An aggregation equation with degenerate diffusion: qualitative property of solutions

Lincoln Chayes, Inwon Kim and Yao Yao

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Abstract

We study a nonlocal aggregation equation with degenerate diffusion, set in a periodic domain. This equation represents the generalization to $m > 1$ of the McKean–Vlasov equation where here the "diffusive" portion of the dynamics are governed by *Porous medium* self–interactions. We focus primarily on $m \in (1, 2]$ with particular emphasis on $m = 2$. In general, we establish regularity properties and, for small interaction, exponential decay to the uniform stationary solution. For $m = 2$, we obtain essentially sharp results on the rate of decay for the entire regime up to the (sharp) transitional value of the interaction parameter.

1 Introduction

In this paper we study weak solutions of the equation:

$$\rho_t = \Delta(\rho^m) + \theta L^{d(2-m)} \nabla \cdot (\rho \nabla (V * \rho)) \quad \text{in } \mathbb{T}_L^d \times [0, \infty), \tag{1.1}$$

where $*$ stands for convolution, and the space domain is the d-dimension torus with scale L, defined as $\mathbb{T}_L^d := [-\frac{L}{2}, \frac{L}{2}]^d$ with periodic boundary condition. We assume that V smooth and integrable (for precise conditions, see **(V1)**-**(V2)** in Section 3), and that θ is a positive constant. The primary focus of this work concerns the cases $m \in (1, 2]$ – especially $m = 2$. In addition, we remark that a goal of interest (not always achieved) is to acquire results uniform in L for $L \gg 1$. We point out that, in the absence of the aggregation term (i.e., when $V = 0$) our equation becomes the well-known *Porous medium equation* (PME):

$$\rho_t - \Delta(\rho^m) = 0.$$

Note that, formally (and in actuality) the mass of the solution to Eq.(1.1) is preserved over time:

$$\int \rho(x, 0) dx = \int \rho(x, t) dx \text{ for all } t > 0.$$

Without loss of generality, we can thus assume $\int \rho(x, 0) dx = 1$ and results for other normalizations can be obtained by scaling.

In the context of biological aggregation, ρ represents the population density which locally disperses by the diffusion term, while V is the sensing (interaction) kernel that models the long-range attraction; Eq.(1.1) is relevant for models which have been introduced by [BCM] and [TBL], and further studied by [BCM2] and [BF]. The above equation can also be regarded as the evolution equation for a strongly interacting fluid: The V represents the long distance component of the interaction while short distance interactions – and entropic effects – are accounted for by the *degeneracy* ($m > 1$)

in the diffusion term. Mathematically, the equation exhibits an interesting competition between degenerate diffusion and nonlocal aggregation.

When V satisfies $V(x) = V(-x)$, Eq.(1.1) is a gradient flow of the following energy with respect to the Wasserstein metric:

$$\mathcal{F}_\theta(\rho) := \int_{\mathbb{T}_L^d} \frac{1}{m-1}(\rho^m - \rho) + \frac{1}{2}\theta L^{d(2-m)}\rho(V * \rho)dx. \qquad (1.2)$$

Note that as $m \to 1$, the first term in the integrand of $\mathcal{F}_\theta$ converges to $\rho \log \rho$ which we refer to as the $m = 1$ case. Using above energy structure, the existence and uniqueness properties of Eq.(1.1), in some appropriate Sobolev space, has been obtained in [BS] (also see [S] and [BRB] for relevant results).

Compared to the well–posedness theory based on energy methods, few results has been known for pointwise behaviors of solutions, due to the lack of regularity estimates: the difficulty for regularity analysis lie mainly in the fact that the solutions are not necessarily positive (i.e., *strictly* positive) due to the degenerate diffusion. This is what we address in the first part of our paper. In addition, in the non–compact setting, the plausible limiting solutions tend to be trivial; here, since mass is conserved, even in the "worst" of cases, there is always the uniform stationary state. Most of the rest of this work is concerned with the approach to the asymptotic state.

• *Regularity properties* Due to the degenerate diffusion, one cannot expect smooth solutions of Eq.(1.1): even for (PME), Hölder regularity is optimal, as verified by the self-similar (Barenblatt) solutions (see [V]). On the other hand, the solution of (PME) is indeed Hölder continuous (again, see [V]), which motivates the question of Hölder regularity of the solution of our problem Eq.(1.1).

Note that, if we choose V as a mollifier approximating the Dirac delta function, formally the nonlocal term approximates

$$\nabla \cdot [\theta L^{d(m-2)}\nabla V * \rho] = \theta L^{d(m-2)}\nabla \cdot [\rho \nabla \rho] = \theta L^{d(m-2)}\Delta(\rho^2).$$

Therefore it is plausible that, at least when ρ is bounded from above, diffusion dominates when $m < 2$ and the aggregation dominates when $m > 2$. Indeed we will show that, when $m < 2$, the effect of the aggregation term is weak enough that it is possible to locally approximate solutions of Eq.(1.1) with those of (PME). As a result, Hölder regularity of solutions of Eq.(1.1) for $m < 2$ follows. As for $m \geq 2$, we show that solutions are continuous "uniformly in time", based on the result of Dibenedetto ([Dib]). For all $m > 1$, we also show that the L^∞ norm of solution is uniformly bounded from above depending on the L^1 and L^∞ norm of the initial data (see Theorem 2.1) which is of independent interest.

• *Asymptotic behavior* Our next result, partly an application of the first result, is on the asymptotic behavior of solutions of Eq.(1.1) in the periodic domain $\mathbb{T}_L^d$. We work in a periodic domain because, primarily, we are interested in finite volume problems and $\mathbb{T}_L^d$ provides the most convenient boundary conditions. Even though asymptotic behavior for $m < 2$ has been studied before in various references (e.g., [S], and presumably others) this is one of the first such result for these type of domains to the best of the authors' knowledge. One difficulty specific to the periodic setting is that the radial symmetry is not preserved over time, and thus exact (non–constant) solutions – always useful in these contexts – are not readily available. We also point out that in the case $m \geq 2$, there exist solutions which assumes zero value, possibly with compact support. Asymptotic behavior of such solutions are, in general, an interesting and difficult question, even for radial solutions in $\mathbb{R}^d$ (see [KY]).

Intuitively, one expects that when the diffusion term is "dominant" in Eq.(1.1), the solutions would converge to the constant solution as time goes to infinity. We show that this is indeed the

case when $1 < m < 2$ and θ is sufficiently small. However, with most interactions (specifically, V being *not* of positive type) there is a linear instability that sets in at some $\theta^\sharp = \theta^\sharp(m) < \infty$ which is determined by the minimal coefficient in the Fourier series of V (see Section 4). It is not hard to show that for all m, when $\theta > \theta^\sharp$, the functional in Eq.(1.2) has non–constant minimizers (and the constant solution is not a minimizer – in fact, not even a *local* minimizer). However as has been shown explicitly for $m = 1$ under reasonable conditions – pertinently $d \geq 2$ – this "transition" occurs at some $\theta_\mathrm{T} < \theta^\sharp$ [CP]. Presumably, this argument holds in great generality. It is therefore somewhat surprising that for $m = 2$ the transition occurs exactly at $\theta = \theta^\sharp$.

