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Ballistic behavior for biased self-avoiding walks

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Abstract

For self-avoiding walks on the d-dimensional cubic lattice defined with a positive bias in one of the coordinate directions, it is proved that the drift in the favored direction is strictly positive.

Keyword: Biased self-avoiding walks

1. Introduction

From the mathematical perspective, much progress has been made in the understanding of self-avoiding walks (SAW). Specifically, in [1], diffusive behavior was established in dimensions (somewhat) higher than the upper critical dimension; a result that was extended and generalized in [9] and, more recently, improved in [6]; cf. [6] and references therein. From an alternative perspective, the problem was treated as a system approaching a phase transition in [2]. The second-order nature of the transition was established and the non-critical behavior was characterized. However, some of the most basic questions concerning physically "obvious" properties of the SAW remain open; e.g. general arguments which, for $d \geq 2$, preclude subdiffusive behavior and/or demonstrate subballistic behavior.

Related to the aforementioned is the problem of *biased* self-avoiding walks (BSAW). For current purposes these may be loosely defined as SAWs with a tendency to prefer movement in some particular direction. Like for the above "basics" for the SAW, it is readily argued that such a system actually *should* behave ballistically. Specifically, in an N-step BSAW, the endpoint of the walk should lie a distance $\sim vN$ from its point of origin, in the preferred direction, with $v > 0$. Some advance on this question was achieved in the works [5,8]. In [5], for $d > 4$,

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and for sufficiently strong bias (and for a particular softening of the self-avoidance constraint — also utilized in [1]) ballistic behavior was indeed established using the methods of the *lace expansion* pioneered in [1]. In [8], the one-dimensional case – here, of course, with the constraint softened – has been characterized at the level of central limit theorems. Notwithstanding, it appears that the general question has not yet been addressed. In this note, for all dimensions and any non-vanishing bias, ballistic behavior will be established for the BSAW. In particular, as will be demonstrated, the problem may be easily and entirely understood by consideration of the subcritical SAWs that were analyzed in [3,2].

2. Definitions and pertinent results

2.1. Standard approach to SAW and BSAW

In the normal definition of the BSAW, e.g. the analog of the definition in [5], one must first consider all N-step self-avoiding walks emanating from (say) the origin of the d-dimensional hypercubic lattice, $\mathbb{Z}^d$. For the sake of non-triviality, it is assumed throughout that $d \geq 2$ and, by an SAW, what is meant is a sequence $(x_0, \dots x_N)$ of points in $\mathbb{Z}^d$ with x_j and x_{j+1} neighbors and the N points in the sequence all distinct. For the standard SAW, one is supposed to examine the uniform measure on these walks with the emphasis on the large N properties of this measure. Biased walks, which depend on one or more parameters, are defined as follows: For a given SAW starting from 0 (denoted $w : 0 \to \cdot$) let $N_r(w)$ denote the number of "right-moving" steps in w, i.e. (the number of) successive points of the walk where the first coordinate of x_j increases by $+1$ in the step to x_{j+1}. Similarly, let N_ℓ denote the number of "left-moving" steps. Let $\epsilon_r > 0$ and $\epsilon_\ell < \epsilon_r$. The measure $M_{N,\underline{\epsilon}}$ is now constructed, which assigns a weight to the N-step SAW emanating from the origin, that is given by

$$M_{N,\underline{\epsilon}}(w) \propto \mathrm{e}^{[\epsilon_r N_r(w) + \epsilon_\ell N_\ell(w)]} \tag{2.1}$$

and, for fixed $\underline{\epsilon} = (\epsilon_r, \epsilon_\ell)$, one is supposed to examine the $N \gg 1$ properties of $M_{N,\underline{\epsilon}}$. More generality is of course possible in the construction of the drift but it does not appear that such generality will introduce any significant additional features. For the purposes of this note, let us be content with the case $\epsilon_r = -\epsilon_\ell$ and denote the mutual parameter by ϵ.

