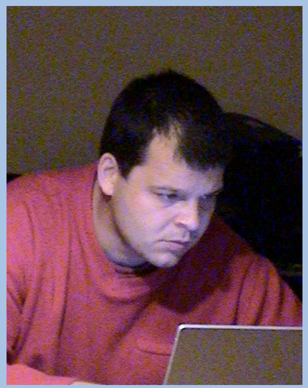
UCLA
Department of Mathematics

Condensed Matter / Mathematics Seminar
September 22nd 2006

On the Absence of Ferromagnetism in Typical 2D Ferromagnets

Joint work:



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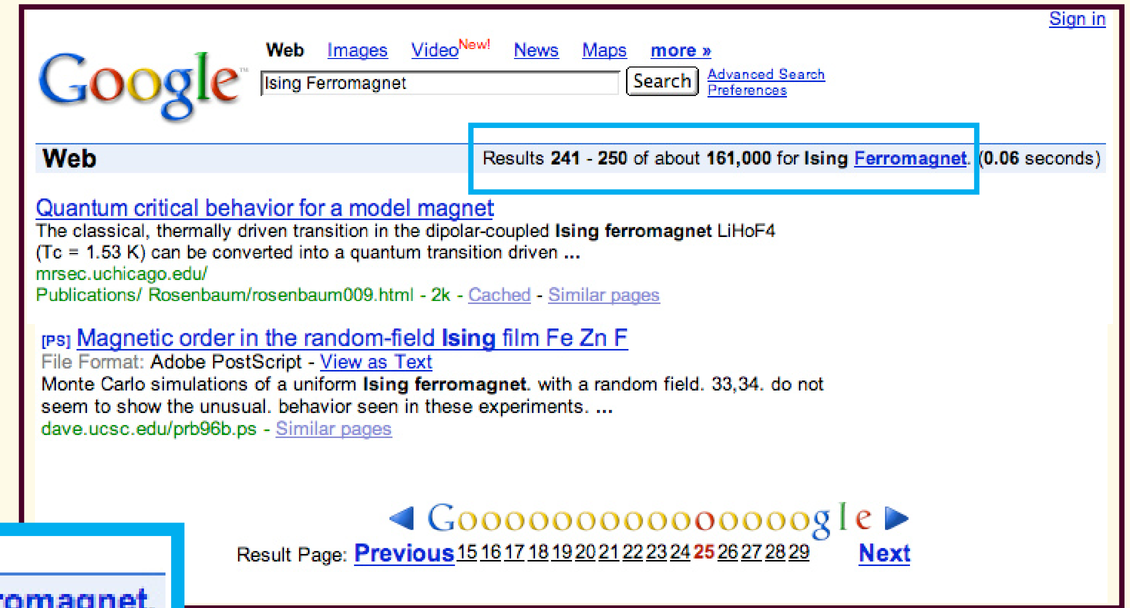
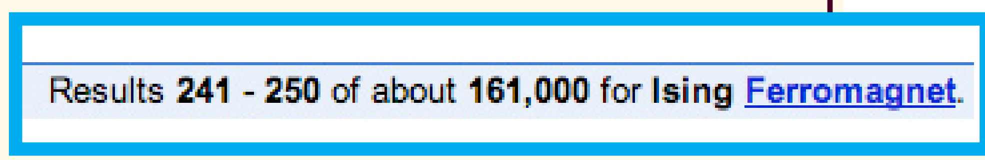
Talk Outline:

