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Non-Perturbative Criteria for Gibbsian Uniqueness

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This paper is dedicated to the memory of R. Dobrushin: A loss that cannot be replaced.

Abstract: For spin-systems with an internal symmetry, we provide sufficient conditions for unicity of the Gibbs state and/or complete analyticity by comparison to random cluster models.

Introductory Remarks

In the realm of statistical mechanics, under the subject headings *high-temperature behavior*, *analyticity* and *uniqueness*, the philosophical and mathematical contributions of R. Dobrushin will remain intact as long as the subject still exists. The usual approach to these questions consists of "expansion techniques" – high temperature expansions, cluster expansions, etc. These expansions have the advantage that they may be applied to virtually any (short-ranged) system, however, they suffer in that they are only functional for extreme values of parameters. As was often stressed by Dobrushin, a peculiar feature of these expansions is that while the formulation and resolution of problems within such a framework constitute definitive probabilistic statements, the intermediate steps do not. Concrete actions towards the repair of this deficiency were taken in [D_2] where a not-cluster expansion was derived. Most of the usual high-temperature results can be obtained by this method (but unfortunately with the same sorts of restrictions) and in addition, certain new problems are suggested.

Carrying the probabilistic attitude to its extreme, we arrive at the other edge of the spectrum: Graphical representations in statistical mechanics. These are *faithful* representations of the problem at hand, leading to stochastic-geometric problems that are well defined for all values of parameters. Prominent examples include the random cluster [FK] and random current [Ai] representations. The above examples are *successful*

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in the sense that phase transitions are characterized by a geometric phase transition in the graphical representation ([ACCN] and [Ai], respectively).[1] The shortcomings of this approach are all too apparent: Such representations have only been found for a very few systems – each new result along these lines represents a separate challenge. The above cited applies, respectively, to the Potts ferromagnets and to Ising-type (Griffiths–Simon class) systems, period. The complete list (to date, to the authors' knowledge) consists of the 2-component Widom-Rowlinson model [CCK, GLM] the cubic (generalized Ashkin–Teller) models and some models with first-order transitions [CM].

In this paper, we will pursue a hybrid approach: we will consider graphical representations for a "wider than usual" class of systems but sacrifice the "successfulness" clause usually associated with such representations. Let us address the specifics of these two points:

(i) The systems that we study consist of interacting spins taking values in a compact (or discrete) group. The group structure is respected by the Hamiltonian and by the single-spin measure – Haar measure. In other words, for a given spin, all spin states are *a priori* equivalent. Thus, we are well away from a statement concerning "all possible spin-systems." However, we are by no means restricted to phase transitions that result from a break down of symmetry. For ease of exposition, we will further restrict to translation invariant nearest neighbor interactions on the d-dimensional hypercubic lattices. By and large, these latter restrictions are far less important. (Related results on non-translation invariant systems, e.g. "disordered" systems, will appear in a future paper.)

(ii) In a successful representation, the usual signal of a phase transition in the underlying spin-system is *percolation* in the graphical problem. In one form or another, this is the case in all the examples mentioned. Here we will find situations where percolation in the graphical representation implies nothing in particular for the spin-system.

On the other hand, the *absence* of percolation in these representations is strongly suggestive of high temperature behavior. Unfortunately, as of yet, these systems are too poorly understood to demonstrate that absence of percolation is, in fact, a sufficient criterion for uniqueness. Nevertheless, these representations can be compared with and coupled to other graphical models, e.g. the independent percolation models. When the comparison models fail to percolate, uniqueness and, under stronger conditions, complete analyticity can be established.

Of course the use of "non-percolation" as a tool for establishing uniqueness or complete analyticity is hardly new. These ideas are implicitly in play when the cluster expansion is shown to converge and, e.g. in the original derivation of Dobrushin [D_1]. Furthermore, the works of [vdBM] and [N] both use (absence of) percolation in a dominating measure to establish complete analyticity/uniqueness. However, as will be discussed to some extent at the end, the results here represent an improvement over the existing (general) sufficient conditions.

[1] A related approach, designed for the study of lattice models that approximate field theories, are the random walk expansions [BFSp, BFSo]. Although it may be that the full statistical mechanics model can be recovered from this expansion, it is difficult to conceive of explicit expressions, e.g. for the probability of cylinder sets in terms of the polymer weights. Nevertheless, it is presumably the case that some version of "percolation" in these expansions corresponds to the multiple phase regime in the lattice system. E.g. in finite volume, the dominant contribution to the two-point function, in finite volume, could come from terms where the polymer fills a fraction of the available space. However, to the authors' knowledge, such a statement has not appeared in the literature.

Derivation of the Expansion

In what follows, we will consider only nearest neighbor interactions on $\mathbb{Z}^d$. The forthcoming is easily generalizable to any system with pair interactions (i.e. any graph) and, with some additional labor, to systems with multi-spin interactions including, e.g. lattice gauge theories.

Let G denote a compact group, let $\mathfrak{h} : G \times G \to \mathbb{R}$ denote a left-invariant function and consider the Hamiltonian described by the formal expression

$$\mathcal{H} = \frac{1}{2} \sum_{\substack{i,j \in \mathbb{Z}^d \\ |i-j|=1}} \mathfrak{h}(s_i, s_j). \tag{1}$$

Remarks and restrictions. (a) Here, the left-invariance of $\mathfrak{h}$ is the mechanism for assuring that the spin-states at a single site are *a priori* equivalent. (b) In the above formula, each neighboring pair is counted twice. We will get rid of this convention – and the $\frac{1}{2}$ – by asserting that $\mathfrak{h}$ is symmetric (which is physically reasonable). For future convenience – but of no physical significance – we will assume that each *bond* of the lattice has some fixed orientation and, for $|i-j| = 1$ use $\langle i, j\rangle$ as notation for the bond pointing from i to j. (c) We will only consider (again on physical grounds) the cases where $\mathfrak{h}$ is continuous, or in the discrete cases, bounded. Without loss of generality, we will set the maximum value to zero.

Throughout this work, we will often consider *graphical* subsets of $\mathbb{Z}^d$; that is, collections of sites and some of the edges (or bonds) connecting nearest neighbor pairs. Although we will often be notationally cavalier regarding the distinction between the bonds and/or the sites of a graph and/or the graph itself, in all instances, the meaning should be clear from context.

Let $\Lambda \subset \mathbb{Z}^d$ and let $|\Lambda|$ denote the number of sites in Λ. The boundary, $\partial\Lambda$, is here defined as the sites in $\mathbb{Z}^d \setminus \Lambda$ with a neighbor in Λ and we will use $\overline{\Lambda}$ to denote $\Lambda \cup \partial\Lambda$. For $|\Lambda| < \infty$, and fixed spin configuration $s_{\partial\Lambda} \in G^{|\partial\Lambda|}$, the Hamiltonian is a well defined function of spin configuration $s_\Lambda \in G^{|\Lambda|}$, that can be inferred from Eq. (1) and will be denoted by $\mathcal{H}(s_\Lambda \mid s_{\partial\Lambda})$. The partition function on Λ at temperature $1/\beta$ with boundary condition $s_{\partial\Lambda}$ is given by

$$\mathcal{Z}_{\mathcal{H},\beta}^{\Lambda, s_{\partial\Lambda}} = \int e^{-\beta\mathcal{H}(s_\Lambda | s_{\partial\Lambda})} d^{|\Lambda|}s, \tag{2}$$

where ds is normalized Haar measure. To avoid cumbersome expressions, we will use $\odot$ as notation for both the generic lattice Λ and the generic boundary condition $s_{\partial\Lambda}$; indeed, we will further extend the notation and allow $\odot$ to stand for superpositions of boundary conditions, periodic boundary conditions, etc. As usual, the integrand in Eq. (2) defines the finite volume Gibbs measures on $G^{|\Lambda|}$; we denote these measures (or their densities) by $g_{\mathcal{H},\beta}^{\odot}(-)$.