A further possibility, which will not be described with precision, is the so-called self-repellant (Domb–Joyce) walk. These walks (a principal object of study in [1,5,8]) contain an additional parameter $\lambda \in [0, 1]$ where $\lambda = 0$ corresponds to the random walk (with or without drift depending on ϵ) and $\lambda = 1$ the SAW (or BSAW for $\epsilon \neq 0$). Intermediate values of λ disfavor, but do not disallow, self-intersections of the walk. It is in the context of the self-repellant walks that ballistic behavior was shown in [5] under the conditions that λ is not too large, ϵ is not too small and $d > 4$. While no specific claims are being made, it is likely that the methods here apply equally well to the self-repellant cases.

The primary result of this note may be stated:

Theorem 2.1. *Let $\epsilon > 0$ and consider the N-step BSAWs as described above with $\epsilon_r = \epsilon = -\epsilon_\ell$. Let X_N denote the displacement of the endpoint in the first coordinate direction. Then there is a $v(\epsilon) > 0$ such that*

$$\frac{X_N}{N} \Longrightarrow v, \tag{2.2}$$

i.e. $\frac{X_N}{N}$ converges, in probability, to a positive, deterministic drift.

2.2. Alternative approach to SAW and BSAW

As an alternative to the definitions of the previous subsection, here the objects of interest are the so-called *generating functions* (or Green's functions). There is some variety in the species but for present purposes let us define the fundamental object:

Definition 2.2. Let x and y denote distinct points in $\mathbb{Z}^d$ and $\beta > 0$ a real number. We consider the (formal) sum

$$G_{x,y} = G_{x,y}(\beta) = \sum_{w:x\to y} \mathrm{e}^{-\beta|w|}. \tag{2.3}$$

In the above, the sum is over all SAW that start at x and end at y and $|w|$ denotes the length (number of steps) of the walk w.

The above object is formal in the sense that, depending on β, it may or may not be a convergent sum. However, it is not hard to see that there is a β_c above which $G_{x,y}(\beta)$ tends to zero exponentially fast as $|x - y| \to \infty$ and below which it does not. The usual model for SAW, namely uniform measure on various N-step walks emanating from the origin, is presumed to be equivalent to the study of the $G_{x,y}$ at $\beta = \beta_c$. Unfortunately, a general argument to this effect seems to be lacking, in particular below the upper critical dimension. (For $d \geq 5$, equivalence can be established: For cases with softened constraint this was achieved in [7] and, recently, for the standard SAW model this was derived in [6].) However, as we shall see, the full task is quite manageable for the BSAW. First let us attend to some preliminaries:

In addition to $G_{x,y}$, let us (re)introduce some notation: $G_{L;a}$ represents the weighted sum over walks that start at the origin and end at a distance L on the x_1 axis with a representing the other $(d-1)$ coordinates of the endpoint, i.e. $G_{L;a} = G_{0,(L,a)}$. In addition, let $G_L = G_{L;0}$ and finally $\mathbb{G}_L = \sum_a G_{L;a}$. For these various functions, if $\beta \neq \beta_c$, it is not difficult to show that, e.g.,

$$m(\beta) = \lim_{L\to\infty} -\frac{1}{L}\log G_L = \lim_{L\to\infty} -\frac{1}{L}\log \mathbb{G}_L \tag{2.4}$$

exists. It has been demonstrated by elementary means [3] (see also [4]) that $m(\beta)$ is a concave, non-decreasing function on (β_c, ∞). Furthermore, the Legendre transform of m has, at least formally, some geometric significance. Indeed, denoting the dual variable by κ and the Legendre transform by $\zeta(\kappa)$, then $\kappa(\beta)$ has the interpretation that walks of length κL are the principal contributors to $\mathbb{G}_L(\beta)$ while $\zeta(\kappa)$ is the *entropy* associated with walks of this length. These matters are bolstered with the detailed properties of $m(\beta)$ that were proved in [2,3]:

Proposition 2.3. *For the function $m(\beta)$ associated with the SAW on $\mathbb{Z}^d$; $d \geq 2$:*

(i) *For $\beta < \beta_c$, $m(\beta) \equiv -\infty$.*
(ii) *For $\beta > \beta_c$, $m(\beta)$ is analytic.*
(iii) *The function $m(\beta) \downarrow 0$ as $\beta \downarrow \beta_c$.*