I. Background Ising Discussion
II. Real Magnets
III. Physical Arguments
IV. Mathematical Arguments

I. Background Discussion

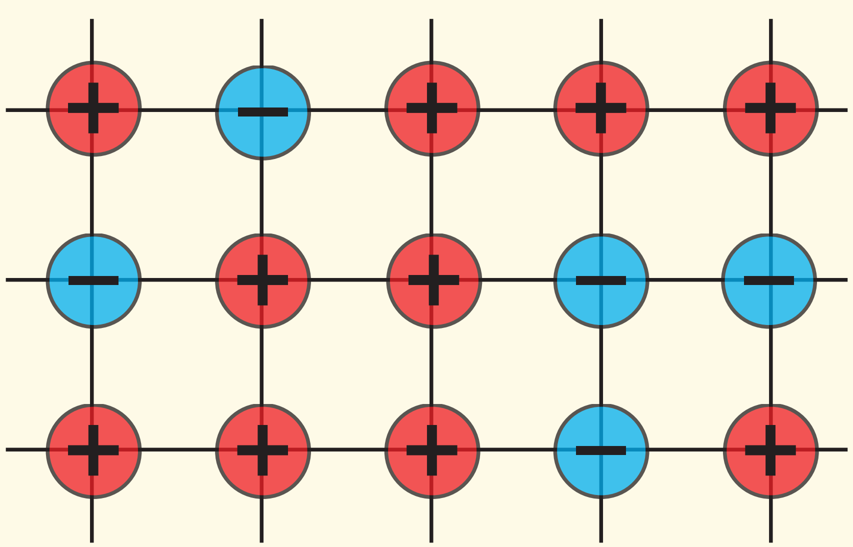


As usual: 2D Ising Magnet



Simplest example of *phase transition* with genuine cooperative phenomena.

Standard NN model: Each site $\vec{r}$ of the square lattice



$$\sigma_{\vec{r}} = \pm 1$$

$$-\mathcal{H} = \sum_{\langle \vec{r}, \vec{r}' \rangle} \sigma_{\vec{r}} \sigma_{\vec{r}'} .$$

Rules:

$\underline{\sigma}$ = spin configuration,

$$\mathbb{P}(\underline{\sigma}) \propto e^{-\beta \mathcal{H}(\underline{\sigma})}$$ "boundary conditions".

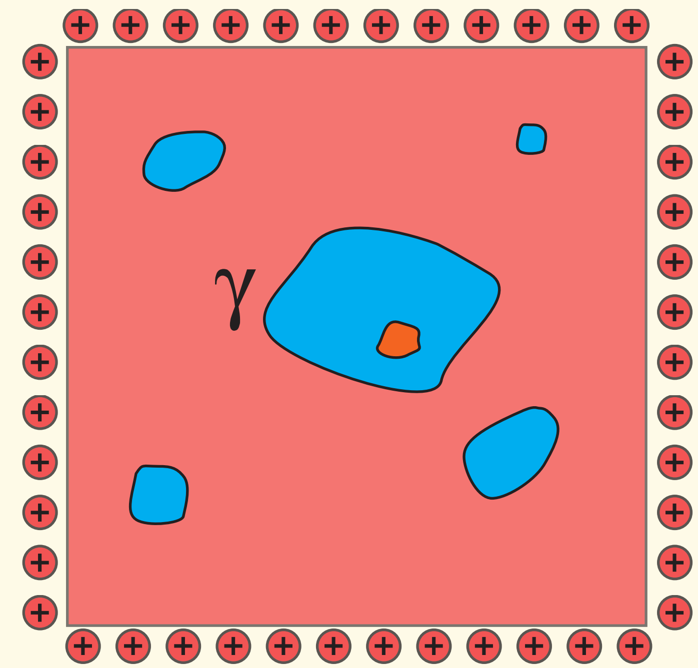
What happens when volume $\longrightarrow \infty$?

d = 1, (Landau) $\varnothing$.

Extension of Landau's result to $d > 1$ ...

– Peierls Argument –

$$\mathbb{P}(\gamma) \leq e^{-2\beta|\gamma|}$$



In $d = 2$, not so many contours, β large, infinite volume state with + b.c. has statistical bias for $\sigma_0 = +1$. **Magnetization is positive.**

Remarks:

(1) Need to prove that for $\beta << 1$, unique state (independent of b.c.).

(2) Original argument difficult to follow.

