succcurlyeq \nu$, if for all increasing functions $f: \Omega \to \mathbb{R}$ the expectation values satisfy

$$\mu(f) \geqslant \nu(f) \tag{2.12}$$

The above condition implies, of course, a similar inequality for probabilities of increasing events, and reversed inequalities for decreasing functions and events.

4. A measure μ on Ω is said to have the *FKG property* if increasing events are positively correlated; i.e., μ is an *FKG measure* iff, for every pair of increasing (measurable) events $A, B \subset \Omega$,

$$\mu(A \cap B) \geqslant \mu(A) \cdot \mu(B) \tag{2.13a}$$

An equivalent statement to (2.13a) is that for all increasing functions f that are nonnegative with $\mu(f) < \infty$, the probability measure whose density (i.e., the Radon–Nikodym derivative) with respect to μ is $f(\cdot)/\mu(f)$ dominates μ:

$$\frac{\mu(\cdot f)}{\mu(f)} \succcurlyeq \mu(\cdot) \tag{2.13b}$$

This property is possessed by any product probability measure on Ω, as originally discovered by Harris.[32] A much more general sufficiency condition for a measure to be FKG was later established in Ref. 24 and is known as the FKG lattice condition. A closely related sufficient condition (but for measures on $\mathbb{R}^n$ with a density) was actually obtained earlier in Ref. 48. The version of Ref. 24 is as follows:

Theorem 2.1.[24] Let $\mu(dn)$ be a probability measure on Ω $(=\{0,1\}^{\tilde{\Lambda}})$ of the form $\mu(dn)=f(n)\,\rho(dn)$, where $\rho(\cdot)$ is a product (Bernoulli) measure and f is a (nonnegative) function which, for all pairs $n, n' \in \tilde{\Lambda}$, satisfies

$$f(n \vee n')\, f(n \wedge n') \geqslant f(n)\, f(n') \tag{2.14}$$

where the (lattice operations) $\vee$ and $\wedge$ are defined by

$$n \vee n'_b = \max\{n_b, n'_b\}, \qquad n \wedge n'_b = \min\{n_b, n'_b\} \tag{2.15}$$

for all $b \in \tilde{\Lambda}$. Then μ is an FKG measure (in the sense of Definition 4 above).

Remarks. 1. Measures satisfying the conclusions of the FKG inequality (2.12), but not the lattice condition, are not referred to as FKG measures in certain circles. Such situations will not be encountered in this work.

2. It is easy to see that measures that satisfy the lattice condition stated in Theorem 3.1 also enjoy the property that if we condition on the event that a prescribed set of bonds is occupied and/or that another set is vacant (i.e., a cylinder event), then the resulting conditional measure is itself an FKG measure. Such measures are sometimes referred to as *strong FKG measures*,[7] a convention to which we will adhere. We note, however, that it has recently been shown that the strong FKG property is actually equivalent to the FKG lattice condition.[12]

3. It is clear that infinite-volume or other distributional limits of FKG measures are FKG measures.

Our primary concern in these matters is the following result:

Theorem 2.2.[22] For $q \geqslant 1$, the random cluster measures defined according to the weights (1.14) are strong FKG.

Proof. Since the random cluster measures given by (1.14) have the basic structure assumed in Theorem 3.1, it suffices to show that the function

$$f(n) = q^{c(n)} = \exp[\log q \cdot c(n)] \tag{2.16}$$

satisfies the lattice condition (2.14). For $\log q \geqslant 0$, this is implied by the statement that for all $n, m, k \in \Omega$, if $m \succ k$, then

$$c(n \vee m) - c(m) \geqslant c(n \vee k) - c(k) \tag{2.17}$$

(To see the implication, choose $k = n \wedge n'$ and $m = n'$.) Using a telescopic decomposition of (2.17), it is easy to see that it suffices to prove it for the case in which the configuration n has only one occupied bond.

For the special case in which n has only the bond $b = \{x, y\}$, we get

$$\mathcal{c}(n \vee m) - \mathcal{c}(m) = \begin{cases} -1 & \text{if } x \text{ and } y \text{ are not connected in } m \\ 0 & \text{otherwise} \end{cases} \tag{2.18}$$

The right side is easily seen to be an *increasing* function of m, and hence (2.17) is satisfied. ∎

Remarks. 1. The above proof applies directly to the random cluster or Potts models with *free* (or even *periodic*) boundary conditions. However, since the lattice was arbitrary, the theorem can also be applied to the case of a *wired* boundary condition, the latter being regarded as a free boundary problem on a slightly different lattice (where Λ^c is collapsed to a point). It should, however, be noted that not all boundary conditions for q-state Potts ferromagnets on finite lattices transform into random cluster measures that enjoy the FKG property.

2. It is no accident that the proof only applies only to $q \geqslant 1$. Elementary calculations on a small lattice (e.g., an "equilateral triangle") will produce pairs of positive events for which (2.12) is violated whenever $q < 1$.

3. While the above argument was concerned with a *convexity*-type property of the function $\mathcal{c}(n)$, it is worthwhile to note the more elementary fact [seen explicitly in (2.18)] that $\mathcal{c}(n)$ is a *decreasing* function of n.

In the next subsection we use the above FKG property to establish some of the most basic properties of the Gibbs states of any ferromagnetic Potts model.

2.3. The Basic States for Potts and Random Cluster Models

We now present, or review, some basic results on the properties of both spin models and random cluster systems in the infinite-volume limit. For translation-invariant systems obeying the summability condition $|J| < \infty$, the existence of the limit for the *free energy density* $(1/|\Lambda|) \log Z$ and its independence of the boundary conditions are implied by standard arguments (see Ref. 46). Typically, the questions of existence and uniqueness of the limit for *states* are somewhat more delicate. Our main purpose here is to establish that for certain canonical boundary conditions the limits always exist, and to clarify the relations of their properties with the order parameter.

When the order parameter $M = M_q(\{\beta J\})$ does not vanish, for q integer, then the Gibbs states of the Potts spin models exhibit (at least) a q-fold symmetry-breaking, since

$$\langle \hat{\mathbf{e}}_k \cdot \boldsymbol{\sigma}_x \rangle_m = M(q\delta_{k,m} - 1)/(q-1)$$

This nonuniqueness is eliminated by the passage to the induced random cluster measure (where the long-range order parameter is manifested by percolation; see below). Can the random cluster model itself still manifest more than one phase? The answer is—yes! Such a phenomenon is observed at the critical point of nearest neighbor models in $d \geqslant 2$ dimensions with q large enough, where the Potts models have $q+1$ phases in coexistence: q ordered and 1 disordered.[38,42] The existence of a disordered phase, which is distinct from a q-fold average of ordered phases, is manifested by the existence of a difference for the random cluster model between the wired and the free boundary conditions. Hence, even for that reduced model the infinite-volume limit requires some attention.

The basic facts on the existence of the infinite-volume limits are summarized in the following statement, parts of which appear in Ref. 22. We refer here to ferromagnetic models, *without* any assumption of translation invariance or other restrictions on $\{J_{x,y}\}$. By the infinite-volume limit we mean that the subset $\Lambda \subset \mathbb{Z}^d$ for which the corresponding states are defined eventually covers any finite region in $\mathbb{Z}^d$. The states are probability measures on $\{1,..., q\}^{\mathbb{Z}^d}$ for the Potts models, and on $\{0, 1\}^{\tilde{\mathbb{Z}}^d}$ for the random cluster measures, where $\tilde{\mathbb{Z}}^d$ is the set of bonds of $\mathbb{Z}^d$.