More precisely, for $m = 2$, we show that for $\theta < \theta^\sharp$ (the *subcritical* case), the constant solution is the only minimizer and is stable. Indeed we can actually show that for all bounded initial data $\rho(x, 0)$, the dynamical $\rho(\cdot, t)$ will converge to the constant solution ρ_0 exponentially fast in L^2-norm. See Section 4 for detailed discussion on critical and supercritical case. When $1 < m < 2$, the energy is no longer in the form of an L^2–norm, and our Fourier-transform based approach does not generate a transitional value for θ. However, when θ is sufficiently small, similar approach used by one of the authors in [CP] yields that the constant solution is the global minimizer. Moreover, we show when θ is sufficiently small, the solution uniformly and exponentially converges to the constant solution.

Below we sketch an outline of our paper: In Section 2 we first give a uniform upper bound for the weak solution to porous medium equation with a drift for $m > 1$, then prove Hölder continuity of the weak solution when $1 < m < 2$. In Section 3 we apply the Hölder continuity result to a nonlocal aggregation equation. In Section 4 we use Fourier transform approach to study the nonlocal aggregation equation when $m = 2$, and prove the exponential convergence of the weak solution in the subcritical case. Analogous results for $1 < m < 2$ is established in Section 5. When $1 < m < 2$, for θ sufficently small, we prove there is also exponential convergence.

2   Hölder Continuity of the Solution of PME with a Drift

In this section, we study the regularity of the porous medium equation with a drift, where the drift potential may depend on time:

$$\rho_t = \Delta(\rho^m) + \nabla \cdot (\rho \nabla \Phi) \quad \text{in } \Omega, \tag{2.1}$$

with Neumann boundary condition on $\partial\Omega$. Here we may assume Ω is a bounded open set in $\mathbb{R}^d$, where $d \geq 1$, but all the results in this section certainly hold for periodic domain $\mathbb{T}_L^d$ as well. We assume $1 < m < 2$, the initial data $\rho(x, 0) \in L^\infty(\Omega) \cap L^1(\Omega)$, the potential $\Phi(x, t) \in C(\Omega \times \mathbb{R}^+)$, and that $\Phi(\cdot, t) \in C^2(\Omega)$ for all $t \geq 0$.

Before even stating the main result, we will first prove that $\rho \in L^\infty(\Omega \times \mathbb{R}^+)$. When Φ does not depend on t, Bertch and Hilhorst in [BH] proved a uniform L^∞ bound of ρ by comparing ρ with an explicit supersolution which does not depend on t. When Φ is a function of both x and t, using arguments similar to those in [KL], we aquire an L^∞ bound for ρ which doesn't depend on t:

Theorem 2.1. *Suppose $m > 1$. Let ρ be the unique weak solution of Eq.(2.1) with Neumann boundary condition, with initial data $\rho(x, 0) \in L^\infty(\Omega) \cap L^1(\Omega)$. We assume that the potential $\Phi(x, t)$ satisfies $\Phi(x, t) \in C(\Omega \times \mathbb{R}^+)$, and $\Phi(\cdot, t) \in C^2(\Omega)$ for all t with uniformly bounded norm. Then there exists $M > 0$, such that $\|\rho(\cdot, t)\|_{L^\infty(\Omega)} \leq M$ for all t, where M depends on $\|\rho(x, 0)\|_{L^\infty(\Omega)}$, $\|\rho(x, 0)\|_{L^1(\Omega)}$, $\sup_{t \in [0, \infty)} \|\Phi(\cdot, t)\|_{C^2(\Omega)}$, and m.*

Proof. We begin with implementing the following scaling: Let

$$\tilde{\rho}(x, t) = a^{\frac{1}{m-1}} \rho(x, at),$$

where $0 < a < 1$. Let us choose a sufficiently small such that

$$a < \min\Big\{\big(\frac{1}{\|\rho(x,0)\|_{L^\infty(\Omega)}}\big)^{m-1}, \big(\frac{c_0}{\|\rho(x,0)\|_{L^1(\Omega)}}\big)^{m-1}, \frac{1}{\|\Phi\|_{C^2(\Omega)}}\Big\}, \qquad (2.2)$$

where c_0 is a sufficiently small constant – certainly less than 1 – depending only on m and d and whose precise value will be determined later. By choosing a in this way, we have both $\|\tilde\rho(x,0)\|_{L^1(\Omega)} \le c_0 < 1$ and $\|\tilde\rho(x,0)\|_{L^\infty(\Omega)} \le 1$ and, moreover, that $\tilde\rho$ is a viscosity solution to the following PDE:

$$\tilde\rho_t = \Delta\tilde\rho^m + \nabla\cdot(\tilde\rho\nabla\tilde\Phi), \qquad (2.3)$$

where $\tilde\Phi := a\Phi$. From the definition of a we know $\|\tilde\Phi(\cdot,t)\|_{C^2(\Omega)} \le 1$ for all t.

Our preliminary goal is to show $\|\tilde\rho(x,1)\|_{L^\infty(\Omega)} \le 1$; then we can take $\tilde\rho(x,1)$ as the new initial data and iterate the argument to get a uniform bound for all time.

We will introduce another variable v, which is bigger than $\tilde\rho$ and is of order unity in $\Omega \times [0,1]$. Let v be the viscosity solution to the following equation

$$v_t = \nabla\cdot(mv^{m-1}\nabla v + v\nabla\tilde\Phi), \qquad (2.4)$$

with initial data $v(x,0) = \tilde\rho(x,0) + \frac{1}{2}e^{-1}$. Since v solves the same equation as $\tilde\rho$ with bigger initial data, we can apply the comparison principle for the porous medium equation with drift, which was established in Theorem 2.21 of [KL]. This comparison principle immediately implies $v(x,t) \ge \tilde\rho(x,t)$ for all (x,t), hence it suffices to show $\|v(\cdot,1)\|_{L^\infty(\Omega)} \le 1$.