- Onsagar (40's)
- Griffiths, Dobrushin (60's)

(3) Real spins? $O(2)$ or $O(3)$ symmetry.

No magnetization ($d = 2$); Mermin–Wagner.

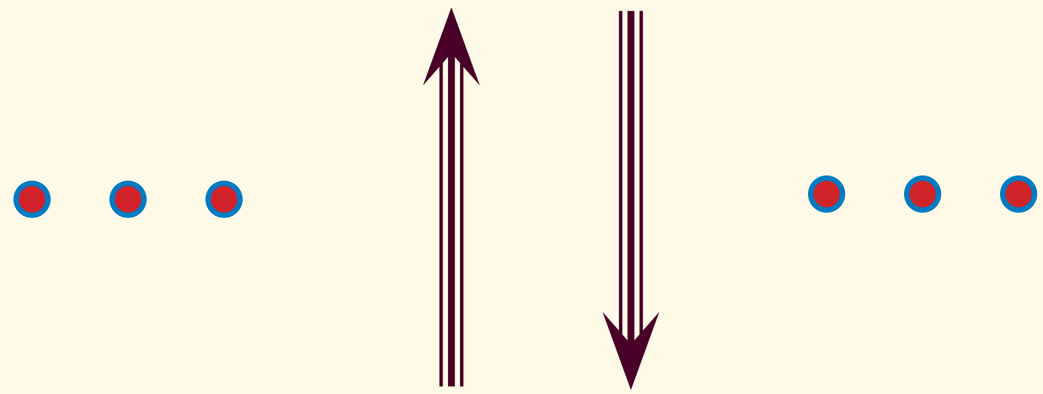
But: Ligand–field effects break $O(N)$ symmetry.

Ising approximation good enough. (Can prove this).

(4) Great model of attractive cooperative phenomena.

Binary alloys, adsorbed gasses, math, other fields of science.

Can imagine electron spins (e.g. in a plane) quantized so as to point up/down.



Origin of ferromagnetic force: Quantum exchange
(usually antiferromagnetic).

– Mysterious –

But, *magnetic* system had genuine long–range antiferromagnetic interaction. [Dipole–dipole interaction.]

If $\vec{S}_{\vec{r}_i}$ & $\vec{S}_{\vec{r}_j}$ genuine spins,

$$\varepsilon \frac{\vec{S}_{\vec{r}_i} \cdot \vec{S}_{\vec{r}_j} - 3(\vec{r}_{ij} \cdot \vec{S}_{\vec{r}_i})(\vec{r}_{ij} \cdot \vec{S}_{\vec{r}_j})}{\left|\vec{r}_{ij}\right|^3}.$$

II. Real 2D Magnets.

$$\varepsilon \frac{\vec{S}_{\vec{r}_i} \cdot \vec{S}_{\vec{r}_j} - 3(\vec{r}_{ij} \cdot \vec{S}_{\vec{r}_i})(\vec{r}_{ij} \cdot \vec{S}_{\vec{r}_j})}{\left|\vec{r}_{ij}\right|^3}.$$

"Small." (ε)

"Not long range." ($\left|\vec{r}_{ij}\right|^3$)

Today:

$$-\mathcal{H} = \sum_{\langle \vec{r},\vec{r}' \rangle} \sigma_{\vec{r}} \sigma_{\vec{r}'} - \varepsilon \sum_{(\vec{r},\vec{r}')} \frac{\sigma_{\vec{r}} \sigma_{\vec{r}'}}{\left|\vec{r} - \vec{r}'\right|^3}$$

For all β, any $\varepsilon > 0$, magnetization is zero.

Remarks: Known to greater/lesser extent in physics community.

[Kivelson, Spivak (2004)]

[1], [5], [6], [8], [9], [12], [15], [16], [18], [19], [20], [23].

Actually, big open question.

"Stripes"

(Giuliani, Lebowitz, Lieb)

Many similar sounding results from early 80's.