The derivation of the expansion follows closely the derivation of the random cluster representation in [FK]; a less compressed version (for the discrete cases) of what is to follow can be found in [CM]. For $t \in G$, let $\mathcal{E}(t) = \mathfrak{h}(e, t)$ (where e is the identity) and define $R_t = R_t(\beta) = e^{\beta|\mathcal{E}(t)|} - 1$. The partition function admits the expression

$$\mathcal{Z}_{\mathcal{H},\beta}^{\odot} = \int d^{|\Lambda|}s \prod_{\langle i,j\rangle} (R_{s_i^{-1}s_j}(\beta) + 1), \tag{3}$$

where we assume that a single spin configuration provides the boundary condition on $\partial\Lambda$ or there are free boundary conditions on Λ. (Otherwise, a boundary spin integral would be required.) Let $\mathbb{B}_\Lambda$ denote the set of bonds of Λ – including those connecting Λ with $\partial\Lambda$. Let $\omega \subset \mathbb{B}_\Lambda$ and for $b \in \mathbb{B}_\Lambda$, define $\omega_b = 1$ (or "occupied ") if $b \in \omega$ and 0 (or "vacant") if $b \notin \omega$. Expanding the product in Eq. (3), we may identify each term in the expansion with an $\omega \subset \mathbb{B}_\Lambda$: ω_b is occupied if the "R" term is selected and is vacant otherwise. This defines a set of graphical weights:

$$W^{\odot}_{\mathcal{H},\beta}(\omega) = \int d^{|\Lambda|}s \prod_{\langle i,j\rangle\in\omega} R_{s_i^{-1}s_j}(\beta) \tag{4}$$

which will be our principal tool. We will denote the corresponding finite volume graphical measures (measures on $\{0,1\}^{\mathbb{B}_\Lambda} \equiv \Omega_{\mathbb{B}_\Lambda}$) by $\mu^{\odot}_{\mathcal{H},\beta}(-)$ and we will refer to these as the *grey* measures.

The configuration ω divides the lattice into connected components – isolated sites and *clusters* (components that contain bonds). We will denote the total number of clusters by $k(\omega)$ and, for future reference, the total number of components by $c(\omega)$. Obviously, the isolated sites can be integrated away which allows us to express the weights as a product over clusters:

$$W^{\odot}_{\mathcal{H},\beta}(\omega) = \prod_{\ell=1}^{k(\omega)} \int d^{|K_\ell|}s \prod_{\langle i,j\rangle\in K_\ell} R_{s_i^{-1}s_j}(\beta), \tag{5}$$

where K_ℓ is the ℓ^{th} cluster of ω and $|K_\ell|$ denotes the number of sites within this cluster. Already Eq. (5) hints at a conditional independence for the behavior of spins residing in disjoint clusters; this matter will be discussed in greater depth after the following paragraph.

At this point it is worth pausing to make contact with the familiar random cluster representation for the Potts model. Here, $s \in \{1,\dots q\}$, the group structure is of no particular significance (we may take $G = \mathbb{Z}_q$) and $R_s(\beta) = e^\beta - 1$ if $s = e$ and is zero otherwise (which serves to define the Potts Hamiltonian). We will consider, for simplicity, the case of free boundary conditions on Λ. Examining Eq. (5) for this case, it is seen that the "integral" over any cluster vanishes unless all spins of the cluster are in the same state. In the q cases where this happens, the result is a factor of $R_e^{||K_\ell||}(\beta)$, where $||A||$ denotes the number of bonds in the set A. Multiplying in the normalization constant of $1/q$ for the single-spin measure at each site we obtain the factor of $qR_e^{||K_\ell||}(\beta)q^{-|K_\ell|}$ for the cluster K_ℓ. Now, multiplying the total weight for any configuration by an irrelevant factor of $q^{|\Lambda|}$, and using [components] = [clusters] + [isolated sites], the weight of the configuration ω is given by $q^{c(\omega)}R_e^{||\omega||}(\beta)$. This is equivalent to the usual random cluster weights (with free boundary conditions), i.e. $W^{FK;f}_{q,p} \propto p^{||\omega||}(1-p)^{[||\Lambda||-||\omega||]}q^{c(\omega)}$ with $R_e(\beta) = p/(1-p)$.

Although information about the spin-system is clearly lost in going to the grey representation, this can, in principal, be recovered or "built back". Let $\omega \in \Omega_{\mathbb{B}_\Lambda}$ denote a bond configuration and s_Λ a spin configuration and assume, for simplicity, that the boundary condition has been provided by a single spin configuration. Consider the function

$$g^{\odot}_{\mathcal{H},\beta}(s_\Lambda \mid \omega) = \frac{1}{W^{\odot}_{\mathcal{H},\beta}(\omega)} \prod_{\langle i,j\rangle\in\omega} R_{s_i^{-1}s_j}. \tag{6}$$

This is clearly positive and integrates to one. We claim that $g^{\odot}_{\mathcal{H},\beta}(s_\Lambda \mid \omega)$ has the interpretation of the conditional Gibbs density given the configuration ω. Indeed,

$$\sum_{\omega\in\Omega_{\mathbb{B}_\Lambda}} \mu^{\odot}_{\mathcal{H},\beta}(\omega) g^{\odot}_{\mathcal{H},\beta}(s_\Lambda \mid \omega) = \frac{1}{Z^{\odot}_{\mathcal{H},\beta}} \sum_{\omega} \prod_{\langle i,j\rangle\in\omega} R_{s_i^{-1}s_j}$$

$$= \frac{1}{Z^{\odot}_{\mathcal{H},\beta}} \prod_{\langle i,j\rangle\in\mathbb{B}_\Lambda} (R_{s_i^{-1}s_j} + 1), \tag{7}$$

which (cf. Eq. (3)) is exactly the Gibbsian probability density for the configuration s_Λ. Furthermore, a brief examination of Eq. (6) – written as a product over clusters as in Eq. (5) – clearly exhibits the conditional independence mentioned previously.

There are several ways to define percolation in the grey representation. The following is the least stringent definition in the sense that if the system does not satisfy the forthcoming criterion for percolation, it certainly cannot percolate by any other definition. Let $\Lambda \subset \mathbb{Z}^d$ with $|\Lambda| < \infty$ and $0 \in \Lambda$. Let $\mathcal{T}_{0,\partial\Lambda}$ denote the event that the origin is connected to the boundary by a path of occupied bonds and define

$$P_\Lambda(\beta) = \max_{s_{\partial\Lambda}} \mu^{\Lambda, s_{\partial\Lambda}}_{\mathcal{H},\beta}(\mathcal{T}_{0,\partial\Lambda}). \tag{8}$$

If (Λ_k) is any sequence of boxes satisfying $\Lambda_{k+1} \supset \Lambda_k$ and $\Lambda_k \nearrow \mathbb{Z}^d$ it is not hard to see that

$$P_\infty(\beta) = \lim_{k\to\infty} P_{\Lambda_k}(\beta) \tag{9}$$

exists and is independent of the sequence (Λ_k). We say that there is percolation if $P_\infty > 0$.

There are a few circumstances where percolation in the grey representation is known to coincide with a phase transition in the spin-system. In particular, this is the case for the Potts models [ACCN], the cubic models (generalized Ashkin–Teller models) – for a certain region of parameters [CM] and, in some generality, systems with discontinuous transitions [CM]. But this is certainly not always the case. For example, it is possible to show that for the 4-state clock model on $\mathbb{Z}^2$, percolation in the grey representation occurs well above the critical temperature [C].

On a less ambitious tack, it seems that the absence of percolation in the grey measure should imply uniqueness of the limiting Gibbs measure. Along these lines, a considerably weaker statement was established in [CM]: if there is no percolation, then all Gibbs states are invariant under the action of G. To date, a full theorem to the effect that non-percolative behavior in a grey representation implies the uniqueness of the corresponding Gibbs measure has required the additional ingredient of a monotonicity property, e.g. the FKG property of the former. Although such monotonicity properties are plausible under some general condition of "ferromagnetism" of the Hamiltonian, the FKG property has only been established in a handful of cases. Notwithstanding the lack of monotonicity, some progress is possible when the graphical measure is dominated by a (non-percolating) measure that *does* have the FKG property. This is exactly the strategy that was used for the case considered in [N] and is operating implicitly in the derivation of [vdBM]. It is therefore worthwhile to consider comparison inequalities between the grey measures and other graphical problems such as the FK random cluster model. Via such comparisons, non-perturbative statements about high temperature behavior are possible.