Theorem 2.3. (a) For random cluster models with $q \geqslant 1$ the infinite-volume limit exists for states with both the wired and the free boundary conditions.

(b) For Potts models, with integer q, the infinite-volume limit for the states $\langle \cdots \rangle_1$ and the free boundary condition states $\langle \cdots \rangle_{\mathrm{f}}$ exist.

(c) For integer q, the magnetization equals (at each site x) the percolation probability:

$$\langle \hat{\mathbf{e}}_1 \cdot \boldsymbol{\sigma}_x \rangle_1 = \mu^{\mathrm{w}}(x \leftrightarrow \infty) \tag{2.19}$$

where $\mu^{\mathrm{w}}(\cdot)$ is the wired state of the corresponding random cluster model and $x \leftrightarrow \infty$ means that the cluster of x is infinite.

Proof. (a) For each subset $\Lambda \subset \mathbb{Z}^d$, let us extend the free and the wired states associated with Λ to the entire collection of bonds, by declaring (or constraining) in the wired case all the bonds of $\mathbb{Z}^d$ with both ends in Λ^c as occupied, and in the free-b.c. case all the bonds with at least one bond in Λ^c as vacant. For any pair of ordered subsets $\Lambda \subset \Lambda'$ the free and the wired measures associated with Λ can be obtained from those of Λ' by relaxing these constraints. Since both the wired and the free states are FKG measures (by Theorem 2.2), we find [see (2.13b)] that the measures have the following monotonicity properties:

$$\mu^{\mathrm{w}}_{\Lambda'}(\cdot) \preccurlyeq \mu^{\mathrm{w}}_{\Lambda}(\cdot), \qquad \mu^{\mathrm{f}}_{\Lambda'}(\cdot) \succcurlyeq \mu^{\mathrm{f}}_{\Lambda}(\cdot) \tag{2.20}$$

The existence of the infinite-volume limits, for either the free or the wired b.c., follows now by standard arguments: the expectations of monotone functions (in the sense discussed in Section 2.2) converge by monotonicity, and the collection of such functions generates through finite linear combinations all the local functions (of the bond variables $\{n_b\}$).

(b) To prove the convergence of the states of the Potts model, we use the previous result and the relations based on (2.7). The finite-volume states with either the free or the $\hat{\mathbf{e}}_1$ (or more generally $\hat{\mathbf{e}}^{\text{b.c.}} = \hat{\mathbf{e}}_k$, $k = 1,..., q$) boundary conditions can be described as averages over very simple measures, parametrized by the bond configurations $n = \{n_b\}$. For a given n the spins are fixed on each connected cluster, their values for different clusters being independent random variables which are uniformly distributed on the q values, except for the "marked" cluster connected to the complement of Λ, where the spin is $\hat{\mathbf{e}}^{\text{b.c.}}$ (for the free b.c. in Λ no site is connected to Λ^c). If we now focus on any finite subset $A \subset \mathbb{Z}^d$, the expected value of any local function can be computed by first conditioning it on the decomposition of A into connected clusters (one of which is "marked"), which depend on n in all of $\tilde{\Lambda}$. Since the set of bond variables that determine the partition of A keeps changing with Λ, it is convenient to introduce a new system of bond and site variables, which are increasing functions of $\{n_b\}$, indicating whether a given site is connected to Λ^c and whether a given pair belongs to the same cluster. The "conditional" expectations $E^*_n(\cdot)$ of observables depending on the spins in A [see (2.7)] can be expressed (*independently* of Λ) as functions of the new variables associated with the set A. While the distribution of these variables does depend on Λ, it inherits the monotonicity properties of the distribution of $\{n\}$ [expressed by (2.20)], and hence converges as $\Lambda \to \mathbb{Z}^d$, by similar arguments to those used in part (a). It follows that the states of the spin systems also converge.

(c) At first sight, it may seem that (2.19) requires no justification beyond (2.8) and the above results on the existence of the infinite-volume limits. However, hidden in the statement is an interchange of limits, which does require a justification. The point is that convergence of the measure does not immediately imply the continuity (as $n \to \infty$) of the probabilities of nonlocal events like $\{x \leftrightarrow \infty\}$. Thus, an additional argument is required.

(Physically, the problem is known as that of the short-long-range order versus the long-long-range order.)

Let now $\Lambda_n \to \mathbb{Z}^d$ be an increasing sequence of finite regions. The percolation density in the limiting state is

$$\mu^{\mathrm{w}}(x \leftrightarrow \infty) = \lim_{m \to \infty} \lim_{n \to \infty} \mu^{\mathrm{w}}_{\Lambda_n}(x \leftrightarrow \Lambda^c_m) \tag{2.21}$$

while the order parameter is

$$\langle \hat{\mathbf{e}}_1 \cdot \boldsymbol{\sigma}_x \rangle_1 = \lim_{n \to \infty} \mu^{\mathrm{w}}_{\Lambda_n}(x \leftrightarrow \Lambda^c_n) \tag{2.22}$$

Letting $a_{m,n} = \mu^{\mathrm{w}}_{\Lambda_n}(x \leftrightarrow \Lambda^c_m)$, we have for all $m \leqslant n$

$$a_{n,n} \leqslant a_{m,n} \leqslant a_{m,m} \tag{2.23}$$

Taking the limit $n \to \infty$ and then $m \to \infty$ in (2.23) gets us

$$\langle \hat{\mathbf{e}}_1 \cdot \boldsymbol{\sigma}_x \rangle_1 \leqslant \mu^{\mathrm{w}}(x \leftrightarrow \infty) \leqslant \langle \hat{\mathbf{e}}_1 \cdot \boldsymbol{\sigma}_x \rangle_1 \tag{2.24}$$

which proves (2.19). ▌

The significance of the order parameter $M_q(\{\beta J\})$ is summarized in the following statement, whose proof is given in the Appendix, along with some related results.

Theorem 2.4. For translation-invariant Potts models, with integer q:

(a) The expressions for $M_q(\{\beta J\})$ in (1.4) are all equal; i.e., the magnetization in the state $\langle \cdots \rangle_1$ agrees with the thermodynamically defined order parameter.

(b) $M_q(\{\beta J\}) = 0$ if and only if for the given values of $\{\beta J\}$ the Gibbs state is unique.

Moreover, for random cluster models with real values of $q \geqslant 1$:

(c) If $M_q(\{\beta J\}) = 0$, then the free and the wired boundary condition states $\mu^{\mathrm{f}}(\cdot)$ and $\mu^{\mathrm{w}}(\cdot)$ coincide in the infinite-volume limit.

3. PROOFS OF THE MAIN RESULTS FOR $1/x^2$ MODELS

3.1. The Dichotomy for *M*

In this section we establish one of the main results of this paper:

Theorem 3.1. For translation-invariant Potts and random cluster models in dimension $d = 1$, with $q \geqslant 1$, the order parameter $M_q(\{\beta J\})$ satisfies the dichotomy (1.6).