Thus, on the basis of property (ii), and some straightforward considerations, $m(\beta)$ is *strictly* increasing and, indeed, has a smooth derivative. Further, as a consequence of some elementary analyses when $\beta \gg 1$, this derivative is not identically a constant on any open interval. Hence each $\beta \in (\beta_c, \infty)$ "selects" a (unique) $\kappa(\beta)$ which is given by

$$\kappa(\beta) = \frac{\partial m}{\partial \beta} \tag{2.5}$$

and it is indeed possible to conclude that for any fixed $\tilde{\kappa} \neq \kappa(\beta)$, as $L \to \infty$, walks with $\tilde{\kappa}L$ steps make only an exponentially small contribution to $\mathbb{G}_L(\beta)$.

As we shall see, from a certain perspective, the chief difference between the BSAW and the SAW is item (i). In particular, the analog of m (which will be denoted by μ) is finite below the critical β and analytic even while it passes through zero. Thus, from the viewpoint of traditional statistical mechanics, the results here are analogous to a Lee–Yang theorem for ferromagnetic spin systems with the bias playing the role of the magnetic field: A non-vanishing bias spoils the non-analyticity of μ when it is supposed to exhibit critical behavior. However, it should be cautiously noted that the analogy to an external field in a magnetic system cannot be pushed too far. In particular, it does not appear that the introduction of *drift* provides an independent critical exponent for walk-type systems. Indeed, on the basis of Theorem 2.5 and elementary scaling considerations, it follows that if $m(\beta) \sim (\beta - \beta_c)^{\nu}$ then $v(\epsilon) \sim \epsilon^{\frac{1-\nu}{\nu}}$. This is in sharp contrast to the case for "analogous" circumstances in ferromagnets where an additional (independent) exponent is required to describe the approach to criticality as the field is removed.

The biased walks which will be studied in this note may be defined via generating functions in much the same spirit as for Definition 2.2.

Definition 2.4. Let $\beta > 0$ and $\epsilon > 0$ (with ϵ not necessarily "small") and let $D_{x,y}$ denote the formal sum

$$D_{x,y} = D_{x,y}(\beta, \epsilon) = \sum_{w:x \to y} \mathrm{e}^{-\beta|w|} \mathrm{e}^{\epsilon N_r(w) - \epsilon N_\ell(w)}. \tag{2.6}$$

The notation from the ordinary SAW will be adapted. Thus $\mu(\beta, \epsilon)$ will stand for the limit of $-\frac{1}{L} \log D_L$ (or $-\frac{1}{L} \log \mathbb{D}_L$) — the existence of which is readily verified and, in any case, will be dispensed with later on. Further let us define $b_c(\epsilon)$ to be the value of β above which $\mu(\beta, \epsilon)$ is positive and below which it is not.

The foundational result of this note is the following:

Theorem 2.5. *Consider the BSAW as described above and let $\epsilon > 0$ be fixed. Then:*

(0) *The limit $\mu(\beta, \epsilon)$ as described above exists (in $\{-\infty\} \cup [-\epsilon, \infty)$).*

(i) *There is a $\theta < b_c(\epsilon)$ such that $\mu(\beta, \epsilon) > -\infty$ for all $\beta > \theta$.*

(ii) *The function $\mu(\beta, \epsilon)$ is analytic for all $\beta > \theta$; in particular at $\beta = b_c$.*

(iii) *The derivative $\kappa(\beta, \epsilon) = \frac{\partial \mu}{\partial \beta}$ evaluated at $\beta = b_c$ is the inverse of the speed described in Theorem 2.1:*

$$v(\epsilon) = \frac{1}{\kappa(b_c, \epsilon)}. \tag{2.7}$$

(iv) *All the above quantities pertaining to $\mu(\beta, \epsilon)$ and its derivatives may be calculated from the ordinary SAW at the parameter value β^*_ϵ that satisfies $m(\beta^*_\epsilon) = \epsilon$. In particular, $\forall \beta > \beta_c$,*

$$\mu(\beta, \epsilon) = m(\beta) - \epsilon. \tag{2.8}$$

3. Proofs

Proof of Theorem 2.5 (*Items (0), (i), (ii) and (iv)*). The seminal identity, perhaps already obvious to the reader, is that $\forall L$,