A Comparison Inequality

For finite $\Lambda \subset \mathbb{Z}^d$, $p \in (0,1)$ and $q > 0$, let $\nu_{q,p}^{FK;\odot}(-)$ denote the random cluster measures with boundary conditions (appropriate to a random cluster model) as specified in $\odot$ (Cf. the description in the statement of Proposition 1). For $\mathfrak{h}(s_1, s_2)$ of the form described in the remarks and restrictions following Eq. (1), we define $\mathcal{E}_0 = \min_{s_1,s_2} \mathfrak{h}(s_1, s_2)$, the quantity

$$R_0 = R_0(\beta) = e^{\beta|\mathcal{E}_0|} - 1 \equiv \max_{t \in G} R_t(\beta) \tag{10}$$

and

$$\overline{R} = \overline{R}(\beta) = \int ds R_s(\beta). \tag{11}$$

The following domination bound is elementary:

Proposition 1. *For a finite lattice, consider the random cluster measures $\nu_{q,p}^{FK;\odot}(-)$, where $\odot$ indicates a boundary condition in which various subsets of the boundary are considered to be "preconnected" (i.e. they act as a single site) and the rest are left free. (This includes free, wired and periodic.) For $\mathcal{H}$ of the type that has been described, let $\mu_{\mathcal{H},\beta}^{\odot}(-)$ denote the grey graphical measures with the same boundary condition. Then*

$$\mu_{\mathcal{H},\beta}^{\odot}(-) \underset{\text{FKG}}{\leq} \nu_{Q,P}^{FK;\odot}(-),$$

where $P = R_0/(1 + R_0) \equiv 1 - e^{-\beta|\mathcal{E}_0|}$ and $Q = R_0/\overline{R}$.

Proof. For boundary conditions of the type stated, the weights of the random cluster measure have the expression

$$\nu_{q,p}^{FK;\odot}(-) \propto [\frac{p}{1-p}]^{||\omega||} q^{c_\odot(\omega)}, \tag{12}$$

where $c_\odot(\omega)$ counts the number of connected components according to the rules specified by the boundary conditions in $\odot$. As is well known, these are FKG measures. For convenience, let us express these measures in "loop form": Let $\ell_\odot(\omega)$ denote the minimum number of bonds in ω that must be removed until what remains is a tree. (As is the case for the number of components, this depends on boundary conditions.) Using $c_\odot(\omega) = \ell_\odot(\omega) - ||\omega||$ + constant, we get

$$\nu_{q,p}^{FK;\odot}(-) \propto [\frac{p}{q(1-p)}]^{||\omega||} q^{\ell_\odot(\omega)}. \tag{13}$$

To establish our claim, we show that the weights $W_{\mathcal{H},\beta}^{\odot}(\omega)$ may be expressed in the form

$$W_{\mathcal{H},\beta}^{\odot}(\omega) \propto [\nu_{Q,P}^{FK;\odot}(\omega)] F^{\odot}(\omega) \propto [\overline{R}^{||\omega||} (\frac{R_0}{\overline{R}})^{\ell_\odot(\omega)}] F^{\odot}(\omega), \tag{14}$$

where $F^{\odot}$ is a decreasing function.

Defining $F^{\odot}(\omega)$ to be the ratio of the right-hand side of Eq. (5) to the quantity $\overline{R}^{||\omega||}[R_0/\overline{R}]^{\ell_\odot(\omega)}$, let $\omega \subset \mathbb{B}_\Lambda$ and $b \in \mathbb{B}_\Lambda \setminus \omega$. Consider $F^{\odot}(\omega)$ versus $F^{\odot}(\omega \vee b)$: The bond b either joins two components of ω or closes a loop in ω. We claim that in the former case, $W_{\mathcal{H},\beta}^{\odot}(\omega \vee b) = \overline{R} W_{\mathcal{H},\beta}^{\odot}(\omega)$ (and hence $F^{\odot}(\omega \vee b) = F^{\odot}(\omega)$). To see this, consider an integration over one component as in the right-hand side of Eq. (5):

$$\tilde{W}^{\odot}_{\mathcal{H},\beta}(C) \equiv \int \prod_{\langle i,j\rangle \in C} R_{s_i^{-1}s_j} d^{|C|}s, \tag{15}$$

where here one of the "sites" may include a boundary component and if C is a single site, the integrand is taken to be unity. For any site $a \in C$, if the integration is performed so that s_a is integrated last, it is seen that the final integrand is a constant independent of s_a. Indeed, this follows directly from the invariance of Haar measure: For fixed value of s_a, let us compare this last integrand with its value at gs_a, $g \in G$. Noting that the s_a dependence always comes in the form $s_a^{-1}s_j$, we perform the other integrations after the change of variables $s_j \to gs_j$, $j \neq a$. The result, after the first $|C|-1$ integrations, is thus manifestly independent of g so indeed the final integrand is a constant. Now consider two disjoint components, C_x and C_y of the configuration ω and suppose that b joins an $x \in C_x$ with a $y \in C_y$. It follows immediately that $\tilde{W}^{\odot}_{\mathcal{H},\beta}(C_x \cup C_y \cup b) = \overline{R}\,\tilde{W}^{\odot}_{\mathcal{H},\beta}(C_x)\tilde{W}^{\odot}_{\mathcal{H},\beta}(C_y)$ because here, saving the s_x and s_y integrations for last, we get $\tilde{W}^{\odot}_{\mathcal{H},\beta}(C_x)\tilde{W}^{\odot}_{\mathcal{H},\beta}(C_y) \times \int ds_x ds_y R_{s_x^{-1}s_y}$ by the previous argument.

In the case where b closes a loop in ω, the derivation is simple: If b joins x to y, as an upper bound we replace the new factor of $R_{s_x^{-1}s_y}$ that appears in the integrand defining the weight for $\omega \vee b$ with R_0. This results in $F^{\odot}(\omega \vee b) \leq F^{\odot}(\omega)$. □

Remark. Following the same derivation, it is easily shown that for virtually any boundary condition $\odot$ in the grey system, we get the above sort of dominations if we compare to the random cluster measure with wired boundary conditions. In particular (and of particular importance) are the $\odot$'s that come from a fixed spin configuration at the boundary. Here the argument is identical if the "new bond" is not connected to the boundary. If the new bond attaches a previously isolated cluster to the boundary, the derivation is the same as when two isolated clusters are joined: the grey weight gets multiplied by $\overline{R}$ and, the number of loops has not changed. Finally, if the new bond joins two clusters that are already attached to the boundary, then, by the definition according to wired boundary conditions, the number of loops increases by one, and, as in the previous loop case, the new weight factor does not exceed R_0.

Let us also observe that these dominations are identities for the Potts models and therefore expected to be fairly sharp for models that are "close" to the Potts models: systems with a significant energy gap and relatively few low-lying states that occur only when the spins are nearly aligned. As an example, suppose the spins take values on the unit circle and are parameterised by θ, $0 \leq \theta \leq 2\pi$ and, using additive notation, a pair interaction given by $V(\theta_i - \theta_j)$. If $V(\theta) = -(1+\cos\theta)$, this is the usual XY model, let us consider the case where $V(\theta) = -1$ if $|\theta| < \epsilon$ and is zero otherwise. Here we have $Q = 1/\epsilon$ and, $P = 1 - e^{-\beta}$. If $\epsilon \ll 1$ then, using the well known results for the Potts model (cf. the discussion before Theorem 3) we can show uniqueness – and exponential decay of correlations down to temperatures β^{-1} satisfying $e^{\beta} \geq \text{const.}(\epsilon^{-1/d})$.