As was pointed out in the introduction (Section 1.1.1), this dichotomy implies:

1. The order parameter vanishes unless the inverse temperature β and the long-range couplings satisfy $\beta J^+ > 1$ [hence $\beta^* \geqslant 1$, β^* being defined by (1.7)].

2. If a system with $1/x^2$ coupling (i.e., with $0 < J^+ < \infty$) has a phase transition, then the order parameter is discontinuous at the critical point, with a jump of at least $(J^+\beta_c)^{-1/2}$. As was discussed in the introduction, the *existence* of phase transitions for all such systems follows from the results of Ref. 25 or Ref. 44 and the comparison inequalities presented in the next section.

Actually, all we shall do here is show that Theorem 3.1 is implied by the general percolation result of Ref. 7, by using the framework presented in the preceding section. For a physical explanation of the phenomenon, the reader is referred to the brief summary in the introduction and to the more detailed discussion in Ref. 7.

The main result of Ref. 7, whose simplest application is to the independent models, is formulated so as to apply to all bond percolation models that have the *strong FKG property.* To achieve the generality, the theorem is stated in terms of quantities that have a somewhat cumbersome appearance. (In essence, the bond densities and the order parameter M are replaced by their highest *conditional values.*) Following are their definitions, for a translation-invariant model on $\mathbb{Z}$, which is described by a probability measure $\mu(dn)$ on the space of bond configurations $n = \{n_b\}$.

1. For each z, we denote

$$K_z^+ = \sup_{\{m_b\}} \{\mu(n_{\{0,z\}} = 1 \mid n_b = m_b \text{ for all } b \neq \{0, z\})\} \tag{3.1}$$

where $\mu(\cdot \mid \cdots)$ is a conditional probability. The model is said to be *regular* if $K_z^+ < 1$ for all z. The quantity β^+ is defined by

$$\beta^+ = \limsup_{|z| \to \infty} K_z^+ \cdot |z|^2 \tag{3.2}$$

2. For $H > 0$, let $\Lambda_H = [-H, H]$ and let

$$M_H = \sup_{\{m_b\}} \{\mu(0 \leftrightarrow \Lambda_H^c \mid n_b = m_b \text{ for all bonds with both ends in } \Lambda_H^c)\} \tag{3.3}$$

and let the quantity M_+ be defined as

$$M_+ = \lim_{H \to \infty} M_H \tag{3.4}$$

Following is the result of Ref. 7.

Theorem 3.2.[7] If a one-dimensional bond percolation model has the strong FKG property, is regular, and satisfies

$$\beta^+ \cdot M_+^2 < 1 \tag{3.5}$$

then the percolation probability in this model vanishes (i.e., $M = 0$).

In order to apply the above result to the models discussed here, we need to identify the quantities β^+ and M_+.

Lemma 3.1. For the wired state $\mu^{\mathrm{w}}(\cdot)$ of a translation-invariant random cluster model with $q \geqslant 1$

$$\beta^+ = \beta \cdot J^+ \tag{3.6}$$

with J^+ defined by (1.5), and

$$M_+ = \mu^{\mathrm{w}}(0 \leftrightarrow \infty) \equiv M_q(\{\beta J\}) \tag{3.7}$$

Proof. Ignoring a certain subtlety, to which we return immediately, the relations (3.6) and (3.7) can be easily explained by noting that the suprema in the expressions (3.1) and (3.3) are attained by the configurations $m_b = 1$, i.e., the corresponding finite-volume wired boundary conditions. The reason for this is that the events whose conditional probabilities define K_z^+ and M_H are monotone-increasing, while the state has the strong FKG property. Hence, the conditional probabilities of these events are monotone-increasing functions of $\{m_b\}$.

To apply the above argument, one should first clarify the structure of the conditional probabilities. The reason for caution is that the measure we are concerned with has itself only been defined as a limit. Since

$$\mu^{\mathrm{w}}(\cdot) = \lim_{L \to \infty} \mu^{\mathrm{w}}_{\Lambda_L}(\cdot)$$

a correct way to regard the conditional expectation of a local function $f(n)$, conditioned on the values of $\{n_b\}$ for b in A^c (i.e., on what is often denoted as the σ-algebra $\mathscr{B}_{A^c}$), is in terms of the following limit:

$$\mu^{\mathrm{w}}(f \mid n_{A^c}) = \lim_{k \to \infty} \lim_{L \to \infty} \mu^{\mathrm{w}}_{\Lambda_L}(f \mid n_{A^c \cap \Lambda_k}) \tag{3.8}$$

That is, the conditioning on all the bonds of the infinite region Λ^c may naturally be regarded as conditioning on a very large region in a finite but much larger system. [Technically, the right-hand side of Eq. (3.8) is a.s. well defined, and the equality holds, by virtue of Doob's Martingale convergence theorem.] The important point now is that the above FKG domination argument does apply to the finite-volume wired b.c. measures which appear on the right side of (3.8).

Of direct concern to us are the following two cases:

1. $f(n) = I[n_{\{0,z\}} = 1]$, $A = \{\{0, z\}\}$
2. $f(n) = I[0 \leftrightarrow \Lambda_{H^c}]$, $A = \tilde{\Lambda}_H$

where $I[\cdot]$ are indicator functions. In case 1, even without the domination argument, the explicit formula (2.10) implies

$$K_z^+ = 1 - e^{-\beta J_z} \tag{3.9}$$

which immediately gives us (3.6). For case 2, the above domination argument easily yields

$$M_H = \mu^{\mathrm{w}}_{\Lambda_H}(0 \leftrightarrow \Lambda_{H^c}) \tag{3.10}$$

As we saw in the proof of Theorem 2.3, the quantity on the right converges (when $H \to \infty$) to the infinite-volume order parameter, i.e., M_+ is given by (3.7). ∎

Proof of Theorem 3.1. The claim made in the theorem is now a direct consequence of Theorem 3.2 and Lemma 3.1. ∎

In view of the discontinuity, one naturally wants to know the value of the order parameter at the critical point. For completeness, let us mention the following general result, proven by a well-known argument.

Lemma 3.2. For (ferromagnetic) systems with $q \geqslant 1$, at any β, M is continuous from above, i.e.,

$$M_q(\{\beta J\}) = \lim_{\varepsilon \to 0^+} M_q(\{(\beta + \varepsilon)J\}) \tag{3.11}$$

Proof. The semicontinuity expressed by (3.11) follows, by a classical argument, from the fact that $M(\beta)$ is representable as an infimum of a family of continuous increasing functions: $M(\{\beta J\}) = \inf_\Lambda M^{\mathrm{w}}_\Lambda(\{\beta J\})$ (where Λ are finite subsets of $\mathbb{Z}^d$). ∎

3.2. The High-Temperature Behavior of Correlation Functions

For any translation-invariant Ising ferromagnet it is known that the susceptibility $\chi(\beta) = \sum_x \langle \sigma_0 \sigma_x \rangle(\beta)$ diverges as $\beta \uparrow \beta_c$ (i.e., χ^{-1} is continuous at β_c). In light of this, Theorem 3.1 proves that the transition in the $1/|x-y|^2$ Ising model is neither purely discontinuous (i.e., not first order) nor purely continuous. Here we restrict attention to the Ising case ($q=2$) and examine the decay of the two-point function for $\beta<\beta_c$. We prove that if $J_x \approx c/x^s$ with $s>1$, then $\langle \sigma_0 \sigma_x \rangle \approx c'(\beta)/|x|^s$ throughout the high-temperature regime. For long-range interactions, such a decay is the analog of the exponential decay found in the high-temperature phase of short-range models. This result means that the $s>1$ Ising models have no high-temperature intermediate phase analogous to the low-temperature intermediate phase discussed in Refs. 13, 55 for which the exponent of power-law falloff changes with temperature.