R. Israel, A. Sokal

$$\mathcal{H} = \mathcal{H}_0 + \mathfrak{G}$$

$\mathfrak{G}$ is "generic interaction" (from the Banach space of all things that can be).

Then, generically, no magnetization.

But, result easy to understand: $\mathfrak{G}$ has *very* long–range interactions. In language of charges, system cannot support non–neutral configurations.

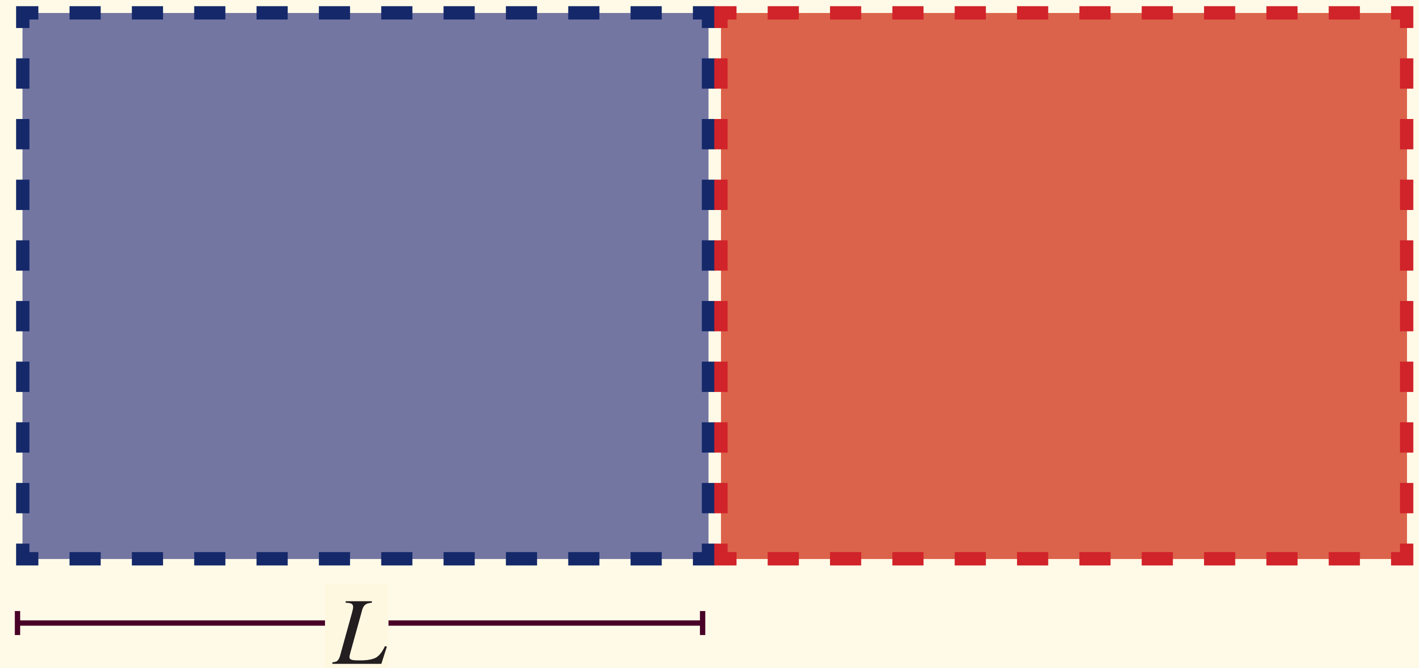
Van Enter (1981). Provided examples of (non–generic) interactions which do this. Also conjectured present result (for powers less than 3).

Here: *Not* a bulk effect.
It is a surface effect.

Argument of Kivelson–Spivac (simplest version):

(1) Suppose $m > 0$. Then states with $+m$ and $-m$.