An FKG Decomposition

To implement our strategy, we need some standard terminology from percolation theory. In what follows, we will focus on a fixed $U \subset \Lambda$. Consider a minimal set of bonds that separates U from $\partial\Lambda$. Such an object is often better envisioned on the dual lattice and will be referred to as a separating surface. If C is such a surface, we will denote the interior graph – sites and bonds with both endpoints that are inside C – by $I(C)$. Similarly, the

exterior graph will be denoted by $E(C)$. Since $|\Lambda| < \infty$, we may consider the entire collection $C_1, \ldots, C_N$ of such separating surfaces. We let $\mathbf{C}_j \subset \Omega_{\mathbb{B}_\Lambda}$ denote the event that all the bonds in C_j are vacant. Finally, for future reference, let us observe that the surfaces $\{C_j\}$ have a natural partial order by containment of interiors. Let $\mathbb{C}_j \subset \mathbf{C}_j$ denote the event

$$\mathbb{C}_j = \{\omega \in \mathbf{C}_j \mid C_j \text{ is the outermost vacant surface separating } U \text{ from } \partial\Lambda\}. \quad (16)$$

We remark on two standard features of these $\mathbb{C}_j$: First, any configuration in which U is disconnected from $\partial\Lambda$ belongs to a unique $\mathbb{C}_j$ and second, the event $\mathbb{C}_j$ is determined exclusively by the bonds in $C_j \cup E(C_j)$.

Since, for $\mathcal{H}$ of the type described, the weight factor $W^{\odot}_{\mathcal{H},\beta}$ is given by a product over clusters, it follows that restriction of the conditional graphical measure to the set U, $\mu^{\odot}_{\mathcal{H},\beta}(- \mid \mathbf{C}_j) \mid_U$, is identical to (the restriction of) the measure on $I(C_j)$ with free boundary conditions on C_j:

$$\mu^{\odot}_{\mathcal{H},\beta}(- \mid \mathbf{C}_j) \mid_U = \mu^{I(C_j),f}_{\mathcal{H},\beta}(-) \mid_U . \quad (17)$$

Indeed, the above holds for the restrictions to $I(C_j)$. Furthermore, considering the Gibbsian (built back) viewpoint, it is clear that under the condition $\mathbf{C}_j$, the spins on the inside of C_j have the same distribution as the spin-system with free boundary conditions on C_j. Explicitly,

$$g^{\odot}_{\mathcal{H},\beta}(- \mid \mathbf{C}_j) \mid_{I(C_j)} = g^{I(C_j),f}_{\mathcal{H},\beta}(-), \quad (18)$$

where $g^{\odot}_{\mathcal{H},\beta}(- \mid \mathbf{C}_j) \equiv \sum_\omega \mu^{\odot}_{\mathcal{H},\beta}(\omega \mid \mathbf{C}_j) g^{\odot}_{\mathcal{H},\beta}(- \mid \omega)$.

Next, let us define a version of FKG dominance that is slightly stronger than usual. Suppose that ν and μ are probability measures on some finite $\Omega_\Sigma = \{0,1\}^\Sigma$ and that ν FKG-dominates μ in the usual sense and in addition, for any $\Gamma \subset \Sigma$ and any configuration η_Γ on Γ, then for all $\omega_\Gamma \prec \eta_\Gamma$, we have $\mu(- \mid \omega_\Gamma) \underset{\text{FKG}}{\leq} \nu(- \mid \eta_\Gamma)$. Then we will call such a relationship *extended* FKG dominance and express this relationship by the symbol $\underset{\text{FKG}}{\leq}_{\mathbf{e}}$. We remark in passing that in the above definition and in the decomposition that will follow below, there is no requirement that either measure have the FKG property in its own right.

Of importance in the present context is the fact that for systems of the type described, if $\odot$ is any boundary condition in the spin system, the corresponding grey measure satisfies

$$\nu^{FK;\Lambda,w}_{Q,P}(-) \underset{\text{FKG}}{\leq}_{\mathbf{e}} \mu^{\odot}_{\mathcal{H},\beta}(-), \quad (19)$$

where P and Q are as described in Proposition 1. First, if $\omega_\Gamma \in \Omega_\Gamma$, we claim that $\nu^{FK;\Lambda,w}_{Q,P}(- \mid \omega_\Gamma) \underset{\text{FKG}}{\leq} \mu^{\odot}_{\mathcal{H},\beta}(- \mid \omega_\Gamma)$. Indeed, the vacant bonds of ω_Γ are accounted for by considering a graph with these edges deleted. The occupied clusters of ω_Γ that are detached from the boundary constitute (part of) a boundary condition on $\Lambda \setminus \Gamma$ of the type described in Proposition 1, while the bonds in ω_Γ that attached to $\partial\Lambda$ are equivalent to a superposition of spin-state boundary conditions on this portion of $\partial(\Lambda \setminus \Gamma)$. But then, if $\eta_\Gamma \succ \omega_\Gamma$, we automatically have $\nu^{FK;\Lambda,w}_{Q,P}(- \mid \eta_\Gamma) \underset{\text{FKG}}{\leq} \nu^{FK;\Lambda,w}_{Q,P}(- \mid \omega_\Gamma)$ (and hence $\nu^{FK;\Lambda,w}_{Q,P}(- \mid \eta_\Gamma) \underset{\text{FKG}}{\leq} \mu^{\odot}_{\mathcal{H},\beta}(- \mid \omega_\Gamma)$) by the strong FKG property of the random cluster measures with $Q \geq 1$.

Whenever ν and μ are measures on an Ω_Σ with $\nu \underset{\text{FKGe}}{\leq} \mu$, we claim that μ admits a family of decompositions that are analogous to decompositions into conditional measures for cylinder events. However here the measure ν influences the nature of the "conditional measures" and completely determines the coefficients of the decomposition. We will illustrate with the simplest example: Let $\Gamma \subset \Sigma$. Then we claim that μ may be expressed as

$$\mu(-) = \sum_{\eta_\Gamma \in \Omega_\Gamma} \nu(\eta_\Gamma)\mu_{\eta_\Gamma}(-), \tag{20}$$

where each $\mu_{\eta_\Gamma}(-)$ is a probability measure which itself is a convex sum of measures obtained by conditioning on configurations ω_Γ that are below η_Γ:

$$\mu_{\eta_\Gamma}(-) = \sum_{\omega_\Gamma : \omega_\Gamma \prec \eta_\Gamma} \lambda_{\eta_\Gamma}(\omega_\Gamma)\mu(- \mid \omega_\Gamma) \tag{21}$$

with $0 \leq \lambda_{\eta_\Gamma}(\omega_\Gamma) \leq 1$ and $\sum_{\omega_\Gamma} \lambda_{\eta_\Gamma}(\omega_\Gamma) = 1$. Similar decompositions occur for a wider variety of partitioning events. In the general case, the conditioning events are not always situated on the same set; indeed Γ may be random but should be constructed via a growth algorithm. Of immediate importance is when Γ coincides with $C_j \cup E(C_j)$ whenever the event $\mathbb{C}_j$ occurs in the η-configuration.

One can formulate these decomposition in terms of "couplings" as follows: The pair $(\omega_\Gamma, \eta_\Gamma)$ are constrained in such a way that ω_Γ always lies below η_Γ and the conditional distribution of ω_Λ depends only on ω_Γ. The measures μ_{η_Γ} and λ_{η_Γ} can then be interpreted as the conditional distributions given η_Γ. Expressed in this language, the decomposition is not dissimilar to the couplings used in [vdBM] and in [N].

A precise statement of the generalization and a proof of the existence of such decompositions will be provided in the Appendix. On this basis, we have the following:

Proposition 2. *Let $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$, $U \subset \Lambda$, and let $\mathcal{H}$ be a Hamiltonian of the type described in Eq. (1) and the remarks that follow. Let $\mu^{\odot}_{\mathcal{H},\beta}(-)$ denote a grey measure with $\odot$ denoting boundary conditions coming from a single spin configuration or combinations thereof and let $\nu^{FK;\Lambda,w}_{Q,P}(-)$ denote the random cluster measure with wired boundary conditions on $\partial\Lambda$ and with parameters $Q = Q(\mathcal{H},\beta)$ and $P = P(\mathcal{H},\beta)$ as described in Proposition 1. Then $\mu^{\odot}_{\mathcal{H},\beta}(-)$ admits the decomposition*

$$\mu^{\odot}_{\mathcal{H},\beta}(-) = \sum_{j=0}^{N} \nu^{FK;\Lambda,w}_{Q,P}(\mathbb{C}_j)\mu^{\odot}_{\mathbb{C}_j;\mathcal{H},\beta}(-),$$

where for $j = 1 \ldots N$,

$$\mu^{\odot}_{\mathbb{C}_j;\mathcal{H},\beta}(-) \mid_{I(C_j)} = \mu^{I(C_j),f}_{\mathcal{H},\beta}(-),$$

and the measure $\mu^{\odot}_{\mathbb{C}_0;\mathcal{H},\beta}(-)$ (with $\mathbb{C}_0$ denoting the configurations where there is a connection between U and $\partial\Lambda$) is a certain combination of μ's that have been conditioned on cylinder events defined outside of U.