Our discussion will again rely on the results presented in Ref. 7 for the case $q=1$. The present restriction to $q=2$ is dictated by the fact that the basic properties of the model that are used in the proof have been established for only these two cases. The properties are:

1. The two-point function $\tau(x, y) = \langle \sigma_x \sigma_y \rangle$ obeys the Hammersley–Simon inequality[31,50]

$$\tau(x, y) \leqslant \sum_{u \in \Lambda, v \in \Lambda^c} \tau(x, u)\, J_{u,v}\, \tau(v, y) \tag{3.12}$$

2. The susceptibility $\chi(\beta)$ is finite for all $\beta<\beta_c$.

The first condition is expected to hold for all random cluster models with $1 \leqslant q \leqslant 2$ (though so far it has been proven only for $q=1, 2$); but it presumably fails for q large enough. A relevant observation here is that a natural extension of (3.12) implies that χ diverges as $\beta \uparrow \beta_c$.[8] However, it is known for nearest neighbor models that for integer q sufficiently large (presumably $q>2$ is sufficient in high dimensions) χ^{-1} is discontinuous at the transition point.[38,42] It is worth noting that the special case of (3.12) with $\Lambda=\{x\}$ (as in Griffiths' third inequality[28]) has been proven by Sokal[52] for all $1 \leqslant q \leqslant 2$. The second condition for Ising and percolation models with $J_x \leqslant C/|x|^2$ is implied by either the analysis of Ref. 7 or by the more general proofs of Refs. 2 and 3.

In contrast to the previous subsection, where in order to apply the previous results we had to rely on some work to demonstrate the connection between percolation and magnetization, here we can just quote some lemmas from Ref. 7, which are applicable to any model obeying the above two assumptions. (Nevertheless, it may still be amusing to note that, by

[8] The argument is presented in Ref. 6. In an earlier work, Simon[50] showed that such a conclusion follows, for finite-range models, from Lieb's[39] improved version of (3.12).

(2.9), the two-point correlation is identical, in the high-temperature phase, with the connectivity function.)

Without any further ado, let us state here the result.

Theorem 3.3. For the translation-invariant, one-dimensional ferromagnetic Ising model with couplings satisfying

$$J_x \, |x|^s \to \text{const} \tag{3.13}$$

with a constant in $(0, \infty)$ and $s > 1$, whenever $M(\beta) = 0$, then

$$c(\beta)/|x|^s \leqslant \langle \sigma_0 \sigma_x \rangle \leqslant c'(\beta)/|x|^s \tag{3.14}$$

with $c(\beta) > 0$ and $c'(\beta) < \infty$. In models for which (3.13) can be replaced by only a single-sided inequality, the corresponding part of (3.14) is satisfied.

Proof. As mentioned above, once it is known that the susceptibility stays finite for all inverse temperatures β below the threshold for spontaneous magnetization, this theorem is implied by an extension of Simon's analysis[50] to long-range interaction. The entire argument is given in Section 5 of Ref. 7 (see especially Lemmas 5.1–5.3 there). The distinction made there between the cases $s \geqslant 2$ and $1 < s < 2$ is irrelevant for us, since the recent result of Ref. 3 provides an independent proof of the above condition 2, which along with the Simon inequality (3.12) is all that is required for the argument of Ref. 7. ∎

Finally, it may be mentioned that at the transition point, for a $1/|x|^2$ system with $q \geqslant 1$, the two-point function does not decay to zero (in the state $\langle \cdots \rangle_1$), since by a simple FKG argument

$$\langle \boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_x \rangle_1 \geqslant M_q(\{\beta J\})^2 \tag{3.15}$$

and $M_q > 0$ by our main discontinuity result and the semicontinuity of Lemma 3.2.

4. COMPARISON METHODS FOR POTTS AND RANDOM CLUSTER MODELS

In this section we turn to the second part of the paper, presenting various applications of the comparison inequalities discussed in Section 1.2.

4.1. The Comparison Principles

Let us start with the derivation of the comparison principles of Fortuin. The following result is actually more general than the inequalities

(1.16) and (1.17). For completeness, we enclose here a proof (which is essentially that of Ref. 22, Lemma 3).

Theorem 4.1.[(22)] Let $\mu_\Lambda^*(\cdot)$ and $\mu_\Lambda'^*(\cdot)$ be two free or two wired-b.c. states of random cluster models, with parameter values $\langle q, \{\beta J_b\}\rangle$ and $\langle q', \{\beta' J_b'\}\rangle$, in some (finite or infinite[9] subset) $\Lambda \subset \mathbb{Z}^d$. Then the following domination relations (in the sense of FKG) apply:

$$\text{(a)} \qquad \left\{ \begin{array}{l} q' \geqslant q,\ q' \geqslant 1 \\ \beta' J'_{x,y} \leqslant \beta J_{x,y} \end{array} \right\} \Rightarrow \mu_\Lambda^*(\cdot) \succcurlyeq \mu_\Lambda'^*(\cdot) \tag{4.1}$$

and

$$\text{(b)} \qquad \left\{ \begin{array}{c} q' \geqslant q,\ q' \geqslant 1 \\ [\exp(\beta' J'_{x,y}) - 1]/q' \geqslant [\exp(\beta J_{x,y}) - 1]/q \end{array} \right\} \Rightarrow \mu_\Lambda^*(\cdot) \preccurlyeq \mu_\Lambda'^*(\cdot) \tag{4.2}$$

where the conditions on $\{J_{x,y}\}$ in (4.1) and in (4.2) are assumed to hold for all bonds. We note that the condition in (4.2) is implied by the simpler requirements

$$\beta' J'_{x,y}/q' \geqslant \beta J_{x,y}/q \tag{4.3}$$

Proof. We give the proof for the case of free boundary conditions and Λ finite. The wired-b.c. case reduces to the free one by identifying all sites in Λ^c, and the infinite Λ case is handled by the obvious limit procedure.

Noting that the probability measure described by (1.13)–(1.14) can be written in the form

$$\mu(\cdot) = q^{c(n)} \prod_{\{x,y\}} (e^{\beta J_{x,y}} - 1)^{n\{x,y\}} \text{ /normalization} \tag{4.4}$$

we find that $\mu(\cdot)$ can be represented as

$$\mu(\cdot) = \mu'(\cdot f)/\mu'(f) \tag{4.5}$$

with

$$f(n) = \left[\frac{q}{q'}\right]^{c(n)} \prod_{\{x,y\}} \left[\frac{\exp(\beta J_{x,y}) - 1}{\exp(\beta' J'_{x,y}) - 1}\right]^{n\{x,y\}} \tag{4.6}$$

[9] If $q < 1$ and Λ is infinite, $\mu_\Lambda^*(\cdot)$ should be understood as some subsequence limit of finite-volume states. The standard argument for the uniqueness of the limit is inapplicable due to the failure of the FKG inequalities for $q < 1$.