$$\mathbb{D}_L = \mathrm{e}^{\epsilon L} \mathbb{G}_L(\beta). \tag{3.1}$$

Indeed, to prove this formally, if $(L, a) \in \mathbb{Z}^d$ (with L denoting the first and a denoting the remaining $d-1$ coordinates) and $w : 0 \to (L, a)$ then $N_r(w) - N_\ell(w) = L$. The identity in Eq. (3.1) now follows walk by walk. The existence of μ – which could easily have been proved by the conventional subadditive methods – and the identity

$$\mu(\beta, \epsilon) = m(\beta) - \epsilon \tag{3.2}$$

follow from the existence of m.

For fixed ϵ the strict monotonicity properties of $m(\beta)$ along with item (iii) of Proposition 2.3 allow us to find a β^*_ϵ such that $m(\beta^*_\epsilon) = \epsilon$ and thereby identify

$$b_c(\epsilon) = \beta^*_\epsilon. \tag{3.3}$$

Since $\epsilon = m(\beta^*_\epsilon) > 0$ we have $\beta^*_\epsilon > \beta_c$ thus m – and hence μ – is analytic at $\beta = b_c$. Indeed, $\mu(\beta, \epsilon)$ "lives" until $\beta = \beta_c < b_c(\epsilon)$; the quantity θ is just β_c. The claims (0), (i), (ii) and (iv) of Theorem 2.5 have all been established. □

The next – and penultimate – result is of some æsthetic interest:

Proposition 3.1. *For $\epsilon > 0$, let*

$$Z_N(\epsilon) = \sum_{\substack{w:0\to\cdot\\|w|=N}} \mathrm{e}^{\epsilon X_N(w)} = \sum_{\substack{w:0\to\cdot\\|w|=N}} \mathrm{e}^{\epsilon N_r(w) - \epsilon N_\ell(w)} \tag{3.4}$$

where, it will be recalled, $X_N(w)$ denotes the x_1 component of the endpoint. Let β^*_ϵ denote the value of β such that $m(\beta^*_\epsilon) = \epsilon$. Then

$$Z_N^{\frac{1}{N}} \to \mathrm{e}^{\beta^*_\epsilon}. \tag{3.5}$$

Proof. Following [3] let us use the notation $\Gamma_\kappa(L)$ to count walks of length κL that start at the origin and end on the plane $x_1 = L$. However as in [3], these objects only count walks that are restricted to the strip $0 \le x_1 \le L$ along with the stipulation that the first and last steps are in the x_1 direction. As this will not affect the lower bound we shall start with these quantities. For rational numbers, $[\Gamma_\kappa(L)]^{\frac{1}{L}} \to \mathrm{e}^{\zeta(\kappa)}$ but due to (easily verified) concavity – and hence continuity – the limiting function is well defined for all $\kappa \ge 1$. As mentioned previously, $\zeta(\kappa)$ is the Legendre transform of $m(\beta)$, i.e.

$$-m(\beta) = \sup_\kappa(\zeta(\kappa) - \beta\kappa). \tag{3.6}$$

The lower bound follows easily; for any (rational) κ let L_N denote a sequence of lengths so that κL_N is an integer N. Then

$$Z_N(\epsilon) \ge \Gamma_\kappa(L_N)\mathrm{e}^{\epsilon L_N} \tag{3.7}$$

and, as is not hard to see,

$$\liminf_{N\to\infty} Z_N^{\frac{1}{N}} \ge \exp\left\{\frac{\zeta(\kappa) + \epsilon}{\kappa}\right\}. \tag{3.8}$$

To prove the lower bound, any value of κ may (by continuity) be inserted so let us use $\kappa = \kappa^*_\epsilon$ where κ^*_ϵ is the value "selected" at temperature parameter β^*_ϵ to get $m = \epsilon$:

$$-m(\beta^*_\epsilon) = -\epsilon = \zeta(\kappa^*_\epsilon) - \beta^*_\epsilon\kappa^*_\epsilon. \tag{3.9}$$