One can check easily that $\tilde v(x,t) := [\|v(\cdot,0)\|_{L^\infty(\Omega)}]e^{Kt}$ – where $K := \sup_{t\in[0,\infty)}\|\tilde\Phi(\cdot,t)\|_{C^2(\Omega)}$ – is a classical supersolution to Eq.(2.4) and hence also a viscosity supersolution. Noting that the initial data of v satisfies, for all x, $\frac{1}{2}e^{-1} \le v(x,0) \le 1 + \frac{1}{2}e^{-1}$, the comparison principle gives the following upper bound for v:

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \le [\|v(\cdot,0)\|_{L^\infty(\Omega)}]e^{Kt} \le (1 + \frac{1}{2}e^{-1})e^t.$$

Similarly we can find a classical subsolution which gives the lower bound

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \ge [\|v(\cdot,0)\|_{L^\infty(\Omega)}]e^{-Kt} \ge \frac{1}{2}e^{-1}e^{-t}.$$

Combining the two inequalities above, we have

$$v(x,t) \in [\frac{1}{2}e^{-2}, e + \frac{1}{2}] \quad \text{for all } x \in \Omega, t \in [0,1].$$

We would like to refine the estimate above and get a better estimate at $t = 1$. By treating the diffusion coefficients mv^{m-1} in Eq.(2.4) as an *a priori* function, – which we denote by $b(x,t)$ – then we may say that v solves a linear equation of divergence form, where the diffusion coefficient is of (the order of) size unity:

$$v_t = \nabla\cdot(b(x,t)\nabla v + v\nabla\tilde\Phi), \qquad (2.5)$$

where $b(x,t) := mv^{m-1}(x,t) \in [m(\frac{1}{2}e^{-2})^{m-1}, m(e + \frac{1}{2})^{m-1}]$ for all $x \in \Omega$, $t \in [0,1]$.

In particular, since Eq.(2.5) is linear, we can decompose v as $v_1 + v_2$, such that v_1 solves Eq.(2.5) with initial data $v_1(x,0) = \tilde\rho(x,0)$, and v_2 solves Eq.(2.5) with initial data $v_2(x,0) = \frac{1}{2}e^{-1}$. We claim that $v_1(x,1)$ and $v_2(x,1)$ are both bounded by $\frac{1}{2}$, for all $x \in \Omega$.

For v_1, first note that due to the divergence form of Eq.(2.5), the L^1 norm of v_1 is conserved, i.e. $\|v_1(\cdot,1)\|_{L^1(\Omega)} = c_0$. Since b is bounded above and below away from zero, then by [LSU] (see

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Theorem 10.1, pp. 204), $v_1(\cdot, 1)$ is Hölder continuous, where the Hölder exponent and coefficient do not depend on c_0, as long as $c_0 < 1$. So if we choose c_0 to be sufficiently small, we have $v_1(x, 1) < \frac{1}{2}$ for all $x \in \Omega$.

For v_2, we can directly evaluate the necessary L^∞ bounds:

$$\sup_x v_2(x, 1) \le e^{\|\Delta \tilde{\Phi}\|_\infty} \sup_x v_2(x, 0) \le e \frac{1}{2} e^{-1} = \frac{1}{2}$$

(where again, on the basis of continuity, we may now talk about the supremum).

Combining the two estimates together, we have $\sup_x v(x, 1) \le 1$, which implies $\sup_x \tilde{\rho}(x, 1) \le 1$ from our discussion above. Also, for $0 < t < 1$ we have $\rho(x, t) \le v(x, t) \le e + 1/2$. Then by treating $\tilde{\rho}(x, 1)$ as initial data and iterating the same argument, we get $\sup_x \tilde{\rho}(x, t) \le e + 1/2$ for all t, i.e.,

$$\rho(x, t) \le (e + \frac{1}{2}) a^{-\frac{1}{m-1}} \quad \text{for all } x \in \Omega, t \ge 0.$$

Now plugging in the definition of a in the above and the bound becomes

$$\rho(x, t) \le (e + \frac{1}{2}) \max \left\{ \|\rho(x, 0)\|_{L^\infty(\Omega)}, \ \frac{\|\rho(x, 0)\|_{L^1(\Omega)}}{c_0}, \ \|\Phi\|_{C^2(\Omega)}^{\frac{1}{m-1}} \right\} \quad \text{in } \Omega \times [0, \infty).$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Remark 2.2. In the statement of Theorem 2.1, we assumed that Ω is a bounded open set, with Neumann boundary conditions. The same proof also applies to Dirichlet boundary condition. Indeed, the L^∞ bound we obtained is independent with the size of Ω, and the same proof works as well when $\Omega = \mathbb{R}^d$. However, ostensibly, the L^∞ norm of ρ should be of the order L^{-d} and, even if true in the initial data, we cannot establish that this order is preserved at later times.

Since $\rho(x, t)$ is uniformly bounded for all (x, t), DiBenedetto has shown in [Dib] that $\rho(\cdot, t)$ is continuous uniformly in t:

Theorem 2.3 ([Dib])**.** *For any $m > 1$, let ρ be the weak solution to Eq.(2.1) with initial data $\rho(x, 0) \in L^\infty(\Omega) \cap L^1(\Omega)$. Let the potential $\Phi(x, t)$ satisfy $\Phi(x, t) \in C(\Omega \times \mathbb{R})$, $\Phi(\cdot, t) \in C^2(\Omega)$ for all t, moreover $\sup_t \|\Phi(\cdot, t)\|_{C^2(\Omega)} < \infty$. Then for all $\tau > 0$, $\rho(x, t)$ is uniformly continuous in $\Omega \times [\tau, \infty)$, and the continuity is uniform in x and t.*

Now we want to show when $1 < m < 2$, for all $\tau > 0$, $\rho(x, t)$ is uniformly Hölder continuous in space and time in $\Omega \times [\tau, \infty)$. Our main theorem of this section is stated as following:

Theorem 2.4. *Let $1 < m < 2$. Let ρ be a viscosity solution of Eq.(2.1), with initial data $\rho(x, 0)$. We make the following assumptions on $\rho(\cdot, 0)$ and Φ:*

1. *$\|\rho(\cdot, 0)\|_\infty \le M_1$ and $\int_\Omega \rho(x, 0) dx \le M_1$.*

2. *$\Phi(x, t) \in C(\Omega \times \mathbb{R})$, and $\|\Phi(\cdot, t)\|_{C^2(\Omega)} \le M_2$ for some $M_2 > 0$ for all $t \ge 0$.*

Then for any $0 < \tau < \infty$, u is Hölder continuous in $\Omega \times [\tau, \infty)$, where the Hölder exponent and coefficient depends on τ, m, d, M_1 and M_2.

Proof. To prove the Hölder continuity of ρ, our goal is to show that for any $(x_0, t_0) \in \Omega \times [\tau, \infty)$,

$$\text{osc}_{B(x_0, a^2) \times [t_0, t_0 + a^4]} \rho \le C a^\gamma \tag{2.6}$$

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for some $C, \gamma > 0$ not depending on a, for a satisfying $0 < a < \min\{\dfrac{2-m}{2c}, \sqrt{\tau}\}$ (where c is a constant to be determined soon).