The number of connected clusters $c(n)$ has the following monotonicity properties:

1. $c(n)$ is monotone-*decreasing* (meaning nonincreasing) in n (since the addition of a bond can only decrease the number of connected clusters).
2. The function $g(n) = c(n) + \sum_b n_b$ is monotone-*increasing* (since the addition of a bond can decrease $c(n)$ by not more than one).

Using the above two statements, one can easily see that in the first case, i.e., under the assumptions in (4.1), the function $f(\cdot)$ is monotone-increasing, while in the case (b) it is decreasing, as a function of n. Since for $q' \geqslant 1$ the measure $\mu'(\cdot)$ has the FKG property (by Theorem 2.2), the domination relations claimed in (4.1) and (4.2) follow now, by the principle embodied in the inequality (2.13b).

The fact that for $q' \geqslant q$ the inequality (4.3) implies the one seen in (4.2) is an elementary observation. The quickest way to see it is by employing the positivity of the coefficients in the power expansion of e^x (one may also deduce it from the convexity of e^x).

Both of the above comparisons hinged on the fact that at least one of the measures had a q value exceeding unity (and thus had the FKG property). However, by using (4.1) and (4.2) in combination, one can obtain FKG dominance relations also for other combinations.

Example. For q and q' both less then 1,

$$\left.\begin{cases} 1 \geqslant q', q \\ \beta' J'_{x,y} \geqslant \beta J_{x,y}/q \text{ for all bonds} \end{cases}\right\} \Rightarrow \mu'^*_\Lambda(\cdot) \succcurlyeq \mu^*_\Lambda(\cdot) \tag{4.7}$$

where the second condition could also be replaced by

$$\exp(\beta' J'_b) - 1 \geqslant [\exp(\beta J_b) - 1]/q$$

[The proof is by comparison to $\mu''(\cdot)$ with $\beta'' J''_{x,y} = \beta' J'_{x,y}$ and $q'' = 1$.]

4.2. The Phase Structure of q-State Potts Spin Systems

In this subsection, we apply the domination principles (4.1) and (4.2) for results (some of which were previously approached by separate and more complex arguments) on the phase structure of various Potts spin models and independent percolation. One important conclusion can be summarized in the following sentence: The existence, or absence, of distinct high- and low-temperature phases in any q-state Potts (ferromagnetic) spin model or Bernoulli percolation implies that the same is true for any other value of q in the corresponding system. This statement extends also to the random cluster problems with noninteger values of q, which offer an interpolation between different Potts models.

The following is a more complete expression of the above principle.

Theorem 4.2. (o) In any ferromagnetic q-state Potts model on $\mathbb{Z}^d$, with a Hamiltonian $\mathscr{H}$ of the form (1.3), if for some value of β the order parameter $M(\beta) = \langle \hat{\mathbf{e}}_1 \cdot \boldsymbol{\sigma}_0 \rangle_1(\beta)$ does not vanish, then for all $\beta' > \beta$, $M(\beta') \geqslant M(\beta) > 0$. Hence, a transition temperature β_c^{-1} is well-defined.

(i) For a fixed set of ferromagnetic couplings $\{J_{x,y}\}$, denote the values of the inverse transition temperatures for Potts models with different values of q by $\beta_c(q)$ (allowing for 0 and ∞). Then

$$\beta_c(q') \geqslant \beta_c(q) \geqslant \frac{q}{q'} \beta_c(q') \qquad \text{for all} \quad q' \geqslant q \geqslant 1 \tag{4.8}$$

(ii) Analogous results hold for independent percolation and for the random cluster models with all real values of $q \geqslant 1$.

Proof. The first statement is, of course, well known for the Ising systems ($q = 2$), for which it follows from the Griffiths inequalities.[26,37] Its analog for percolation ($q = 1$) is obvious; it is in this spirit that the result can be established for general Potts models.

By Theorem 2.3 the magnetization equals the percolation probability in the wired state of the corresponding random cluster model. Since the event $\{0 \leftrightarrow \infty\}$ is increasing, its probability follows the monotonicity properties of the state. The rest of the argument consists of just straightforward applications of the domination inequalities (4.1) and (4.2).

Remarks. It may be noted that the above argument does not involve any hypothesis of translation invariance. Similar (but somewhat weaker) results can be obtained for the random cluster models when $0 < q < 1$ [using, e.g., (4.7)].

We devote the remainder of this section to three amusing applications of Theorem 4.2.

4.2.1. One-Dimensional Models with $J_{x,y} \approx C/|x-y|^s$. As explained in the introduction, Theorem 4.2 allows one to relate the results previously derived for Ising models by Fröhlich and Spencer[25] and for percolation by Newman and Schulman[44] (the relation is discussed in Section 1.2.3). Adding to those the converse bounds on the percolation problem of Aizenman and Newman,[7] we get, by means of Theorem 4.2, the following conditions for the existence of spontaneous magnetization.

Theorem 4.3. For the translation-invariant, one-dimensional Ising or Potts ferromagnets at inverse temperature β with (finite) couplings satisfying

$$\limsup_{x \to \infty} \beta \cdot x^2 J_x \leqslant 1 \tag{4.9}$$

there is no long-range order, regardless of the individual J_x values. However, if

$$\liminf_{x \to \infty} \beta \cdot x^2 J_x > q \tag{4.10}$$

then the spontaneous magnetization is made positive by sufficiently increasing any single coupling J_z (say J_1).

Remark. The independent percolation ($q = 1$) case of the second part of the theorem was explicitly stated in Ref. 44 only for $z = 1$. However, the result for general z follows easily from the renormalization group methods used there. In particular, we note that the probability $\lambda(L, J_z)$ that all sites in a block of length L are connected by paths within that block may be driven arbitrarily close to one by choosing *both* J_z and L large.

4.2.2. Dilute and Random Ferromagnets. The most amusing application of the comparison inequalities is to the study of the phase diagrams of dilute and random Ising and Potts ferromagnets. While for models with interactions the dilution introduces new effects and nontrivial shifts in the transition points, for independent percolation the effect of (bond) dilution is just an easily calculable shift of the critical density. In Ref. 5 we apply these ideas and obtain simple derivations of rather sharp bounds on the critical temperatures of dilute and random ferromagnets when the dilution density is close to the natural threshold, at which the lattice is effectively divided into finite noninteracting clusters.

4.2.3. Stingy Potts' Pies. Grimmett[29] addressed the following question. Let $\mathscr{S}_f$ denote the "pizza slice"

$$\mathscr{S}_f = \{x \in \mathbb{Z}^2 \mid x_1 > 0,\ |x_2| \leqslant f(x_1)\} \tag{4.11}$$

for some (strictly positive) function f, and consider the percolation problem restricted to $\mathscr{S}_f$. For what functions f can we hope to find percolation?