Put two together and calculate *surface tension.*



$$\tau \sim \frac{1}{L}\left[\kappa L - I(L)\right]$$

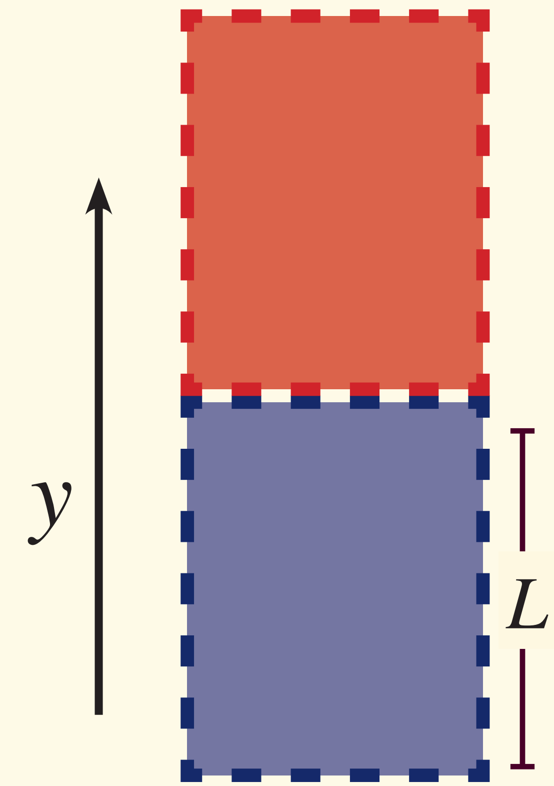
short range term

$$I(L) \sim \varepsilon m^2 \int \frac{d^2 r\, d^2 r'}{\left|\vec{r} - \vec{r}'\right|}$$

Start with $2 < s < 3$.

III. Physical Arguments.

$$\int\limits_{\substack{+L>y'>0\\ -L<y<0}} \frac{dx\,dy\,dx'\,dy'}{[(x-x')^2+(y-y')^2]^{s/2}}$$



Integral difficult, but scale by L.

$$L^{4\text{-}s} \int\limits_{\substack{|x|,|y|<1\\ |x'|,|y'|<1}} \frac{dx\,dy\,dx'\,dy'}{[(x-x')^2+(y-y')^2]^{s/2}}$$

Claim: Legit (@ short distance) even without lattice cutoff.

III. Physical Arguments.

Check: $x - x' = \Delta$, do x' integration; order 1.

Now: $y - y' = \theta$; fixed θ, integrate y' has range $\sim \theta$.

Got: $$\int d\Delta \int_{\theta>0} \frac{\theta d\theta}{(\theta^2 + \Delta^2)^{s/2}}$$, scale out Δ;

$$\int_{\theta>0} \frac{\theta^2 d\theta}{\theta^s} \int d\left(\frac{\Delta}{\theta}\right) \frac{1}{(1 + \frac{\Delta^2}{\theta^2})^{s/2}}.$$

Left with $$\int_{\theta>0} \frac{\theta^2 d\theta}{\theta^s}$$ which is fine if $s < 3$.

If $s = 3$, must actually do integral (with lattice cutoff).

III. Physical Arguments.

Conclusion: $\tau \sim [\kappa L - (\text{const.})\varepsilon m^2 L^{4-s}]\ (s < 3)$.

Negative surface tension, cannot be two such states; $m = 0$.

Similar situation for $s = 3$. Put in cutoff a, $L^{4-s} \longrightarrow \log L/a$

Remarks. Hard to refute but ...

(•) Assumed homogeneity; $s = 3$, all scales contribute.
What if interfacial region weird, non localized.

(•) Mathematically, difficult place to start.

[Need to assume/establish properties of states which you aim to prove do not exist.]

Mathematical approach; slightly different perspective.

But closely related. First:

Theorem (Thermodynamic statement – as strong as possible)

$$f(\beta,h) = -\frac{1}{\beta}\lim_{L\to\infty}\frac{1}{L^2}\log Z_L(\beta,h)$$

$$m(\beta,h) = -\frac{\partial f}{\partial h}\bigg|_{h^+} \qquad m_*(\beta) = -\frac{\partial f}{\partial h}\bigg|_{0^+}$$

for all β, any $\varepsilon > 0$, $m_*(\beta) = 0$.