While this immediately establishes the lower bound, for future reference it is worthwhile to demonstrate that this is in fact the optimal value of κ for the right hand side of Eq. (3.8). Indeed, for any value of κ, by adding and subtracting,

$$\zeta(\kappa) + \epsilon = [\zeta(\kappa) - \beta_\epsilon^* \kappa] + \epsilon + \beta_\epsilon^* \kappa. \tag{3.10}$$

Since the supremum in Eq. (3.6) is a maximum, if $\kappa \neq \kappa_\epsilon^*$, then $[\zeta(\kappa) - \beta_\epsilon^* \kappa] \leq -\epsilon$ so, indeed, κ_ϵ^* is the optimal choice. (Later it will be demonstrated that the maximum is in fact strict.)

With the above in mind, the upper bound is now almost immediate save for the fact that one must count *all* walks — not just the ones counted by $\Gamma_\kappa(L)$. To this end, let $\tilde{\Gamma}_\kappa(L)$ denote the function that counts all the SAW of length κL beginning at the origin and ending at $x_1 = L$. The next claim is that these objects also enjoy

$$\lim_{L\to\infty} \frac{1}{L} \log \tilde{\Gamma}_\kappa(L) = \zeta(\kappa) \tag{3.11}$$

along with a workable (uniform) upper estimate. Indeed, let us make the observation that while any such walk may spend a fraction λ^I of its time prior to its final entrance into the pertinent region and an additional fraction λ^F of its time after its first exit, during the interim, it behaves like a restricted walk with a reduced value of κ. Thus, neglecting the constraint that the various pieces of the walk are supposed to self-avoid, it is seen that

$$\tilde{\Gamma}_\kappa(L) \leq \sum_{\lambda^I, \lambda^F} \mathcal{N}(\lambda^I \kappa L)\mathcal{N}(\lambda^F \kappa L)\Gamma_{(1-\lambda)\kappa}(L) \tag{3.12}$$

where $\mathcal{N}(n)$ denotes the number of SAW beginning at the origin that are of length n and $\lambda = \lambda^I + \lambda^F$. Letting λ_L^I and λ_L^F denote maximizers for a particular L, it follows that

$$\tilde{\Gamma}_\kappa(L) \leq (\kappa L)^2 \mathcal{N}(\lambda_L^I \kappa L)\mathcal{N}(\lambda_L^F \kappa L)\Gamma_{(1-\lambda_L)\kappa}(L). \tag{3.13}$$

Now in general, the $\Gamma_\kappa(L)$ have an upper bound of $[\kappa L]^{\frac{d-1}{2}} \mathrm{e}^{\zeta(\kappa)L}$ (see, e.g., [3] Proposition 5.1 and Eq. (5.19) — with an unfortunate switch of some Γ-notation). The *a priori* bounds of $\mathrm{e}^{\beta_c n}$ for $\mathcal{N}(n)$ go in the wrong direction but it is still possible to say that for any $\eta > 0$, there is an $A(\eta) < \infty$ such that

$$\mathcal{N}(n) \leq A(\eta) e^{(\beta_c + \eta)n}. \tag{3.14}$$

Putting all of this together and, if necessary, taking a subsequence which ensures the convergence of $\lambda_L^I + \lambda_L^F$ we arrive at

$$\tilde{\zeta}(\kappa) = \limsup_{L\to\infty} \frac{1}{L} \log \tilde{\Gamma}_\kappa(L) \leq \lambda \beta_c \kappa + \zeta((1-\lambda)\kappa) \tag{3.15}$$

where λ denotes the limit of $\lambda_L^I + \lambda_L^F$. However, it is claimed that by a convexity argument, the inequality

$$\zeta(\kappa) \geq \lambda \beta_c \kappa + \zeta((1-\lambda)\kappa) \tag{3.16}$$

holds for all $\lambda \in [0, 1]$. Indeed, the above is equivalent to

$$\frac{\zeta(\kappa) - \zeta((1-\lambda)\kappa)}{\lambda\kappa} \geq \beta_c. \tag{3.17}$$

But, by concavity, the left side of the above is already larger than the left derivative of ζ at κ and since the function $\zeta(\kappa)$ is asymptotic to $\beta_c\kappa$ ([3], Corollaries of Theorem 5.4) then β_c is the *limiting* value of the derivative and is hence smaller.