Grimmett showed that if $f(z)$ has the asymptotic behavior

$$f(z) \cong a \log z \tag{4.12}$$

then percolation does occur in the slices, but the threshold bond density $p^*(a)$ tends to 1 as $a \to 0$. This result was extended to higher dimensions by

Hammersley and Whittington (Ref. 33, Section 7). Although the above seemed to be quite a definitive statement, when Hammersley and Whittington studied the analogous question for self-avoiding walks, they found that, provided $\lim_{z \to \infty} f(z) = \infty$, for this system the relevant notion of a critical temperature, i.e., the connectivity constant, was unchanged from that of the full lattice.[33] The curious contrast between these two facts was resolved by Chayes and Chayes,[17] who also extended the results to Ising systems. The resolution amounts to the statement that the logarithmically growing slices exhibit an intermediate phase characterized by the absence of spontaneous magnetization (or percolation) and nonexponential falloff of correlations. Indeed, Grimmett had been looking at the "low-temperature" transition point, while Hammersley and Whittington, in examining the connectivity constant, had pinpointed the high-temperature transition.

With the FK representation and the above domination lemmas, these results are now easily extended to the q-state ferromagnetic nearest neighbor models on stingy wedges in all dimensions.

Theorem 4.4. For the wedges of $\mathbb{Z}^d$ $(d \geqslant 2)$,

$$\mathscr{S}_f = \{x \in \mathbb{Z}^d \mid x_1 \geqslant 0, 0 \leqslant |x_2|,..., |x_d| \leqslant f(x_1)\} \tag{4.13}$$

the ferromagnetic q-state Potts models, with the nearest neighbor couplings $J_{x,y} = \delta_{|x-y|,1}$, have vanishing spontaneous magnetization at all temperatures if

$$\lim_{z \to \infty} [f(z)]^{d-1}/\log z = 0 \tag{4.14}$$

On the other hand, if the limit in (4.14) is positive, i.e., $f(z) \cong a[\log z]^{1/(d-1)}$ as $z \to \infty$, then there is spontaneous magnetization at low enough temperatures, with the transition point satisfying $\beta^*(a) \to \infty$ for $a \to 0$. Regardless of the previous considerations, provided that $\lim_{z \to \infty} f(z) = \infty$, the asymptotic rate of exponential decay of untruncated correlations is identical to that of the free boundary full lattice systems (at the same temperature). This implies, for small enough a, the existence of an intermediate phase in the logarithmically growing wedges.

Remarks. 1. Before we delve into the proof, a few of the above terms should be defined.

(a) Since not all the boundary disappears in the infinite-volume limit, the boundary conditions that are naturally used for the definition of the spontaneous magnetization consist of free boundary at the edge of the wedge and the $\hat{\mathbf{e}}_1$ boundary condition at the intersection of the wedge with the hyperplanes $x_1 = n$. Magnetization, or lack thereof, was established by obtaining estimates on this $\langle \sigma_0 \rangle_1^n$ that were uniform in n.

(b) By untruncated correlations, we simply mean $\langle \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_y \rangle$ computed with free boundary conditions. The asymptotic rate of exponential decay is described by the correlation length ξ, defined as

$$\xi = \left(\lim_{z \to \infty} \frac{-1}{|z|} \log \langle \boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_{\tilde{z}} \rangle \right)^{-1}, \tag{4.15}$$

where $\tilde{z} = (z, 0, ..., 0)$. On the full lattice $\mathbb{Z}^d$, standard arguments, based on the inequality

$$\langle \boldsymbol{\sigma}_{x_0} \cdot \boldsymbol{\sigma}_{x_n} \rangle \geqslant \prod_{k=0}^{n-1} \langle \boldsymbol{\sigma}_{x_k} \cdot \boldsymbol{\sigma}_{x_{k+1}} \rangle \tag{4.16}$$

imply that the limit in (4.15) exists, and that furthermore

$$\langle \boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_{\tilde{z}} \rangle \leqslant e^{-|z|/\xi} \tag{4.17}$$

(These arguments can be found in many references; e.g., Refs. 6 and 14.) By the representation (2.9) and the FKG property, (4.16), and hence also its consequences, hold for all ferromagnetic Potts models.

2. For translation-invariant systems, it has now been established that such intermediate phases do not occur in any Ising ferromagnet or independent percolation models (for $d = 2$ nearest neighbor models by Ref. 35 for percolation, and by the exact solution for the Ising model; and for more general cases by Refs. 2 and 43 for percolation and Refs. 1 and 3 for Ising systems).

Proof. Applying the domination lemmas to the results of Refs. 17 and 29 yields the low-temperature behavior of the wedges immediately. However, the lemmas cannot be applied to extend the high-temperature results; these, though quite straightforward, must be established separately. Thus, essentially all that is needed to prove Theorem 4.4 is to show that if the pizza slice, or wedge, is not infinitely stingy [i.e., if $f(z) \to \infty$], then the correlation length inside the slice is the same as in the full lattice. While the previous results do not apply here, their proof does.

Let $\langle \cdots \rangle_{\mathscr{S}}$ and $\langle \cdots \rangle$ denote the free-b.c. states in the wedge $\mathscr{S}$ and, correspondingly, in $\mathbb{Z}^d$. Using the inequality (4.16) for the state $\langle \cdots \rangle_{\mathscr{S}}$, we have

$$\liminf_{z \to \infty} \langle \boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_{\tilde{z}} \rangle_{\mathscr{S}}^{1/z} \geqslant \inf_{z \geqslant v} \langle \boldsymbol{\sigma}_{\tilde{z}} \cdot \boldsymbol{\sigma}_{\tilde{z}+\tilde{u}} \rangle_{\mathscr{S}}^{1/u} \tag{4.18}$$

for all $v \geqslant 0$ and $u > 0$. Taking in (4.18) first the limit $v \to \infty$ (using the monotonicity in volume of the free-b.c. state) and then $u \to \infty$, one finds that the lower bound converges to the full lattice quantity $e^{-\xi}$ [provided $f(z) \to \infty$ as $z \to \infty$]. Since also $\langle \boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_x \rangle_{\mathscr{S}} \leqslant \langle \boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_x \rangle$, it easily follows that the wedge correlation functions also satisfy (4.15) and (4.17) with the correlation length ξ of the full lattice! Evidently, the high-temperature critical points coincide.

The previous results for independent percolation can be used to control the temperature where spontaneous magnetization occurs. Indeed, if

$$f(z)/[\log z]^{1/d-1} \to 0 \tag{4.19}$$

then the absence of a percolative phase[33] rules out the possibility of spontaneous magnetization for the q-state models. On the other hand, if the left side of (4.19) is uniformly bounded away from zero, the fact that percolation can occur in the wedge (this follows from the $d=2$ result of Ref. 29 by the simple embedding argument of Ref. 33) implies, by Theorem 4.2, positive spontaneous magnetization below a certain temperature. Finally, for wedges with $f(z)/(\log z)^{1/d-1}$ tending to a small constant (with percolation threshold in the wedge pushed close to unity[33]), the domination bounds ensure that the spontaneous magnetization for a q-state Potts model does not occur until well below the transition temperature of the full lattice. That ensures the existence of an intermediate phase.