IV. Mathematical Arguments.

- In any translation invariant state, $\langle \sigma_0 \rangle = 0$.

- In *any* state, block magnetization,

$$\lim_{L\to\infty} \frac{1}{L^2} \sum_{|\vec{r}|<L} \sigma_{\vec{r}}$$

goes to zero.

- $\mathbb{P}\left(\frac{1}{L^2} \sum_{|\vec{r}|<L} \sigma_{\vec{r}} > \eta L^2 \,|\text{outside} \right) < e^{\delta L^2}$.

IV. Mathematical Arguments.

Starting point: Find state which has purported magnetization.

(Pure ferromagnet, use limiting state of + b.c. -- no guarantee here.)

Take limit of $h > 0$ (limiting) torus states.

(a) Can do this (e.g. Israel's book, Simon's book).

(b) Will have "magnetization" (e.g. average or block) = $m_*(\beta)$.

(c) Translation invariant.

(But no guarantee of decay of correlations.)

$$\langle - \rangle_{\mathbb{T}}$$

Idea: Take $L \times L$ block, Λ_L.

$$\mathbf{T}_L = \sum_{\substack{\vec{r} \text{ inside } \Lambda_L \\ \vec{r}' \text{ outside } \Lambda_L}} \frac{\sigma_{\vec{r}} \sigma_{\vec{r}'}}{|\vec{r} - \vec{r}'|^s}$$

← random variable.

Represents "long distance" contribution to interaction between inside and outside of Λ_L.

Somehow, $\mathbf{T}_L \sim \mathrm{c}(m_*)^2 L^{4-s} \quad (s < 3)$

$\mathbf{T}_L \sim \mathrm{c}(m_*)^2 L \log L \quad (s = 3)$.

But, if this is true, Boltzman factor would "motivate turnover" of the block.

Magnetization has to be zero.

Hard days: Try to show *if* $m_* > 0$ then

Prob.($\mathbf{T}_L \geq c(m_*)^2 L^{4-s}$) $\longrightarrow$ 1 (exponentially). And similarly...

Difficult enough ($s < 3$); $s = 3$ statement would require multi–scale analysis.

Just deal with the *average* of $\mathbf{T}_L$.

Usually this sort of approach not enough. But, bounded spins, etc. $\mathbf{T}_L$ is going to have (unless $m_* = 0$) an average which is an appreciable fraction of its maximum value.

Define quantity (deterministic) which *is* the maximum value of $\mathbf{T}_L$:

$$T_L = \sum_{\substack{\vec{r} \text{ inside } \Lambda_L \\ \vec{r}' \text{ outside } \Lambda_L}} \frac{1}{|\vec{r} - \vec{r}'|^s}$$

Claim: $2 < s < 3, \quad T_L \sim QL^{4-s}$

$s = 3, \quad T_L \sim AL\log L$

Second one, "a little delicate"

Goal (more later) to show that $\langle \mathbf{T}_L \rangle_{\mathbb{T}} \sim (m_*)^2 T_L$.