It is further clear that the above argument implies an upper bound for $\tilde{\Gamma}_\kappa(L)$; i.e. for any η there is a $B(\eta) < \infty$ and an $\alpha < \infty$ such that

$$\tilde{\Gamma}_\kappa(L) < B[\kappa L]^\alpha \mathrm{e}^{\eta\beta_c\kappa L}\mathrm{e}^{\zeta(\kappa)L}. \tag{3.18}$$

It is remarked that η is devoid of any significance; it is an auxiliary small parameter to be taken to zero *after* the $N \to \infty$ limit. The upper bound is now immediate. For any N,

$$Z_N(\epsilon) = \sum_{L=0}^{N} \exp\left\{\frac{\epsilon N}{\kappa(L,N)}\right\} \tilde{\Gamma}_{\kappa(L,N)}(L) + \sum_{L=1}^{N} \exp\left\{\frac{-\epsilon N}{\kappa(L,N)}\right\} \tilde{\Gamma}_{\kappa(L,N)}(L) \tag{3.19}$$

where $\kappa(L, N)$ is, simply, the ratio of N/L. The second sum can be absorbed by the first at the cost of a factor of two. Using the estimate in Eq. (3.18) and the fact that there are only N terms to consider the result is that for any η,

$$Z_N(\epsilon) \le N^{\alpha+1}B'(\eta)e^{\eta\beta_c N} \sup_\kappa \left[\exp\left\{\frac{1}{\kappa}[\zeta(\kappa)+\epsilon]\right\}\right]^N. \tag{3.20}$$

But the supremum has already been established to be $\mathrm{e}^{\beta_\epsilon^* N}$ and the result follows after the appropriate $N \to \infty$ and $\eta \to 0$ limits. □

As a corollary we now have:

Proof of Theorem 2.1 and Theorem 2.5(iii). For all intents and purposes, these theorems will be established if it is demonstrated that $N^{-1}X_N \Longrightarrow (\kappa_\epsilon^*)^{-1}$. It is first remarked that while it is actually the case that X_N can be negative, it is obvious that this is a negligible possibility as N gets large; this will be dispensed with later on. For the moment, let us focus on the conditional measures $M_{N,(\epsilon,\epsilon)}(\cdot \mid X_N \ge 0)$ which will be denoted by $\mathbb{P}_N(\cdot)$. Furthermore, rather than considering for various values of $\vartheta' \ne 0$, the events $\left\{N^{-1}X_N = (\kappa_\epsilon^*)^{-1}(1+\vartheta')\right\}$, it will prove convenient to consider the equivalent events where NX_N^{-1} differs from κ_ϵ^* by some related $\vartheta(\vartheta', \kappa_\epsilon^*)$. Let us therefore focus on the random variable NX_N^{-1} (which, formally, can equal $+\infty$). For $\vartheta \ne 0$ (and, technically, no less than $1-\kappa_\epsilon^*$) let us estimate $\mathbb{P}_N(NX_N^{-1} = \kappa_\epsilon^* + \vartheta)$.

It is noted that, to within a factor of two this is just $Z_N^{-1}(\epsilon)\tilde{\Gamma}_\kappa(\kappa^{-1}N)\mathrm{e}^{\epsilon\kappa^{-1}N}$ where $\kappa = \kappa_\epsilon^* + \vartheta$. Thus, on the basis of Eq. (3.18) and the conclusion of Proposition 3.1 it is clear that for any $\theta > 0$

$$\mathbb{P}_N(NX_N^{-1} = \kappa_\epsilon^* + \vartheta = \kappa) \le C(\theta)N^\alpha \exp\left\{\frac{N}{\kappa}\left[\zeta(\kappa)+\epsilon-\kappa\beta_\epsilon^*\right] + \theta N\right\} \tag{3.21}$$