Remark. The proof of the equality of $\xi(\mathscr{S})$ and $\xi(\mathbb{Z}^d)$ also implies the following. Let

$$T_L = \{x \in \mathbb{Z}^d \mid x_1 > 0, 0 \leqslant |x_2|, \dots, |x_d| \leqslant L\}$$

Then $\xi(T_L) \to \xi(\mathbb{Z}^d)$ as $L \to \infty$. This is quite different than the situation for critical points, since, e.g., $\beta_c(T_L) = \infty$ for all finite L. Indeed, it is an interesting open problem even for independent percolation[4] to show that $\beta_c([-L, L] \times \mathbb{Z}^d) \to \beta_c(\mathbb{Z}^{d+1})$ as $L \to \infty$ (for $d \geqslant 2$). Further discussion and an affirmative solution of such an issue for the nearest neighbor Ising models and $d > 2$ can be found in Ref. 1.

APPENDIX

At the end of Section 2 we summarized in Theorem 2.4 the significance of the order parameter M as an indicator of the structure of the Gibbs states of Potts models. These facts are certainly well known for the Ising model. For completeness we present here the proofs of these statements in general, and add some useful facts. These results may already be known to various specialists, and some of them are contained in the paper of Fortuin.[22] We split here the proof of Theorem 2.4 into two parts, which are covered by Theorems A.1 and A.3.

Theorem A.1. For translation-invariant Potts models, with integer q, the expressions for $M_q(\{\beta J\})$ in (1.4) are all equal, i.e., the magnetization in the state $\langle\cdots\rangle_1$ agrees with the thermodynamically defined order parameter.

Proof. Since the argument is well known for Ising models, we only present the flow of the key steps. Let $f_m(h)$ be the finite-volume free energy:

$$f_m(h)=\frac{1}{|\Lambda|}\log\left\langle \exp\left(h\sum_{\Lambda}\hat{\mathbf{e}}_1\cdot\boldsymbol{\sigma}_x\right)\right\rangle_1^{\Lambda} \tag{A.1}$$

with $\Lambda=\Lambda_m\equiv[-m,m]^d$. Then

$$\frac{\partial f_m(h)}{\partial h}=\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\hat{\mathbf{e}}_1\cdot\boldsymbol{\sigma}_x\rangle_1^{\Lambda}(h) \tag{A.2}$$

where $\langle\cdots\rangle_1^{\Lambda}(h)$ is the finite-volume equilibrium state with the $\hat{\mathbf{e}}_1$ boundary conditions and an external field $h\hat{\mathbf{e}}_1$.

By standard arguments, the functions $f_m(h)$ are convex in h, and are pointwise convergent, as $m\to\infty$, to $f_\infty(h)$. Restricting h to be positive, and avoiding the at most countable set of values at which $f_\infty(\cdot)$ is not differentiable, we have

$$\begin{aligned}\langle\hat{\mathbf{e}}_1\cdot\boldsymbol{\sigma}_0\rangle_1&\leqslant\frac{\partial f_m(0)}{\partial h}\\&\leqslant\frac{\partial f_m(h)}{\partial h}\xrightarrow[(m\to\infty)]{}\frac{\partial f_\infty(h)}{\partial h}\xrightarrow[(h\to 0^+)]{}\frac{\partial f_\infty}{\partial h}(0^+)\end{aligned} \tag{A.3}$$

For the other direction we note that for $h\geqslant 0$ the states $\langle\cdots\rangle_1^{\Lambda}(h)$ have the same monotonicity properties in Λ as the $h=0$ $\hat{\mathbf{e}}_1$ states. (The simplest way to see that is by representing the external field by a coupling to a Griffiths ghost-spin, as in Ref. 27.) In particular, the finite-volume magnetization is bounded below by that of the infinite-volume state $\langle\cdots\rangle_1(h)$. Thus,

$$\partial f_m(h)/\partial h\geqslant\partial f_\infty(h)/\partial h \tag{A.4}$$

Taking the limit $h \to 0^+$ and then $m \to \infty$, we get

$$\langle \hat{\mathbf{e}}_1 \cdot \boldsymbol{\sigma}_0 \rangle_1 \geqslant \frac{\partial f_\infty}{\partial h}(0^+) \tag{A.5}$$

The two relations (A.3) and (A.5) establish the claimed equality.

Remark. An alternative way to present the argument is to note that by general arugments $\partial f_\infty / \partial h(0^+)$ equals the maximum magnetization over all the Gibbs states. Using our FKG arguments, we can show that the maximum is attained in the state $\langle \cdots \rangle_1$ [see part (i) of Theorem A.2 below].

Before we turn to the second part of Theorem 2.4, let us present a handy construction. Let $\rho(d\sigma)$ be an equilibrium (Gibbs) state for a Potts model in $\mathbb{Z}^d$ with the Hamiltonian (1.3) and a finite $|J|$. By the Dobrushin–Lanford–Ruelle[19,41] characterization of the Gibbs states, for each finite $\Lambda \subset \mathbb{Z}^d$ the restriction of $\rho(\cdot)$ to the spins in Λ is an average, with respect to the probability measure $\rho(d\eta)$, of the finite-volume equilibrium measures $\mathscr{G}_\Lambda(\cdot \mid \eta)$ corresponding to the boundary conditions $\sigma_x = \eta_x$ for all $x \in \Lambda^c$, i.e.,

$$\rho(d\sigma_\Lambda) = \int \mathscr{G}_\Lambda(d\sigma_\Lambda \mid \eta)\, \rho(d\eta) \tag{A.6}$$

Mindful of the above representation, we shall now associate with each Gibbs state ρ the following family of probability measures on the sets of bond configurations in $\tilde{\Lambda}^+ = \{\{x, y\} \mid x \in \Lambda, y \in \mathbb{Z}^d\}$:

$$\tilde{\rho}_\Lambda(dn_{\tilde{\Lambda}^+}) = \int \tilde{\mathscr{G}}^\eta_\Lambda(dn_{\tilde{\Lambda}^+})\, \rho(d\eta) \tag{A.7}$$

where the integration is over η [as in (A.6)], and $\tilde{\mathscr{G}}^\eta_\Lambda(\cdot)$ are the probability distributions of the system of bonds in $\tilde{\Lambda}^+$, which were introduced in Section 2.1 [in the paragraph following (2.7)].

Next, we consider for each Gibbs state ρ the measures $\tilde{\rho}$ on the random cluster system in all of $\mathbb{Z}^d$ obtained as (subsequence) limits of $\tilde{\rho}_\Lambda$ with $\Lambda \to \mathbb{Z}^d$. (Although this will not be needed for our purposes, it can be shown that the family $\tilde{\rho}_\Lambda$ is consistent whenever ρ is an equilibrium spin measure, and hence there is no problem of convergence or of uniqueness of the infinite-volume measure $\tilde{\rho}$.) We shall refer to such measures as equilibrium measures of the random cluster model. It should, however, be noted that the conditional distributions $\tilde{\mathscr{G}}^\eta_\Lambda(dn_{\tilde{\Lambda}^+})$ involve for general η new types of constraints, which are not observed when dealing with either the free or the wired boundary conditions (see Section 2.1).