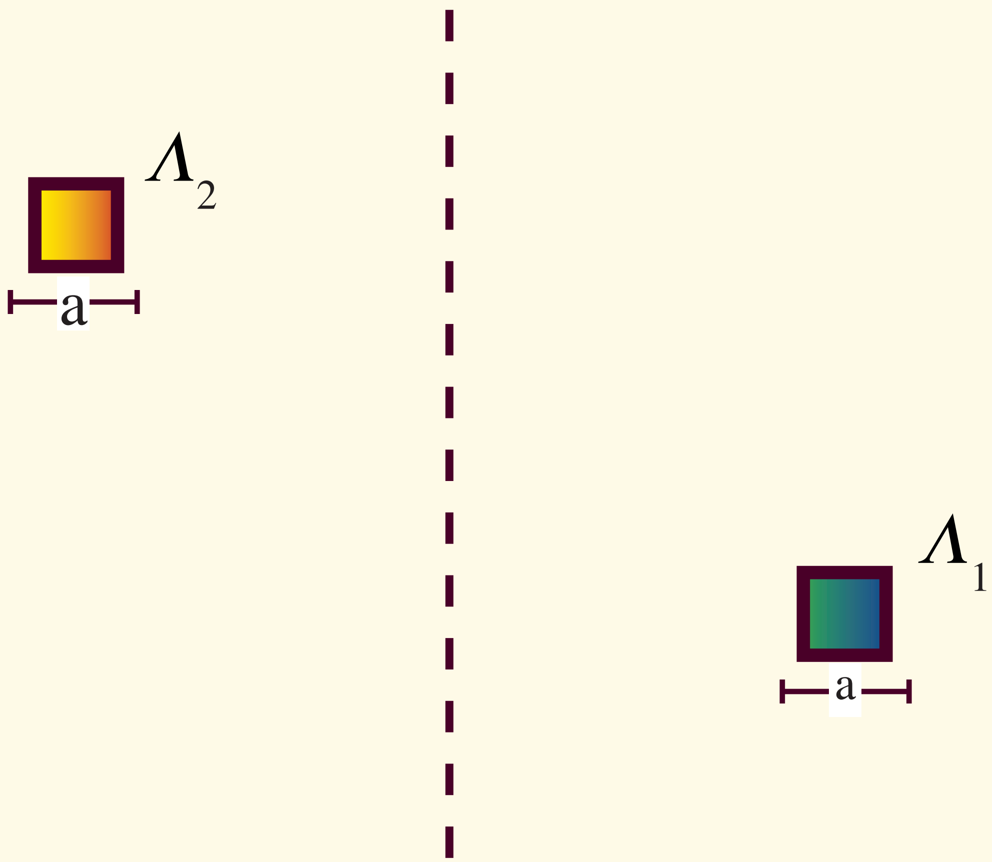
Would be easy if we had clustering of correlations: $\langle \sigma_{\vec{r}} \sigma_{\vec{r}'} \rangle_{\mathbb{T}} \longrightarrow m_*^2$.

Block average property.

IV. Mathematical Arguments.

Proposition: For any λ, $0 < \lambda < 1$,

$$\langle \mathrm{T}_L \rangle_{\mathbb{T}} \geq \lambda (m_*)^2 T_L.$$



(a) From block magnetization property, for any μ, $0 < \mu < m_*$

$$\mathrm{Prob.}\left(\frac{1}{a^2} \sum_{\vec{r} \text{ in } \Lambda_a} \sigma_{\vec{r}} \quad > \mu a^2 \right) \longrightarrow 1$$

Standard from "theory of Gibbs states (actually thermodynamic).

(b) $1 << a << l_0 << L$. If both "good", contribution:

$$\mu^2 \sum_{\substack{\vec{r} \text{ in } \Lambda_1 \\ \vec{r}' \text{ in } \Lambda_2}} \frac{1}{|\vec{r} - \vec{r}'|^s} \qquad (1 - \eta)$$

(c) If either "bad" take negative of this; probability η.

Note: total contribution from sites "right up against the boundary" will be of order

$$l_0 L << T_L.$$

Upshot:

$$\left\langle \sum_{\substack{\vec{r} \text{ in } \Lambda_1 \\ \vec{r}' \text{ in } \Lambda_2}} \frac{\sigma_{\vec{r}}\sigma_{\vec{r}'}}{|\vec{r}-\vec{r}'|^s} \right\rangle_{\mathbb{T}} \geq ([1-\eta]\mu^2 - \eta) \sum_{\substack{\vec{r} \text{ in } \Lambda_1 \\ \vec{r}' \text{ in } \Lambda_2}} \frac{1}{|\vec{r}-\vec{r}'|^s}$$