Proof. This is an immediate consequence of Corollary II to Proposition A.1. (Cf. also the discussion following the proof of Corollary II). □

Principal Results

Our principal results will follow after a few definitions and remarks pertaining to the random cluster models: Consider the random cluster models on $\mathbb{Z}^d$ with parameters $q \geq 1$ and $p \in (0,1)$. Define $P_\infty(q,p)$ as in Eq. (9) – here the optimizing boundary condition is known to be the wired boundary condition. Let $p_c(q)$ denote the percolation threshold:

$$p_c(q) = \inf\{p \mid P_\infty(q,p) > 0\}. \tag{22}$$

For the sequence of hypercubes Λ_L of side L centered at the origin, let $w_c(q)$ be defined as the supremum of the set of p's for which the estimate $P_{\Lambda_L}(q,p) \leq D_1 e^{-D_2 L}$ holds uniformly in L for some $D_1 < \infty$ and $D_2 > 0$.

Remark. Obviously $p_c(q) \geq w_c(q)$. An additional notion of a transition point may be defined: Assume for simplicity that $q \geq 1$ and let $\tau_n(p,q)$ denote the probability, in the limiting wired measure, that the origin and the point $(n,0,\ldots,0)$ are in the same connected cluster. Standard subadditive arguments show that

$$-\frac{1}{\xi} = \lim_{n\to\infty} \frac{\log \tau_n}{n} \tag{23}$$

exists. The point $\pi_c(q)$ is defined by $\pi_c(q) = \sup\{p \mid \xi(q,p) < \infty\}$. By straightforward arguments, $p_c(q) \geq \pi_c(q) \geq w_c(q)$. It is widely believed that for all q and in all dimensions, $p_c(q) = \pi_c(q) = w_c(q)$ and that in $d = 2$, the unique transition point is located at the self dual point $p_D(q) = \sqrt{q}/(1+\sqrt{q})$ (which, *a priori*, lies in $[\pi_c(q), p_c(q)]$). For $q = 1$, these issues have all been settled starting with [K] and ending with [AB, MMS]. For $q = 2$ (starting with [O] and ending with [ABF]) these problems have also been solved. For $q \gg 1$, a variety of techniques can be brought into play: Expansion techniques [LMMsRS] (for general q) or reflection positivity [KS, CM] (for integer q) can be used to show that $p_c(q) = \pi_c(q)$ and that in $d = 2$, these coincide with $p_D(q)$. The stronger result $p_c(q) = w_c(q)$ is established (for large q and general dimension) in [vEFSS]. Using different methods, exclusive to two dimensions, for $q \gtrsim 25.9$ the result $p_c(q) = w_c(q) = p_D(q)$ is proved in [Al_1 and G]. For $d = 2$ and integers (of relevance) between 3 and 25, it can be shown that $w_c(q) \geq p_D(q-1)$ [Al_2]. For $d \geq 3$ and moderately large q, it is known that $w_c(q) \geq [(q-1) - (q-1)^{\frac{d-1}{d}}]/[q-2]$ [Al_2] which agrees, to lowest non-trivial order, with the large q expansion for $p_c(q)$ in [LMMsRS].

Theorem 3. *Consider a spin-system with Hamiltonian $\mathcal{H}$ at inverse temperature β as described in Eq. (1) and in the paragraph that follows and let Q and P denote the quantities defined in Proposition 1. Then if $P < p_c(Q)$, there is a unique limiting measure for the spin-system and if $P < w_c(Q)$, the spin-system has weak mixing.*

Proof. Let $\Lambda \subset \mathbb{Z}^d$, $U \subset \Lambda$ and consider the spin system with two boundary conditions on $\partial\Lambda$ denoted by $\odot$ and $\otimes$. Let $\text{Var}_U(g^{\otimes}_{\mathcal{H},\beta}, g^{\odot}_{\mathcal{H},\beta})$ denote the variational distance between the two measures:

$$\text{Var}_U(g^{\otimes}_{\mathcal{H},\beta}, g^{\odot}_{\mathcal{H},\beta}) = \frac{1}{2}\int d^{|U|}s\, |g^{\otimes}_{\mathcal{H},\beta}(s_U) - g^{\odot}_{\mathcal{H},\beta}(s_U)|. \tag{24}$$

Finally, let $\mathcal{T}_{U,\partial\Lambda} \subset \Omega_{\mathbb{B}_\Lambda}$ denote the event that there is a connection between U and $\partial\Lambda$. (Note that $\mathcal{T}_{U,\partial\Lambda} = \mathbb{C}_0$.) We claim that $\text{Var}_U(g^{\otimes}_{\mathcal{H},\beta}, g^{\odot}_{\mathcal{H},\beta})$ is bounded above by $2\nu^{FK;\Lambda,w}_{Q,P}(\mathcal{T}_{U,\partial\Lambda})$.

Let us start things off with the following elementary consequence of the decomposition described in Proposition 2: For each $\omega \in \Omega_{\mathbb{B}_\Lambda}$, let $J_U(\omega)$ denote those bonds in the connected component of U. If $\zeta \subset \mathbb{B}_\Lambda$ is a set of bonds that are connected to U (and therefore a candidate to be J_U) we claim that

$$\mu^{\otimes}_{\mathcal{H},\beta}(J_U = \zeta) - \mu^{\odot}_{\mathcal{H},\beta}(J_U = \zeta) = \nu^{FK;\Lambda,w}_{Q,P}(\mathcal{T}_{U,\partial\Lambda})[\mu^{\otimes}_{\mathbb{C}_0;\mathcal{H},\beta}(J_U = \zeta) - \mu^{\odot}_{\mathbb{C}_0;\mathcal{H},\beta}(J_U = \zeta)]. \tag{25}$$

Indeed, for $j = 1, \ldots, N$, if ζ pokes through C_j, then both $\mu^{\otimes}_{\mathbb{C}_j;\mathcal{H},\beta}(J_U = \zeta)$ and $\mu^{\odot}_{\mathbb{C}_j;\mathcal{H},\beta}(J_U = \zeta)$ are zero because these measures insist that all the bonds of C_j are vacant. On the other hand, if $C_j \cap \zeta = \emptyset$, the event $J_U = \zeta$ is determined in $I(C_j)$ where these measures agree. Thus, the only surviving term in the difference of the two decompositions from Proposition 2 is the zeroth which is the right-hand side of Eq. (25).