Bearing in mind that we want to zoom in on the profile and look at the oscillation in a small neighborhood, it makes sense to start with a parabolic scaling with scaling factor a. Let

$$\tilde{\rho}(x,t) := \rho(ax, a^2 t + (t_0 - a^2))), \tag{2.7}$$

and our goal Eq.(2.6) would transform into

$$\text{osc}_{B(\frac{x_0}{a}, a) \times [1, 1+a^2]} \tilde{\rho} \leq C a^\gamma. \tag{2.8}$$

Here $\tilde{\rho}(x,t)$ is defined in the domain $\tilde{\Omega} \times [0, \infty)$, where $\tilde{\Omega} := \{x \in \mathbb{R}^d : ax \in \Omega\}$. and, it is noted, the early portion of the time domain had been omitted. We readily see that $\tilde{\rho}$ is the viscosity solution to

$$\tilde{\rho}_t = \Delta \tilde{\rho}^m + \nabla \cdot (\tilde{\rho} \nabla \tilde{\Phi}) \quad \text{in } \tilde{\Omega} \times [0, \infty). \tag{2.9}$$

Here, the initial data reads $\tilde{\rho}(x,0) = \rho(ax, t_0 - a^2)$, which has an *a priori* L^∞ bound depending on m, d, M_1, M_2 due to Theorem 2.1. Moreover, in the above $\tilde{\Phi}(x,t) := \Phi(ax, a^2 t + (t_0 - a^2)))$ and hence $|\nabla \tilde{\Phi}|$ is bounded by $a M_2$. We wish to compare $\tilde{\rho}$ with w, where w is the viscosity solution to the porous medium equation

$$w_t = \Delta w^m \quad \text{in } \tilde{\Omega} \times [0, \infty), \tag{2.10}$$

with initial data $w(\cdot, 0) \equiv \tilde{\rho}(\cdot, 0)$. Since Eq.(2.9) and Eq.(2.10) only differ by the term $\nabla \cdot (\tilde{\rho} \nabla \tilde{\Phi})$, we would expect

$$|\tilde{\rho} - w| \leq C a^\beta \text{ in } \tilde{\Omega} \times [1, 2], \tag{2.11}$$

for some $C > 0, 0 < \beta < 1$ depending on m, d, M_1, M_2.

The main part of this proof will be devoted to proving Eq.(2.11) is indeed true. Without loss of generality, we can assume that $\tilde{\rho}(x,t)$ is a classical solution. First, if the initial data $\tilde{\rho}(x,0)$ is uniformly positive, then $\tilde{\rho}(x,t)$ will be a classical solution for all time. This is because $\tilde{\rho}$ will stay positive for any time period $[0, T]$ (since $\inf_{\tilde{\Omega} \times [0,T]} \tilde{\rho}(x,t) \geq \exp(-t \sup_{t \in [0,T]} \|\Delta \tilde{\Phi}\|_\infty) \inf_{x \in \tilde{\Omega}} \tilde{\rho}(x,0))$, which implies that Eq.(2.9) is uniformly parabolic for $t \in [0, T]$ and hence the weak solution $\tilde{\rho}$ is classical.

For general initial data $\tilde{\rho}(x,0)$, we can use approximation as follows. Let $\tilde{\rho}_n$ and w_n solve Eq.(2.9) and Eq.(2.11) respectively with initial data $\tilde{\rho}(x,0) + 2^{-n}$; n sufficiently large. As discussed above, $\tilde{\rho}_n$ would be a sequence of classical solutions. If we can obtain $|\tilde{\rho}_n - w_n| < C a^\beta$ for all n, (where C, β doesn't depend on n), then Eq.(2.11) would hold for $\tilde{\rho}$ and w as well, since as $n \to \infty$, comparison principle yields $\tilde{\rho}_n(x,t) \searrow \tilde{\rho}$ and $w_n(x,t) \searrow w$ uniformly in x, t.

Note that one cannot directly compare ρ with w, due to the fact that the term $\nabla \cdot (\tilde{\rho} \nabla \tilde{\Phi})$ contains $\nabla \tilde{\rho} \cdot \nabla \tilde{\Phi}$ and hence does not have any *a priori* bound. In order to bound this term, it will help to change from the density variable $\tilde{\rho}$ to the pressure variable $\tilde{u}$. Let

$$\tilde{u} = \frac{m}{m-1} \tilde{\rho}^{m-1},$$

then Eq.(2.9) becomes

$$\tilde{u}_t = (m-1)\tilde{u}\Delta\tilde{u} + |\nabla\tilde{u}|^2 + \nabla\tilde{u} \cdot \nabla\tilde{\Phi} + (m-1)\tilde{u}\Delta\tilde{\Phi}, \tag{2.12}$$

which will enable us to use $|\nabla\tilde{u}|^2$ plus a constant to control the term $\nabla\tilde{u} \cdot \nabla\tilde{\Phi}$: Recall that $|\nabla\tilde{\Phi}| < a M_2$, which gives us the following bound

$$|\nabla\tilde{u} \cdot \nabla\tilde{\Phi}| \leq a M_2 |\nabla\tilde{u}| \leq a[|\nabla\tilde{u}|^2 + \frac{1}{4}(M_2)^2].$$

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Also, due to the fact that $(m-1)\tilde{u}(x,t) \le C_1$ in $\tilde{\Omega} \times [0,2]$, (where C_1, which depends on m, d, M_1 and M_2, is related to the L^∞ bounds on ρ) we obtain

$$|(m-1)\tilde{u}\Delta\tilde{\Phi}| \le a^2 C_1 M_2 \le a C_1 M_2.$$

Putting the above two bounds together, and by choosing c such that $c > C_1 M_2 + (M_2/2)^2$, $\tilde{u}$ will satisfy the following inequality

$$\tilde{u}_t \ge (m-1)\tilde{u}\Delta\tilde{u} + (1-ca)|\nabla\tilde{u}|^2 - ca \quad \text{for all } x \in \tilde{\Omega}, t \in [0,2]. \tag{2.13}$$

Note that we assumed $a < (2-m)/(2c)$ in the beginning of the proof, we have $ca < (2-m)/2$.

In order to make Eq.(2.13) look similar to the porous medium equation in the pressure form, we apply the rescaling $u_1 = (1-ca)\tilde{u}$. Then u_1 satisfies

$$(u_1)_t \ge (m^- - 1)u_1\Delta u_1 + |\nabla u_1|^2 - ca(1-ca) \quad \text{for all } x \in \tilde{\Omega}, t \in [0,2], \tag{2.14}$$

where

$$m^- := \frac{m-1}{1-ca} + 1. \quad \text{(hence } ca < (2-m)/2 \text{ implies that } 1 < m^- < 2) \tag{2.15}$$

Now Eq.(2.14) has the same form as the porous medium equation in the pressure form, minus an extra constant term $ca(1-ca)$. To take advantage of the existence and regularity results for equations with divergence form, we change the pressure variable back to the density variable (however here the power is m^- instead of m), i.e., we define ρ_1 such that

$$(1-ca)\tilde{u} = u_1 = \frac{m^-}{m^- - 1}\rho_1^{m^- - 1}, \tag{2.16}$$

or in other words,

$$\rho_1 = \left(\frac{m}{m^-}\right)^{m^-} \tilde{\rho}^{1-ca} = \left(\frac{1+ca}{1+ca/m}\right)^{m^-} \tilde{\rho}^{1-ca}. \tag{2.17}$$