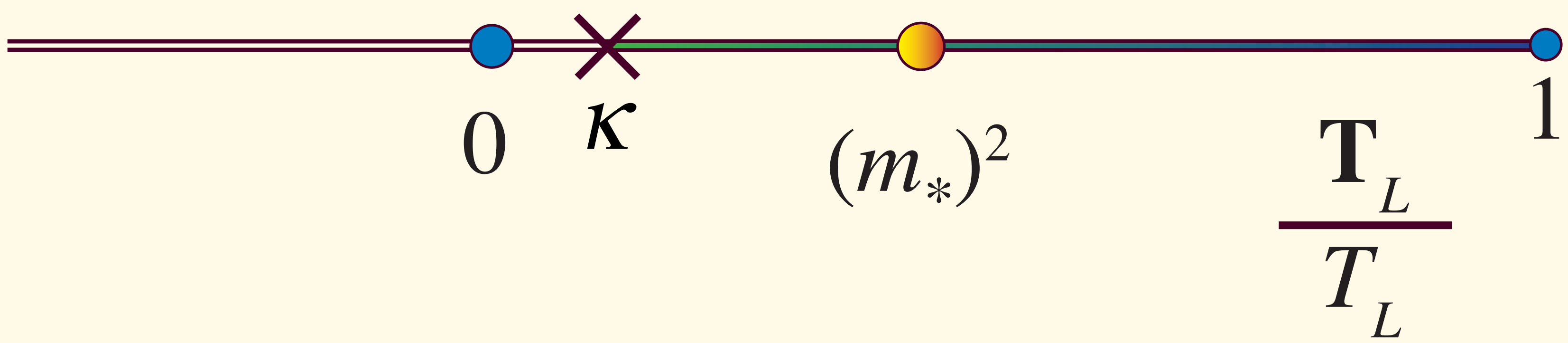
Sum over all boxes -- modulo small details, e.g. O($l_0 L$) @ boundary -- and the proposition is proved. ■

Now, exploit fact that (if $m_* > 0$),

$\langle \mathbf{T}_L \rangle_{\mathbb{T}}$ is a fraction of its max value.

IV. Mathematical Arguments.

On the one hand:



Clear that if $m_* > 0$, then for $\kappa << 1$, will have probability of order unity to exceed κT_L.

But, for any κ, can use Peierls–type argument:

$$\text{Prob.}\left(\mathbf{T}_L > \kappa T_L\right) \leq e^{-2\beta(\kappa T_L - 4JL)} \longrightarrow 0$$

Both cannot be true, 2nd irrefutable. Must have $m_* = 0$.

Can actually avoid "proof by contradiction".

IV. Mathematical Arguments.

$$\langle \mathbf{T_L} \rangle_{\mathbb{T}} \leq \kappa \text{Prob.}(\mathbf{T_L} < \kappa) + T_L \text{Prob.}(\mathbf{T_L} \geq \kappa).$$

-- Solve for Prob.$(\mathbf{T}_L > \kappa T_L)$ --

$$\frac{\frac{1}{T_L}\langle \mathbf{T_L} \rangle_{\mathbb{T}} - \kappa}{(1-\kappa)} \leq \text{Prob.}(\mathbf{T_L} > \kappa T_L) \longrightarrow 0,$$

But we have $\langle \mathbf{T}_L \rangle_{\mathbb{T}} \geq \lambda(m_*)^2 T_L$ (by the proposition). Can bound magnetization above by "arbitrary small #". ■

Conclusions/Open Problems:

(0) In plane quantization axis.

(1) Stripes.

(2) Continuous magnetization @ $h \neq 0$.

(3) Critical power for $O(N)$? Ising: $d < s \leq d+1$, -- $d \geq 2$.

XY, O(3), ¿$d < s < d+2$? -- $d \geq 3$.

(4) Extreme long range: Powers s smaller than d; differentiability wrt “background charge”.