In what follows, we will label a generic ζ with a subscripted G (for good) if no bond of ζ touches $\partial\Lambda$ and with a subscripted B otherwise. Recall, for $\omega \in \Omega_{\mathbb{B}_\Lambda}$, the objects $g^{\odot}_{\mathcal{H},\beta}(s_\Lambda \mid \omega)$ or the similarly defined $g^{\odot}_{\mathcal{H},\beta}(s_U \mid \omega)$ obtained by integrating out the spins in $\Lambda \setminus U$. It is clear that $g^{\odot}_{\mathcal{H},\beta}(s_U \mid \omega)$ depends only on $J_U(\omega)$ so we may write $g^{\odot}_{\mathcal{H},\beta}(s_U \mid J_U(\omega))$ for these conditional densities. However, if $J_U(\omega) = \zeta_G$ for some "good" ζ_G then the density is independent of $\odot$ and we will write $g_{\mathcal{H},\beta}(s_U \mid \zeta_G)$. We thus have

$$\begin{aligned} g^{\odot}_{\mathcal{H},\beta}(s_U) &= \sum_{\omega} \mu^{\odot}_{\mathcal{H},\beta}(\omega) g^{\odot}_{\mathcal{H},\beta}(s_U \mid \omega) \\ &= \sum_{\zeta} \mu^{\odot}_{\mathcal{H},\beta}(J_U(\omega) = \zeta) g^{\odot}_{\mathcal{H},\beta}(s_U \mid \zeta) \end{aligned}$$

$$= \sum_{\zeta_G} \mu^{\odot}_{\mathcal{H},\beta}(J_U(\omega) = \zeta_G) g_{\mathcal{H},\beta}(s_U \mid \zeta_G) + \sum_{\zeta_B} \mu^{\odot}_{\mathcal{H},\beta}(J_U(\omega) = \zeta_B) g^{\odot}_{\mathcal{H},\beta}(s_U \mid \zeta_B). \tag{26}$$

Obviously there is a similar expression for the $\otimes$ boundary condition.

The variational distance may now be estimated:

$$\begin{aligned} 2\mathrm{Var}_U(g^{\otimes}_{\mathcal{H},\beta}, g^{\odot}_{\mathcal{H},\beta}) \le \int d^{|U|}s \sum_{\zeta_G} g_{\mathcal{H},\beta}(s_U \mid \zeta_G) |\mu^{\odot}_{\mathcal{H},\beta}(J_U(\omega) = \zeta_G) - \\ - \mu^{\otimes}_{\mathcal{H},\beta}(J_U(\omega) = \zeta_G)| + \int d^{|U|}s \sum_{\zeta_B} \mu^{\odot}_{\mathcal{H},\beta}(J_U(\omega) = \zeta_B) g^{\odot}_{\mathcal{H},\beta}(s_U \mid \zeta_B) + \\ + \sum_{\zeta_B} \mu^{\otimes}_{\mathcal{H},\beta}(J_U(\omega) = \zeta_B) g^{\otimes}_{\mathcal{H},\beta}(s_U \mid \zeta_B). \end{aligned} \tag{27}$$

In both terms, we may now simply integrate the spins away – these are *probability* densities. The remains of the first term can be estimated using Eq. (25):

$$\begin{aligned} \sum_{\zeta_G} |\mu^{\odot}_{\mathcal{H},\beta}(J_U(\omega) = \zeta_G) - \mu^{\otimes}_{\mathcal{H},\beta}(J_U(\omega) = \zeta_G)| \le \\ \le \nu^{FK;\Lambda,w}_{Q,P}(\mathcal{T}_{U,\partial\Lambda}) \sum_{\zeta} [\mu^{\odot}_{\mathbb{C}_0;\mathcal{H},\beta}(J_U(\omega) = \zeta) + \mu^{\otimes}_{\mathbb{C}_0;\mathcal{H},\beta}(J_U(\omega) = \zeta)] = \\ = 2\nu^{FK;\Lambda,w}_{Q,P}(\mathcal{T}_{U,\partial\Lambda}). \end{aligned} \tag{28}$$

Meanwhile, the second term is just $\mu^{\odot}_{\mathcal{H},\beta}(\mathcal{T}_{U,\partial\Lambda}) + \mu^{\otimes}_{\mathcal{H},\beta}(\mathcal{T}_{U,\partial\Lambda})$ which by the (basic) domination is also bounded by $2\nu^{FK;\Lambda,w}_{Q,P}(\mathcal{T}_{U,\partial\Lambda})$. We have established the claim that followed Eq. (24).

The theorem is now easily proved: if $P < p_c(Q)$ then for fixed U, $\nu^{FK;\Lambda,w}_{Q,P}(\mathcal{T}_{U,\partial\Lambda})$ vanishes as $\Lambda \nearrow \mathbb{Z}^d$ while if $P < w_c(Q)$, it is easily shown that there are positive constants D_3 and D_4 that are independent of U and Λ, such that

$$\nu^{FK;\Lambda,w}_{Q,P}(\mathcal{T}_{U,\partial\Lambda}) \le D_3 \sum_{\substack{x\in\partial U\\ y\in\partial\Lambda}} e^{-D_4|x-y|} \tag{29}$$

which implies weak mixing. □

Corollary. *In two dimensions, for the above systems with a discrete spin-space, the condition $P < w_c(Q)$, implies that the interaction $\beta\mathcal{H}$ has the restricted complete analyticity property.*

Proof. This is an application of [MOS] where it was established that for two dimensional discrete spin-systems, weak mixing implies "strong mixing for squares" ≡ restricted complete analyticity. □

For $d > 2$, the following is of interest:

Theorem 4. *For discrete spin-systems of the type described in the statement of Theorem 3, if $P < p_c(1)$, the interaction $\beta\mathcal{H}$ is completely analytic.*

Proof. We will use condition IIIc of [DS] which for present purposes may be read as follows: Let Λ and U denote sets as described earlier and let $s_{\partial\Lambda}$ and $s'_{\partial\Lambda}$ denote two boundary conditions that differ only at a single site $y \in \partial\Lambda$. Then complete analyticity follows if for all y and for any such $s_{\partial\Lambda}$ and $s'_{\partial\Lambda}$,

$$\mathrm{Var}_U(g^{\Lambda,s_{\partial\Lambda}}_{\mathcal{H},\beta}, g^{\Lambda,s'_{\partial\Lambda}}_{\mathcal{H},\beta}) \le D_5 \sum_{x\in\partial U} e^{-D_6|x-y|} \tag{30}$$

with D_5 and D_6 positive and independent of U, Λ and the boundary conditions $s_{\partial\Lambda}$ and $s'_{\partial\Lambda}$.

The strategy is identical to that used in the proof of Theorem 3 except that here our partitioning events, $\mathbb{A}_1, \ldots, \mathbb{A}_N$ feature the surfaces $A_1, \ldots, A_N$ that separate U from y. Indeed, let $\mathbf{A}_j$ denote the event that all the bonds in A_j are vacant and let $I(A_j)$ denote the region that can be reached by a path inside Λ that starts from U and does not use any bond in A_j. Then $\mu^{\Lambda,s_{\partial\Lambda}}_{\mathcal{H},\beta}(- \mid \mathbf{A}_j)_{|I(A_j)}$ is identical to the grey measure with the boundary condition provided by the configuration $s_{\partial\Lambda}$ restricted to $\partial I(A_j) \cap \partial\Lambda$ and free boundary conditions on A_j. It is evident that this is the same as the restriction of the similar conditional measure with $s_{\partial\Lambda}$ replaced by $s'_{\partial\Lambda}$. With this observation in mind, the derivation is now identical to the one in the previous theorem with the result

$$\mathrm{Var}_U(g^{\Lambda,s_{\partial\Lambda}}_{\mathcal{H},\beta}, g^{\Lambda,s'_{\partial\Lambda}}_{\mathcal{H},\beta}) \le 2\nu^{FK;\Lambda,w}_{Q,P}(\mathcal{T}_{U,y}), \tag{31}$$

where $\mathcal{T}_{U,y}$ is defined similarly to the previous $\mathcal{T}$'s. By the well known domination inequalities, we may replace Q by 1 in which case the stipulation "wired" is meaningless. Finally we have $\nu^{\Lambda}_{1,P}(\mathcal{T}_{U,y}) \le \sum_{x\in U} \nu^{\Lambda}_{1,P}(\mathcal{T}_{x,y}) \le e^{|x-y|/\xi(P)}$, where ξ, the correlation length, is positive for $P < p_c(1)$. Thus, with $D_5 = 2$ and $D_6 = 1/\xi(P)$, complete analyticity is established. □

Brief Comparison to Other Methods

Actual (general) conditions under which expansions converge are often "nearly existential" in their statement and then easily proved at high temperature or low activity. One exception is [D_2] where a long string of equations leading back to Eq. (2.8) allows us to calculate $P < c/2d \Longrightarrow$ CA (complete analyticity) where $c \approx .082$. Of course orthodox enthusiasts will argue that this condition is not optimal. In [KP], one of the better bounds is obtained. Here it is required that $e^{-\tau} < k/2d$ with $k \approx .206$, where $e^{-\tau|\gamma|}$ provides a bound on the weight of the "contour functional" for contours γ of length $|\gamma|$. In simple cases, it can be shown that $e^{-\tau} \sim R_0 \sim P$ (if $P \ll 1$) hence this is similar to the above mentioned with the constant improved by a factor of 2–3. By contrast, substituting the mean-field bound $p_c(1) > \lambda^{-1}(d) > 1/(2d-1)$ with $\lambda(d)$ the connectivity constant of the lattice, we obtain $P < 1/(2d-1) \Longrightarrow$ CA. This represents a substantial improvement if $d \gg 1$ and, even more so in moderate dimensions because λ and/or $p_c(1)$ have improved estimates.