Due to the positivity of $\tilde{u}$, we know ρ_1 is positive as well. Hence when we plug Eq.(2.16) into Eq.(2.14), after canceling a positive power of ρ_1 on both sides, we obtain

$$(\rho_1)_t > \Delta\rho_1^{m^-} - ca(1-ca)\rho_1^{2-m^-} \quad \text{in } \tilde{\Omega} \times [0,2], \tag{2.18}$$

Note that the term $ca(1-ca)\rho_1^{2-m^-}$ has an *a priori* upper bound: since $2 - m^- > 0$ and ρ_1 is given by Eq. (2.17), we have $c(1-ca)\rho_1^{2-m^-} < M$, for some constant M depending on m, d, M_1, M_2. Let us denote by ρ^- the weak solution of

$$(\rho^-)_t = \Delta(\rho^-|\rho^-|^{m^- - 1}) - Ma, \tag{2.19}$$

with initial data the same as $\rho_1(x,0)$, which is

$$\rho^-(x,0) = \left(\frac{m}{m^-}\right)^{m^-} \tilde{\rho}(x,0)^{1-ca} \tag{2.20}$$

Since $Ma \in L^p(\tilde{\Omega})$ for all $p \ge 1$, the existence of weak solution of Eq.(2.19) is guaranteed by Theorem 5.7 in [V]. That theorem also gives us a comparison result that, a.e., $\rho_1 \ge \rho^-$.

Moreover, note that the "a.e." above can in fact be removed, since both $\tilde{\rho}$ and ρ^- are continuous in $\tilde{\Omega} \times [0,2]$: the continuity of $\tilde{\rho}$ is given by Theorem 2.3, and the continuity of ρ^- is given by Theorem 11.2 of [DGV]. Therefore we have the following comparison between ρ^- and $\tilde{\rho}$:

$$\rho^- \le \left(\frac{m}{m^-}\right)^{m^-} \tilde{\rho}^{1-ca} \quad \text{in } \tilde{\Omega} \times [0,2] \tag{2.21}$$

Since $m/m^- = 1 + O(a)$, and $\tilde{\rho}$ is bounded in $\tilde{\Omega} \times [0,2]$, Eq.(2.21) implies that $\tilde{\rho} - \rho^- \geq -C_1 a$, where C_1 depend on m, d, M_1, M_2.

Analogous to the definition to ρ^-, we define ρ^+ to be the weak solution of

$$(\rho^+)_t = \Delta((\rho^+)^{m^+}) + Ma, \tag{2.22}$$

with initial data

$$\rho^+(x,0) = \left(\frac{m}{m^+}\right)^{m^+} \tilde{\rho}(x,0)^{1+ca}, \tag{2.23}$$

where

$$m^+ := \frac{m-1}{1+ca} + 1. \quad (\text{hence } 1 < m^+ < 2) \tag{2.24}$$

Then analogous argument would lead to $\tilde{\rho} - \rho^+ \leq C_1 a$. Summarizing, we have obtained

$$\rho^- - C_1 a \leq \tilde{\rho} \leq \rho^+ + C_1 a \quad \text{in } \tilde{\Omega} \in [0,2], \tag{2.25}$$

where C_1 depends on m, d, M_1, M_2.

To prove Eq.(2.11), it suffices to show $|\rho^\pm - w| \leq O(a^\beta)$ for some $\beta > 0$, which is proved in the following lemma.

Lemma 2.5. *Let $1 < m < 2$. Let w be the viscosity solution of the porous medium equation*

$$w_t = \Delta w^m \quad in \ \tilde{\Omega} \times [0,\infty) \tag{2.26}$$

where the initial data $w(x,0)$ satisfies $w(x,0) = \tilde{\rho}(x,0)$.

Let ρ^- and ρ^+ be the weak solutions to Eq.(2.19) and Eq.(2.22) respectively, where $0 < a < (2-m)/(2c)$ is a small constant, and the initial data is given by Eq.(2.20) and Eq.(2.23). Then

$$|\rho^\pm - w| \leq C a^\beta \quad in \ \tilde{\Omega} \times [1,2], \tag{2.27}$$

where C and β depends on d, m, M_1, M_2.

The proof of Lemma 2.5 is the content of the appendix in Section 6. Putting Lemma 2.5 and Eq.(2.25) together, we obtain Eq.(2.11), and we will use this to (immediately) prove Eq.(2.8).

Since w solves the porous medium equation, Theorem 7.17 in [V] gives us the Hölder continuity of w:

$$\text{osc}_{B(x,a) \times [1,1+a^2]} w \leq C a^\alpha, \ \text{for all } x \in \tilde{\Omega}, \tag{2.28}$$

where C and α depends on $\|w(\cdot,0)\|_\infty$ (and hence depends on m, d, M_1, M_2).

By putting Eq.(2.29) and Eq.(2.11) together, we obtain

$$\text{osc}_{B(x,a) \times [1,1+a^2]} \tilde{\rho} \leq C a^\gamma, \ \text{for all } x \in \tilde{\Omega}, \tag{2.29}$$

where C depends on m, d, M_1, M_2, and $\gamma = \min\{\alpha, \beta\}$ (hence also depends on m, d, M_1, M_2). Hence Eq.(2.8) is proved. $\qquad\square$

Remark 2.6. For $m \geq 2$, Hölder continuity of the solution to Eq.(2.1) is still open. Indeed, concerning the present approach – which closely parallels that of [KL], [K] – when $m > 2$ we have that $m^- = 1 + (m-1)/(1-ca) > 2$. Hence the "inhomogeneous" term in Eq.(2.18), which is proportional to $\rho^{(2-m^-)}$, would actually be divergent in places where $\rho \to 0$. This indicates that another approach will be required.

3 Application to Aggregation Equation with Degenerate Diffusion

In the following two sections, we study Eq.(1.1) in the domain $\mathbb{T}_L^d$, the d-dimension torus of scale L. Here θ is a non-negative constant, and, of course, $*$ denotes convolution in $\mathbb{T}_L^d$. We make the following assumptions on $V(x)$:

(V1) $V(x) = V(-x)$ for all $x \in \mathbb{T}_L^d$.

(V2) $V(x) \in C^2(\mathbb{T}_L^d)$, with $\|V(x)\|_{C^2(\mathbb{T}_L^d)} = C$ for some constant $C < \infty$.

Moreover, we have in mind $V : \mathbb{R}^d \to \mathbb{R}$ compactly supported with the diameter of the support smaller than L. In particular we do not envision "wrapping" effects and $\int_{\mathbb{T}_L^d} |V| dx$ may be regarded as independent of L.

Our goal in this section is to show the Hölder continuity of the weak solution to Eq.(1.1) for $1 < m < 2$, and uniform continuity of the weak solution when $m = 2$. First, we state the definition of weak solution to Eq.(1.1) and a existence theorem from [BS].