for some $C < \infty$. Now, let us investigate the coefficient of N in the exponent (excluding the θ term). Replacing ϵ with $-\zeta(\kappa_\epsilon^*) + \beta_\epsilon^*\kappa_\epsilon^*$ it is seen that the quantity $\frac{1}{\kappa}\left[\zeta(\kappa)+\epsilon-\kappa\beta_\epsilon^*\right]$ is equal to

$$-\left[\frac{\kappa_\epsilon^*-\kappa}{\kappa}\right]\left[\frac{\zeta(\kappa_\epsilon^*)-\zeta(\kappa)}{\kappa_\epsilon^*-\kappa} - \beta_\epsilon^*\right] = -\left[\frac{\kappa-\kappa_\epsilon^*}{\kappa}\right]\left[\beta_\epsilon^* - \frac{\zeta(\kappa)-\zeta(\kappa_\epsilon^*)}{\kappa-\kappa_\epsilon^*}\right] \tag{3.22}$$

where the first form is desirable when $\kappa_\epsilon^* > \kappa$ (i.e. $\vartheta < 0$) and the second when $\kappa_\epsilon^* < \kappa$ (i.e. $\vartheta > 0$). In either case, along with an overall minus sign, we have a product of two terms both of which are non-negative: The "prefactors" are clearly positive and, in the non-trivial looking term(s), non-negativity follows because, as the reader will recall from the properties of Legendre transforms, $\beta_\epsilon^* = \frac{\partial \zeta}{\partial \kappa}(\kappa_\epsilon^*)$. Thus the difference amounts to a pertinent comparison between a finite difference slope and an endpoint derivative of a concave function. Notwithstanding, it has to be demonstrated that these substantive looking terms are non-trivial; let us dispense with this formality. To this end, since $\beta_\epsilon^* > \beta_c$ and m is strictly concave on $[\beta_c, \infty)$ with continuous second derivative that is positive a.e. we have that $\kappa_\epsilon^* \in \text{Int}(\text{Ran}(\frac{\partial m}{\partial \beta}))$. (That is, there is an open interval of values containing κ_ϵ^* which this derivative will take on.) This implies, at least in a neighborhood of κ_ϵ^*, that ζ is *strictly* concave which necessarily implies strict positivity of the terms in question.

Next, it is claimed that – still withholding the minus sign – that both versions of Eq. (3.22) are strictly increasing in $|\vartheta|$. Indeed for the prefactors this is easily checked and on general grounds the substantive looking terms are non-decreasing. But, it is perhaps worth noting that since strict convexity of ζ has been established in a neighborhood of κ_ϵ^* these terms are in fact also strictly increasing.

The upshot – since θ is arbitrary – is that for (any allowed value of) $\vartheta_0 \neq 0$, the probability that $NX_N^{-1} - \kappa_\epsilon^* = |\vartheta_0|$ tends rapidly to zero. Moreover due to the above established monotonicity of the "rate" in $|\vartheta|$ and the fact that there are only of the order of N possible values to begin with, it is also seen that for any $\vartheta \neq 0$,

$$\mathbb{P}_N(|X_N^{-1}N - \kappa_\epsilon^*| > |\vartheta|) \to 0. \tag{3.23}$$

Finally it is clear that, for large N, the amalgamated total of *all* the negative X_N terms in Eq. (3.19) can be safely ignored. Indeed, again using Proposition 3.1 for $Z_N(\epsilon)$ and Eq. (3.18) for the $\tilde{\Gamma}$s it is seen that by the time $\zeta(\kappa)$ builds up to any appreciable value, the term is diminished by the $-\epsilon[\kappa]^{-1}N$ in the exponent. Again using the fact that there are only of the order of N terms it is seen that nothing essential was lost using the conditional measures.

It has now been fully established that $N^{-1}X_N \Longrightarrow [\kappa_\epsilon^*]^{-1} > 0$ which is already more than the claim of Theorem 2.1. Theorem 2.5(iii) follows because *if* $v^{-1} = \kappa_\epsilon^*$ then

$$v^{-1} = \kappa_\epsilon^* = \frac{\partial m}{\partial \beta}(\beta_\epsilon^*) = \frac{\partial \mu}{\partial \beta}(b_c) \tag{3.24}$$

by various earlier identities and previously discussed Legendre transform properties of sufficiently regular functions. □

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