We note further that, as discussed in Section 2.1, for a given q, there is a map associating to each function of the spins in Λ, $f(\sigma_\Lambda)$, a function $\tilde{f}^{\eta}(n)$ of the bonds in $\tilde{\Lambda}^+$ such that the expected value of the former in any equilibrium state $\mathscr{G}_\Lambda(d\sigma_\Lambda \mid \eta)$ coincides with the expectation of the latter in the bond state $\tilde{\mathscr{G}}^{\eta}_\Lambda(dn_{\tilde{\Lambda}^+})$, i.e.,

$$\int f(\sigma_\Lambda)\, \mathscr{G}_\Lambda(d\sigma_\Lambda \mid \eta) = \int \tilde{f}^{\eta}(n_{\tilde{\Lambda}^+})\, \tilde{\mathscr{G}}^{\eta}_\Lambda(dn_{\tilde{\Lambda}^+}) \tag{A.8}$$

for each set of couplings $\{\beta J\}$ and each boundary spin configuration η. In particular, various connectivity functions for the measure(s) $\tilde{\rho}$ are directly expressible in terms of the spin correlation functions of the given equilibrium state ρ.

The following general result expresses the relation of the states introduced above to the two canonical random cluster measures $\mu^{\mathrm{w}}(\cdot)$ and $\mu^{\mathrm{f}}(\cdot)$.

Theorem A.2. Let μ be an equilibrium measure of the random cluster model, corresponding to some integer value of q, and some couplings $\{\beta J\}$, with $\sup_x\{\sum_y J_{x,y}\} < \infty$. Then

(a) $\mu(\cdot) \preccurlyeq \mu^{\mathrm{w}}(\cdot)$ (in the FKG sense) (A.9)

(b) If for all $x \in \mathbb{Z}^d$, $\mu(x \leftrightarrow \infty) = 0$, then $\mu(\cdot) = \mu^{\mathrm{f}}(\cdot)$ (A.10)

Furthermore, (b) is satisfied when μ is the wired state for any real $q \geqslant 1$.

Proof. (a) As an equilibrium state, $\mu(\cdot)$ is a limit of the measures of (A.7) constructed in $\tilde{\Lambda}^+$ by suitable averages over $\tilde{\mathscr{G}}^{\eta}_\Lambda(\cdot)$, while the wired state is constructed as the limit of the finite-volume wired states $\mu^{\mathrm{w}}_\Lambda(\cdot)$ (Theorem 2.3). Since the measures $\tilde{\mathscr{G}}^{\eta}_\Lambda(\cdot)$ are obtained from $\mu^{\mathrm{w}}_\Lambda(\cdot)$ by conditioning on a decreasing event, and $\mu^{\mathrm{w}}_\Lambda(\cdot)$ are FKG measures, for each Λ we have

$$\tilde{\mathscr{G}}^{\eta}_\Lambda(\cdot) \preccurlyeq \mu^{\mathrm{w}}_\Lambda(\cdot) \tag{A.11}$$

The limit $\Lambda \to \mathbb{Z}^d$ leads to the claimed inequality (A.9).

(b) Let $\mu(\cdot)$ be an equilibrium state with a vanishing percolation probability. We shall show that for any local event B, depending on only the bonds in some finite region V, and any $\varepsilon > 0$,

$$|\mu(B) - \mu^{\mathrm{f}}(B)| \leqslant 2\varepsilon \tag{A.12}$$

Let $\Lambda \supset V$ be a finite, but large region (which we later make tend to $\mathbb{Z}^d$). Since there is no percolation in the state μ, there is another finite region $\Lambda' \supset \Lambda$ such that with probability greater than $1-\varepsilon$ none of the connected clusters of sites in $\mathbb{Z}^d \setminus \Lambda'$ reaches Λ. Denoting the above event by D, we can express the probability of B by the following combination of its conditional probabilities:

$$\mu(B) = \mu(B \mid D) + [1 - \mu(D)][\mu(B \mid D^c) - \mu(B \mid D)] \tag{A.13}$$

While the second term on the right side of (A.13) is clearly bounded in magnitude by ε, we claim that the first one is approximately equal to $\mu^{\mathrm{f}}(B)$.

To evaluate the conditional probability of B, conditioned on the event D, it is natural to first condition it further on the connected clusters of all the sites of $\mathbb{Z}^d \setminus \Lambda'$, and in particular on the subset of Λ' connected to $\mathbb{Z}^d \setminus \Lambda'$, which is of the form $\Lambda' \setminus G$, for some $G \supset \Lambda$. The pivotal point is now that the conditional measure is concentrated on all the bond configurations for which no site in G is connected to G^c, and further that the relative weights of the different configurations that satisfy this constraint are such that the resulting probability distribution is exactly the free-b.c. state in G. (A detailed proof of this fact involves the use of another region Λ'' much larger than Λ'.) Thus, $\mu(B \mid D)$ is a mixture (over the G's) of $\mu^{\mathrm{f}}_G(B)$. However, we have seen in Theorem 2.3 that the free-b.c. states converge to a limit. Hence, by choosing Λ large enough, we may guarantee that

$$|\mu^{\mathrm{f}}_G(B) - \mu^{\mathrm{f}}(B)| \leqslant \varepsilon \qquad \text{for all} \quad G \supset \Lambda \tag{A.14}$$

Combining (A.14) with (A.13), we obtain the claimed (A.12). Taking $\varepsilon \to 0$, we see that the measures $\mu(\cdot)$ and $\mu^{\mathrm{f}}(\cdot)$ coincide on all local events, and hence they are equal.

Since the above argument applies (with $\mu = \mu^{\mathrm{w}}$) regardless of whether q is an integer (though it does make use of the condition $q \geqslant 1$), it implies also the last statement of the theorem. ∎

A particularly interesting application of the above theorem is the proof of the following statement, which forms the second part of Theorem 2.4.

Theorem A.3. For Potts models (with integer q) the vanishing of the magnetization in $\langle \cdots \rangle_1$, or of the percolation density in $\mu^{\mathrm{w}}(\cdot)$, implies uniqueness of the Gibbs state and of the equilibrium measure in the corresponding random cluster model.

Proof. Either of the hypotheses implies (by Theorem 2.3) that there is no percolation in the wired state $\mu^{\mathrm{w}}(\cdot)$, and thus also in any of the equilibrium states of the bond model, which are dominated by $\mu^{\mathrm{w}}(\cdot)$. As we say in Theorem A.2, that implies that all the equilibrium states of the bond system are equal to $\mu^{\mathrm{f}}(dn)$. Furthermore, the proof of Theorem 2.3 [part (b); see also (A.8)] also implies that each Gibbs state of the Potts model equals the free-b.c. Gibbs state $\rho^{\mathrm{f}}(d\sigma)$. It follows that both the bond model and the spin system (if q is integer) have unique equilibrium states. ▮

Remark. Theorem A.2 can be extended to a larger class of bond measures μ for both integer and noninteger q. For example, the finite-volume η-b.c. can be replaced by specifying finitely or infinitely many subsets of Λ^c which are not allowed to be connected to each other by the occupied bonds of $\tilde{\Lambda}^+$. For the sake of simplicity, we will not pursue this issue or the corresponding strengthening of part (c) of Theorem 2.4.

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