Definition 3.1 (Weak Solution). *Let $m > 1$, and let us assume that $\rho(x,0)$ is non-negative, with $\rho(x,0) \in L^\infty(\mathbb{T}_L^d)$ and consider a potential V that satisfies the assumptions **(V1)** and **(V2)**. A function $\rho : \mathbb{T}_L^d \times [0,T] \to [0,\infty)$ is a weak solution to Eq.(1.1) if $\rho \in L^\infty(\mathbb{T}_L^d \times [0,T])$, $\rho^m \in L^2(0,T,H^1(\mathbb{T}_L^d))$ (i.e., $\|\rho(\cdot,t)\|_{H^1(\mathbb{T}_L^d)} \in L^2(0,T)$) and $\rho_t \in L^2(0,T,H^{-1}(\mathbb{T}_L^d))$ and for all test function $\phi \in H^1(\mathbb{T}_L^d)$, for almost all $t \in [0,T]$,*

$$< \rho_t(t), \phi > + \int_{\mathbb{T}_L^d} \nabla(\rho^m(t)) \cdot \nabla\phi + \theta L^{d(2-m)} \rho(t)(\nabla V * \rho(t)) \cdot \nabla\phi \, dx = 0. \tag{3.1}$$

In [BS], existence and uniqueness of weak solution are proved:

Theorem 3.2 (Bertozzi-Slepčev). *Let $m > 1$ and consider V that satisfies the assumptions **(V1)** and **(V2)**. Let $\rho(x,0)$ be a nonnegative function in $L^\infty(\mathbb{T}_L^d)$. Then the problem Eq.(1.1) has a unique weak solution on $\mathbb{T}_L^d \times [0,T]$ for all $T > 0$, and furthermore $\rho \in C(0,T,L^p(\mathbb{T}_L^d))$ for all $p \in [1,\infty)$.*

By treating $\theta L^{d(2-m)} \rho * V$ as an *a priori* potential, we can apply our results in Section 2, and obtain L^∞ bound of ρ which does not depend on T, together with uniform continuity of ρ, and Hölder continuity of ρ for $1 < m < 2$.

Theorem 3.3. *Let $m > 1$ and consider V that satisfies the assumptions **(V1)** and **(V2)**. Let $\rho(x,t)$ be the unique weak solution to Eq.(1.1) given by Theorem 3.2, with nonnegative initial data $\rho(x,0) \in C(\mathbb{T}_L^d)$, which satisfies $\int_{\mathbb{T}_L^d} \rho(x,0) dx = 1$. Then $\|\rho(x,t)\|_{L^\infty(\mathbb{T}_L^d \times [0,\infty))}$ is bounded, where the bound only depend on $\sup_x \rho(x,0)$, θ, $\|V\|_{C^2}$ and L.*

Proof. To begin with, note that Theorem 3.2 guarantees the existence and uniqueness of the weak solution to Eq.(1.1), which we denote by by ρ. Now we treat $\Phi := \theta L^{d(2-m)} \rho * V$ as an *a priori* potential, and we obtain the following estimate of Φ assumption **(V2)**:

$$
\begin{aligned}
\|\Phi(\cdot,t)\|_{C^2(\mathbb{T}_L^d)} &\leq \theta L^{d(2-m)} \|\rho(\cdot,t)\|_{L^1(\mathbb{T}_L^d)} \|V\|_{C^2(\mathbb{T}_L^d)} \\
&\leq \theta L^{d(2-m)} \|V\|_{C^2(\mathbb{T}_L^d)} \\
&= \theta L^{d(2-m)} C \quad \text{for all } t \geq 0.
\end{aligned}
$$

We denote by ρ_1 the unique weak solution to the equation

$$(\rho_1)_t = \Delta \rho_1^m + \nabla \cdot (\rho_1 \nabla \Phi) \qquad (3.2)$$

with initial data $\rho_1(\cdot, 0) \equiv \rho(\cdot, 0)$, where the existence and uniqueness is proved in [BH]. Theorem 2.1 implies $\sup_x \|\rho_1(\cdot, t)\|$ is bounded uniformly in t. Moreover, note that ρ also satisfies the weak equation for Eq.(3.2), hence ρ must coincide with ρ_1, which yields a uniform bound of ρ which doesn't depend on time. $\qquad \square$

Applying Theorem 2.3 to Eq.(3.2), we have the continuity of ρ uniformly in t for $m > 1$ – in particular (in light of Theorem 3.5 below) for the case $m = 2$.

Theorem 3.4. *Let $m > 1$ and consider V that satisfies the assumptions **(V1)** and **(V2)**. Let $\rho(x,t)$ be the unique weak solution to Eq.(1.1) given by Theorem 3.2, with nonnegative initial data $\rho(\cdot, 0)$ satisfying $\|\rho(\cdot, 0)\|_{L^\infty(\mathbb{T}_L^d)} < \infty$, and $\|\rho(\cdot, 0)\|_{L^1(\mathbb{T}_L^d)} = 1$. Then for any $\tau > 0$, ρ is continuous in $\mathbb{T}_L^d \times [\tau, \infty)$, where the continuity is uniform in both x and t.*

Proof. Follows immediately from the above reasoning, Theorem 3.3 and Theorem 2.3 $\qquad \square$

Applying Theorem 2.4 to Eq.(3.2), with $\Phi = \theta L^{d(2-m)} \rho * V$ we have the Hölder continuity of ρ for $1 < m < 2$.

Theorem 3.5. *Let $1 < m < 2$ and consider V that satisfies the assumptions **(V1)** and **(V2)**. Let $\rho(x,t)$ be the unique weak solution to Eq.(1.1) given by Theorem 3.2, with nonnegative initial data $\rho(x, 0)$ satisfying $\|\rho(\cdot, 0)\|_{L^\infty(\mathbb{T}_L^d)} < \infty$, and $\|\rho(\cdot, 0)\|_{L^1(\mathbb{T}_L^d)} = 1$. Then for any $\tau > 0$, u is Hölder continuous in $\mathbb{T}_L^d \times (\tau, \infty)$, where the Hölder exponent and coefficient depend on τ, m, d, θ, L and C and the L^∞ norm of the initial condition.*

Proof. Follows immediately from the preceding reasoning, Theorem 3.3 and Theorem 2.4

$\qquad \square$

4 The Case $m = 2$: Analysis Via Normal Modes

In this section, we will use Fourier Transform to study the PDE in Eq.(1.1), and this method works best when $m = 2$. We continue to assume, without loss of generality that $\|\rho(x,0)\|_{L^1(\mathbb{T}_L^d)} = 1$, however from the perspective of *functional analysis*, the homogeneity of the special case $m = 2$ makes even this stipulation redundant.