The second general method involves the calculation of variational norms. The most prominent example is the original result of [D_1] which, in the present context, reads $\rho_D < 1/2d \Longrightarrow$ CA. (See [DS].) Here ρ_D is the maximum variational distance between two single site measures whose neighbors differ at a single site. The more recent results of [vdBM] provides $\rho_{BM} < s_c(d)$, where $s_c(d)$ is the site percolation threshold and ρ_{BM} is similar to ρ_D, but here the boundary conditions are allowed to be different at any or all the neighboring sites. Thus $\rho_D \leq \rho_{BM}$. For highly frustrated systems, it is argued [vdBM] that $\rho_D \approx \rho_{BM}$ and hence the advantage of s_c (where $s_c(d) \geq 1/(2d-1)$) versus $1/2d$. However, for ferromagnetic-type systems, the impact of the full neighborhood is felt and ρ_{BM} is significantly larger than ρ_D. (If $\beta \ll 1$ it is larger by a factor of 2d.) Thus the Dobrushin condition is usually better with the main advantage of [vdBM] coming in $d = 2$ (where $s_c = .59$ may be accepted on faith). Here, if we make certain uncontrolled approximations, we self consistently arrive at $\rho_D \approx HP$ with H a number in the range of 2-5. However, it is difficult to really tell: These variational norms are again easy to estimate as "small" for extreme values of parameters but, in practical situations, they are very difficult to work with, especially for the derivation of general conditions.

In this work, the two-dimensional systems divide into two cases: $Q \gg 1$ and Q of the order of one. In the latter case, it often happens that $Q \geq 2$ and we may compare directly to the Ising case which gives RCA (restricted complete analyticity) for $P < \sqrt{2}/(1+\sqrt{2})$. On the other hand, for large Q (thanks to [vEFSS]) our condition for uniqueness is the same as that of RCA which reads $\overline{R}R_0 < 1 \Longrightarrow$ RCA. Thus, in the example following the proof of Proposition 1 (discretized for convenience) the Corollary to Theorem 3 implies RCA down to temperatures satisfying $e^{\beta} < 1 + (\frac{2\overline{n}}{\epsilon})^{1/2}$ (which is correct to within constants [C]). An unadorned cluster expansion will not pick up the ϵ dependence and an actual variational calculation may even produce the wrong direction of the ϵ dependence (as in [CKS] for another large q model). Now it may be possible that a better variational calculation picks up the correct trend with ϵ. But for large q and no extra symmetry (as in the Potts models) the number of calculations really required is unmanageable. At this moment of writing, we believe that the most tangible asset of these methods is the intrinsic simplicity of the required calculations.

Appendix

Here we establish the decomposition formulas discussed in the text. Let us start off with the one bond case for measures with the usual sort of FKG domination:

Proposition A.1. *Consider measures $\nu(-)$ and $\mu(-)$ defined on some finite $\{0,1\}^\Sigma$ that satisfy $\nu \underset{\text{FKG}}{\leq} \mu$. Assume, without loss of generality, that $\forall b \in \Sigma$, $\nu(b = 1) \neq 0$. Then for any $b \in \Sigma$, we may write*

$$\mu(-) = \sum_{\eta_b=0,1} \nu(\eta_b)\mu_{\eta_b}(-),$$

where $\mu_{\eta_b=0}(-) = \mu(- \mid \omega_b = 0)$ and $\mu_{\eta_b=1}(-)$ is a convex combination of $\mu(- \mid \omega_b = 1)$ and $\mu(- \mid \omega_b = 0)$.

Proof. We write, tentatively,

$$\begin{aligned} \mu(-) = \nu(\eta_b = 0)\mu(- \mid \omega_b = 0) + \\ +\nu(\eta_b = 1)[\lambda\mu(- \mid \omega_b = 1) + (1 - \lambda)\mu(- \mid \omega_b = 0)] \end{aligned} \tag{A.1}$$

and attempt to solve for λ. This is accomplished by directly expanding μ in terms of its conditional measures and equating coefficients. The result is

$$\lambda = \frac{\mu(\omega_b = 1)}{\nu(\omega_b = 1)}; \tag{A.2}$$

this provides a sensible solution since $\nu \underset{\text{FKG}}{\leq} \mu$ implies $\lambda \leq 1$. □

Remark. If it happens that $\nu(b = 1) = 0$ the above decomposition is still valid (and trivial) if the formulas are properly interpreted – indeed, all the ill-defined measures in the decomposition appear with zero coefficient. Hereafter, we will assume this interpretation and omit the provisos analogous to $\nu(b = 1) \neq 0$.

An immediate corollary is the "fixed Γ" decomposition described in the text:

Corollary I. *Let Σ, μ, ν denote the quantities described above but now let us assume that $\nu(-) \underset{\text{FKG}\mathbf{e}}{\leq} \mu(-)$. Then for any $\Gamma \subset \Sigma$, we may write*

$$\mu(-) = \sum_{\eta_\Gamma} \nu(\eta_\Gamma)\mu_{\eta_\Gamma}(-),$$

where the sum runs over all $\eta_\Gamma \in \{0,1\}^\Gamma$ and where the $\mu_{\eta_\Gamma}(-)$ are convex combinations of the μ measure conditioned on configurations $\omega_\Gamma \in \{0,1\}^\Gamma$ that lie below η_Γ:

$$\mu_{\eta_\Gamma}(-) = \sum_{\omega_\Gamma:\ \omega_\Gamma \prec \eta_\Gamma} \lambda_{\eta_\Gamma}(\omega_\Gamma)\mu(- \mid \omega_\Gamma),$$

$0 \leq \lambda_{\eta_\Gamma} \leq 1$ *and* $\sum_{\omega_\Gamma} \lambda_{\eta_\Gamma}(\omega_\Gamma) = 1$.

Proof. Suppose that such a measure can be constructed for any $\Gamma \subset \Sigma$ with k elements, let Γ denote one such example and let $\Gamma' = \Gamma \cup b$ with $b \in \Sigma \setminus \Gamma$. We write the full expansion for $\mu(-)$ with the further expansion of $\mu(- \mid \omega_\Gamma)$ into the two possibilities for ω_b:

$$\mu(-) = \sum_{\eta_\Gamma} \nu(\eta_\Gamma) \sum_{\omega_\Gamma \prec \eta_\Gamma} \lambda_{\eta_\Gamma}(\omega_\Gamma)[\mu(\omega_b = 1 \mid \omega_\Gamma)\mu(- \mid \omega_\Gamma, \omega_b = 1) + \\ + \mu(\omega_b = 0 \mid \omega_\Gamma)\mu(- \mid \omega_\Gamma, \omega_b = 0)]. \tag{A.3}$$

Now we wish to write

$$\mu(-) = \sum_{\eta_\Gamma} [\nu(\eta_\Gamma, \eta_b = 1) \sum_{\omega_\Gamma \prec \eta_\Gamma} \lambda_{\eta_\Gamma, \eta_b = 1}(\omega_\Gamma, \omega_b = 1)\mu(- \mid \omega_\Gamma, \omega_b = 1) + \\ + \nu(\eta_\Gamma, \eta_b = 1) \sum_{\omega_\Gamma \prec \eta_\Gamma} \lambda_{\eta_\Gamma, \eta_b = 1}(\omega_\Gamma, \omega_b = 0)\mu(- \mid \omega_\Gamma, \omega_b = 0) + \\ + \nu(\eta_\Gamma, \eta_b = 0) \sum_{\omega_\Gamma \prec \eta_\Gamma} \lambda_{\eta_\Gamma, \eta_b = 0}(\omega_\Gamma, \omega_b = 0)\mu(- \mid \omega_\Gamma, \omega_b = 0)].$$