The dynamics in Eq.(1.1) is governed by gradient flow for the "free energy" functional

$$\mathcal{F}_\theta(\rho) = \int_{\mathbb{T}_L^d} \rho^2 + \frac{1}{2}\theta\rho(\rho * V)dx. \qquad (4.1)$$

For the analysis of the functional $\mathcal{F}_\theta$, since we are assuming $\rho(x,0)$ integrates to 1, we shall denote by $\mathscr{P}$ the class of probability densities on $\mathbb{T}_L^d$ which also belong to $L^2(\mathbb{T}_L^d)$, i.e.

$$\mathscr{P} := \{f \in L^1(\mathbb{T}_L^d) \cap L^2(\mathbb{T}_L^d) : \|f\|_{L^1(\mathbb{T}_L^d)} = 1\}. \qquad (4.2)$$

Special to the case $m = 2$ is that the functional $\mathcal{F}_\theta(\cdot)$ can be expressed in a simpler form if we express ρ in terms of its Fourier modes. We write

$$\hat{\rho}(k) = \int_{\mathbb{T}_L^d} \rho(x)\mathrm{e}^{-ik \cdot x}dx$$

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where k is of the form $k = \frac{2\pi}{L}\vec{n}$ with $\vec{n} \in \mathbb{Z}^d$. With these conventions we have

$$\rho(x) = \frac{1}{L^d} \sum_k \hat{\rho}(k)\mathrm{e}^{ik\cdot x}$$

and, in terms of these variables, Eq.(4.1) becomes

$$\mathcal{F}_\theta(\rho) = \frac{1}{L^d} \sum_k |\hat{\rho}(k)|^2 (1 + \frac{1}{2}\theta\hat{V}(k)). \tag{4.3}$$

On the basis of Eq.(4.3), a salient value of θ emerges: We denote this value by $\theta^\sharp$, which is defined via

$$[\theta^\sharp]^{-1} := \frac{1}{2} \max_{k \neq 0}\{|\hat{V}(k)|; \hat{V}(k) < 0\}. \tag{4.4}$$

Formally $\theta^\sharp$ may be designated as $+\infty$ in case $\hat{V}(k) \geq 0$ for all $k \neq 0$ – i.e. if V is (essentially) of positive type. For the purposes of the present discussion, we shall assume otherwise. Different values of θ separate our problem into 3 cases:

1. (subcritical) When $\theta < \theta^\sharp$, we have $1 + \frac{1}{2}\theta\hat{V}(k) > 0$ for all $k \in \mathbb{Z}^d$, then under the restriction $\hat{\rho}(0) = 1$, it is manifest that global minimizer for $\mathcal{F}_\theta(\rho)$ in $\mathscr{P}$ is the constant solution

$$\rho_0(x) := \frac{1}{L^d}\int_{\mathbb{T}_L^d} \rho(x,0)dx \equiv \frac{1}{L^d}. \tag{4.5}$$

2. (critical) When $\theta = \theta^\sharp$, we have still have $1 + \frac{1}{2}\theta\hat{V}(k) \geq 0$ for all $k \in \mathbb{Z}^d$ however now there is a set $\mathbb{K}^\sharp$ (containing at least two elements) defined by the condition that for $k \in \mathbb{K}^\sharp$, $1 + \frac{1}{2}\theta^\sharp\hat{V}(k) = 0$. In this case the global minimizers for $\mathcal{F}_\theta(\rho)$ in $\mathscr{P}$ take the form

$$\rho(x) = \rho_0 + \sum_{k \in \mathbb{K}^\sharp} c_k \mathrm{e}^{ik\cdot x}, \tag{4.6}$$

   where $c_{-k} = \bar{c}_k$ and, of course, subject to the restriction that the resultant quantity is non–negative.

3. (supercritical) When $\theta > \theta^\sharp$, we have $1 + \frac{1}{2}\theta\hat{V}(k) < 0$ for some $k \in \mathbb{Z}^d$. In this case the constant solution ρ_0 is not even a local minimizer of $\mathcal{F}_\theta$ in $\mathscr{P}$, let alone global minimizer.

Remark 4.1. The above – which is *manifest* for $m = 2$ – is in sharp contrast to the cases $m \neq 2$. In particular, for general m there is an analogous quantity $\theta^\sharp$ given by

$$[\theta^\sharp]^{-1} := \frac{1}{m} \max_{k \neq 0}\{|\hat{V}(k)|; \hat{V}(k) < 0\}$$

where items (1) – (3) are suggested. However, the following was shown for $m = 1$ and, presumably holds for all $m \neq 2$: While for $\theta < \theta^\sharp$, the constant solution has "some stability" (c.f. [CP] Theorem 2.11 for the case $m = 1$) there is a $\theta_{\mathrm{T}} < \theta^\sharp$ where global considerations come into play. In particular, at $\theta = \theta_{\mathrm{T}}$, there is a non–uniform minimizer for $\mathcal{F}_{\theta_{\mathrm{T}}}(\cdot)$ which is degenerate with the uniform solution. Moreover, for $\theta > \theta_{\mathrm{T}}$ (which implies, in particular, at $\theta = \theta^\sharp$) the uniform solution is no longer a minimizer.

4.1 The subcritical case, when m=2

In the subcritical case, the constant solution ρ_0 is the only global minimizer of $\mathcal{F}_\theta$ in $\mathscr{P}$. Our goal in this section is to show for every non–negative initial data $\rho(x,0) \in L^\infty(\mathbb{T}_L^d)$ which integrates to 1, the weak solution $\rho(x,t)$ converges to ρ_0 exponentially in $L^2(T_L^d)$ as $t \to \infty$, where ρ_0 is as given in Eq.(4.5).

By formally taking the time derivative of the free energy functional, a simple calculation indicates that e.g., at least for classical solutions to Eq.(1.1), the free energy is always non–increasing:

$$\frac{d}{dt}\mathcal{F}_\theta(\rho) = -\int_{\mathbb{T}_L^d} \rho \Big|\nabla(\frac{m}{m-1}\rho^{m-1} + \theta L^{d(2-m)}\rho * V)\Big|^2 dx. \tag{4.7}$$

In [BS], it is proved that Eq.(4.7) is indeed true in the integral sense:

Lemma 4.2 (Bertozzi–Slepčev). *Consider V that satisfies the assumptions (V1) and (V2). Let $\rho(x,t)$ be a weak solution of Eq.(1.1) in $\mathbb{T}_L^d \times [0,T]$. Then for almost all $\tau \in [0,T]$,*

$$\mathcal{F}_\theta(\rho(\cdot,0)) - \mathcal{F}_\theta(\rho(\cdot,\tau)) \geq \int_0^\tau \int_{\mathbb{T}_L^d} \rho|\nabla(\frac{m}{m-1}\rho^{m-1} + \theta L^{d(2-m)}\rho * V)|^2 dxdt \tag{4.8}$$

Remark 4.3. Theorem 3.4 implies that $\rho(\cdot,t)$ is a continuous function of t, hence $\mathcal{F}_\theta(\rho(\cdot,t))$ is continuous in t as well. Therefore Eq.(4.8) indeed holds for all $\tau \in [0,T]$ and, moreover, Eq.(4.7) may be regarded as a *differential inequality*.

In the following lemma, we show when $\theta < \theta^\sharp$, the free energy will decay to the free energy of the global minimizer as $t \to \infty$.