As we will demonstrate, this can be (non-uniquely) accomplished by simply equating coefficients. First off, we are forced with

$$\lambda_{\eta_\Gamma, \eta_b = 1}(\omega_\Gamma, \omega_b = 1) = \frac{\mu(\omega_b = 1 \mid \omega_\Gamma)}{\mu(\eta_b = 1 \mid \eta_\Gamma)} \lambda_{\eta_\Gamma}(\omega_\Gamma) \tag{A.4a}$$

which lies in $[0, 1]$ by the inductive assumption (for $\lambda_{\eta_\Gamma}(\omega_\Gamma)$) and the extended FKG dominance. As for the remainder, there is still a great deal of leeway. A natural choice is

$$\lambda_{\eta_\Gamma, \eta_b = 0}(\omega_\Gamma, \omega_b = 0) = \lambda_{\eta_\Gamma}(\omega_\Gamma), \tag{A.4b}$$

which leaves us with

$$\lambda_{\eta_\Gamma, \eta_b = 1}(\omega_\Gamma, \omega_b = 0) = [1 - \frac{\mu(\omega_b = 1 \mid \omega_\Gamma)}{\mu(\eta_b = 1 \mid \eta_\Gamma)}] \lambda_{\eta_\Gamma}(\omega_\Gamma). \tag{A.4c}$$

This provides the desired decomposition. □

Remark. It is reemphasized that the above constructed $\mu_{\eta_\Gamma}(-)$ is by no means the only possibility. Indeed, the order in which the bonds of Γ are "processed" appears to effect the details of the outcome. We also remark that all of the above results (and those which follow) can be derived from the technically weaker condition than extended dominance. Indeed, all that is needed is the corresponding inequality for single site occupations. However, it is hard to imagine a system that satisfies the weaker condition without enjoying the stronger property.

The generalizations to situations of the sort needed for this work follow from the consideration of partitioning events that can be defined via a *growth algorithm*. Let us start with an informal description: The algorithm starts with some (predetermined) bond b_0. The bond is "checked" to see if η_{b_0} is occupied or vacant. Depending on the outcome, the algorithm picks a new bond $b_1(\eta_{b_0})$. This new bond is checked and, depending on η_{b_0} and η_{b_1}, a new bond b_2 is determined and so forth. The procedure continues until a *stopping* condition is fulfilled (which depending on the algorithm could even happen

on the first step). The algorithm is defined so that any possible choice of outcomes will eventually lead to a stopping condition – this partitions the configuration space.

Let us illustrate this procedure with the "fixed set" rule as featured in the above corollary. The bonds of Γ are deterministically ordered, $b_0, b_1, \ldots, b_N$ and the algorithm dictates that after b_{k-1} has been checked, go to the bond b_k, $k = 1, 2, \ldots N$; after the bond b_N has been checked, stop.

The formal definition in the general case is as follows:

Definition. *Let $\Omega_\Sigma = \{0,1\}^\Sigma$ be a finite space and consider a growth algorithm $\Phi = (\Phi_1, \ldots \Phi_M; b_0)$ defined as follows: The Φ_k are functions with values in $\Sigma \cup$ [stop] and the domains are particular configurations on particular subsets, Γ of Σ. In general, if η_Γ is in the domain of Φ_k and $\Phi_k(\eta_\Gamma) \neq$ [stop], then $\Phi_k(\eta_\Gamma) \in \Sigma \setminus \Gamma$. Thus all possible "alive" sets at the k^{th} stage are of size $k+1$. Let $\Xi_k = \{(\Gamma_\alpha, \eta_{\Gamma_\alpha})\}$ denote the collection of all possible alive sets and configurations at the k^{th} stage. These may be generated as follows: $\Xi_0 = \{(b_0, \eta_{b_0})\}$,*

$$\Xi_{k+1} = \{(\Gamma, \eta_\Gamma) \mid \Gamma = \Gamma' \cup \Phi_{k+1}(\eta_{\Gamma'}); \Phi_{k+1}(\eta_{\Gamma'}) \neq [\text{stop}] \\ \eta_\Gamma = (\eta_{\Gamma'}, \eta_{\Phi_{k+1}(\eta_{\Gamma'})}); (\Gamma', \eta_{\Gamma'}) \in \Xi_k\}. \tag{A.5}$$

It is required that Φ_{k+1} be defined on all η_Γ with $(\Gamma, \eta_\Gamma) \in \Xi_k$. Finally, if relevant, when $k = ||\Gamma|| - 1$ so the first component of each $\Xi_{||\Gamma||-1}$ is all of Σ, it is required that $\Phi_{||\Gamma||-1}(\eta_\Sigma) \equiv$ [stop].

It is evident that such an algorithm partitions Ω_Σ into disjoint events: $\Omega_\Sigma = \cup_y \mathbb{K}_y$ with $\mathbb{K}_y \cap \mathbb{K}_{y'} = \emptyset$ if $y \neq y'$. The $\mathbb{K}_y$ are defined as cylinder events: $\mathbb{K}_y = \{\eta_{\Gamma_\alpha}; (\Gamma_\alpha, \eta_{\Gamma_\alpha}) \in \Xi_k$ for some k, $\Phi_{k+1}(\eta_{\Gamma_\alpha}) =$ [stop]$\}$. With this in mind, the generalization used in the text is another corollary.

Corollary II. *Let Σ, μ, ν be defined as in Corollary I and let Φ denote a growth algorithm as defined above. Then μ can be decomposed:*

$$\mu(-) = \sum_\alpha \nu(\eta_{\Gamma_\alpha}) \mu_{\eta_{\Gamma_\alpha}}(-),$$

where η_{Γ_α} denote the partitioning cylinder events of the algorithm and where

$$\mu_{\eta_{\Gamma_\alpha}}(-) = \sum_{\omega_{\Gamma_\alpha} \prec \eta_{\Gamma_\alpha}} \lambda_{\eta_{\Gamma_\alpha}} \mu(- \mid \omega_{\Gamma_\alpha}),$$

with the $\lambda_{\eta_{\Gamma_\alpha}}$ denoting convex coefficients.

Proof. Fix n and suppose that the corollary is true for any algorithm that always stops by the n^{th} stage. (That is the cardinality of every "alive" set is at most n) This holds if $n = 1$ by Proposition A.1 and now the method of Corollary I readily establishes Corollary II by induction on n. □

Example. For the situation discussed in Proposition 2, the growth algorithm is straightforward: The bonds of Λ are deterministically ordered and b_0 is defined to be the lowest bond that is touching the boundary of Λ. The connected (occupied) cluster of b_0 is explored using the ordering to determine the sequence of bonds checked until this cluster hits the boundary of U or is fully explored (i.e. cutoff by vacant bonds). In the former case, we get a [stop] and in the latter case, we uncover the lowest *unexplored* bond touching the boundary of Λ and repeat. If this procedure exhausts all the bounds touching

the boundary of Λ and never gets to U, then there is evidently no occupied connection between $\partial\Lambda$ and ∂U and we get a [stop] when the cluster of the last bond touching $\partial\Lambda$ has been fully explored. In the latter cases, the exposed configuration defines the outermost separating surface C_j, $j = 1, \ldots, N$. For conceptual convenience, we may take all the η_Γ's that produce the same event $\mathbb{C}_j$ and combine the resulting μ_{η_Γ}'s into $\mu_{\mathbb{C}_j}(-)$, $j = 1, \ldots, N$. What remains – where the connected component of $\partial\Lambda$ succeeded in reaching U, defines the event $\mathbb{C}_0$ and the measure $\mu_{\mathbb{C}_0}(-)$. In a similar fashion, one can define a growth algorithm for the decomposition used in Theorem 4.

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