Lemma 4.4. *Suppose $m = 2$ and consider V that satisfies the assumptions (V1) and (V2). Further suppose that $\theta < \theta^\sharp$, where $\theta^\sharp$ is as given in Eq.(4.4) – including $\theta^\sharp = \infty$ if V is of positive type. Let $\rho(x,t)$ be the weak solution to Eq.(1.1) on $[0,\infty) \times \mathbb{T}_L^d$, with non-negative initial data $\rho(x,0) \in L^\infty(\mathbb{T}_L^d)$ which integrates to 1. Then $\mathcal{F}_\theta(\rho) \to \mathcal{F}_\theta(\rho_0)$ as $t \to \infty$, where ρ_0 is the uniform solution (as given in Eq.(4.5)).*

Proof. By Lemma 4.2, we know $\mathcal{F}_\theta(\rho(t))$ is a continuous and decreasing function of t, whose limit is bounded below by $\mathcal{F}_\theta(\rho_0)$, since ρ_0 is the global minimizer of $\mathcal{F}_\theta$ in $\mathscr{P}$ when $\theta < \theta^\sharp$. Hence we can send τ to infinity in Eq.(4.8), which gives

$$\int_0^\infty \int_{\mathbb{T}_L^d} \rho|\nabla(2\rho + \theta\rho * V)|^2 dxdt < \infty. \tag{4.9}$$

Then there exists an increasing sequence of time $(t_n)_{n=1}^\infty$, where $\lim_{n\to\infty} t_n = \infty$, such that

$$\lim_{n\to\infty} \int_{\mathbb{T}_L^d} \rho(x,t_n)|\nabla(2\rho(x,t_n) + \theta\rho(x,t_n) * V)|^2 dx = 0. \tag{4.10}$$

To avoid clutter, in what follows, we shall abbreviate $\rho(\cdot,t_n)$ by ρ_n. Recall that Theorem 2.1 gives us a uniform bound of $\|\rho_n\|_{L^\infty(\mathbb{R}^d)}$. In addition, by [Dib], (ρ_n) is uniformly equicontinuous, hence Arzelà-Ascoli Theorem enables us to find a subsequence of ρ_n (which we again denote by ρ_n for notational simplicity), and a continuous function ρ_∞, such that

$$\lim_{n\to\infty} \|\rho_n - \rho_\infty\|_{L^\infty(\mathbb{T}_L^d)} = 0, \tag{4.11}$$

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We next claim that $\|\nabla \rho_n^{3/2}\|_{L^2(\mathbb{T}_L^d)}$ is bounded uniformly in n. To prove the claim, we first note that

$$\int_{\mathbb{T}_L^d} \left|\frac{4}{3}\nabla\rho_n^{3/2} + \rho_n^{1/2}\nabla(\theta\rho_n * V)\right|^2 dx = \int_{\mathbb{T}_L^d} \rho_n \left|2\nabla\rho_n + \nabla(\theta\rho_n * V)\right|^2 dx \to 0. \qquad (4.12)$$

To obtain the uniform L^2 bound for $\nabla\rho_n^{3/2}$, due to the triangle inequality, it suffices to prove a uniform L^2 bound for $\rho_n^{1/2}\nabla(\theta\rho_n * V)$, which is true since ρ_n is uniformly bounded in n and $\|V\|_{C^2(\mathbb{T}_L^d)} < \infty$ due to **(V2)**, hence the claim is proved.

As a consequence of the claim, we obtain weak convergence of $\nabla\rho_n^{3/2}$ in L^2 (along another subsequence) And, it is clear, the limit is just $\nabla\rho_\infty^{3/2}$ due to the uniform convergence of the (ρ_n). (Moreover, this places $\nabla\rho_\infty^{3/2} \in L^2(\mathbb{T}_L^d : \mathbb{R}^d)$). Thus:

$$\nabla\rho_n^{3/2} \rightharpoonup \nabla\rho_\infty^{3/2} \text{ as } n \to \infty \text{ weakly in } L^2(\mathbb{T}_L^d : \mathbb{R}^d). \qquad (4.13)$$

Let

$$B_n := \frac{4}{3}\nabla\rho_n^{3/2} + \rho_n^{1/2}\nabla(\theta\rho_n * V).$$

Then Eq.(4.11) and Eq.(4.13) and an additional uniform convergence argument identifying the weak limit of $\rho_n^{1/2}\nabla(\theta\rho_n * V)$, implies that B_n weakly converges to B_∞ in L^2, where

$$B_\infty := \frac{4}{3}\nabla\rho_\infty^{3/2} + \rho_\infty^{1/2}\nabla(\theta\rho_\infty * V).$$

On the other hand, recall that Eq.(4.12) gives us that $B_n \to 0$ strongly in L^2, thus we have B_∞ is indeed 0 i.e.,

$$\int_{\mathbb{T}_L^d} \left|\frac{4}{3}\nabla\rho_\infty^{3/2} + \rho_\infty^{1/2}\nabla(\theta\rho_\infty * V)\right|^2 dx = \int_{\mathbb{T}_L^d} \rho_\infty \left|2\nabla\rho_\infty + \nabla(\theta\rho_\infty * V)\right|^2 dx = 0. \qquad (4.14)$$

In particular, then, $\nabla(\rho_\infty + \frac{1}{2}\theta\rho_\infty * V)$ is zero a.e. on the support of ρ_∞. Now ρ_∞ certainly admits a weak derivative which, clearly, is non–zero only on the support of ρ_∞. Thus, from the preceding, we can write

$$\int_{\mathbb{T}_L^d} \nabla\rho_\infty \cdot \nabla(\rho_\infty + \frac{1}{2}\theta\rho_\infty * V)dx = 0. \qquad (4.15)$$

Now, we wish to express the above as a Fourier sum which requires some additional justification. To this end we claim that ρ_∞ is Lipschitz continuous – i.e., in $W^{1,\infty}(\mathbb{T}_L^d)$ – which places both entities in $L^2(\mathbb{T}_L^d)$ and vindicates the use of explicit formulas.

The equation $\nabla\rho_\infty = -\frac{1}{2}\theta\nabla(V * \rho_\infty)$ valid on the support of ρ shows that in the various *components* where ρ_∞ is positive, it is at least C^2. Indeed, in general, Hypothesis **(V2)** immediately implies $\|\rho_\infty(x) * V\|_{C^2(\mathbb{T}_L^d)} \le \|\rho_\infty\|_{L^1}\|V\|_{C^2(\mathbb{T}_L^d)}$ so whenever ρ_∞ satisfies this ($m = 2$ version of the Kirkwood–Monroe) equation, we have Lipschitz continuity with uniform constant. We shall denote this constant by κ. Now suppose that $x, y \in \mathbb{T}_L^d$ have $\rho_\infty(x)$ and $\rho_\infty(y)$ positive. Let us assume, ostensibly, that x and y belong to different components. On the (shortest) line joining x and y, let z_x denote the first point, starting from x that is encountered on the boundary of the component of x and similarly for z_y. Then

$$\begin{aligned}
|\rho_\infty(x) - \rho_\infty(y)| &= |\rho_\infty(x) - \rho_\infty(z_x) + \rho_\infty(z_y) - \rho_\infty(y)| \\
&\le |\rho_\infty(x) - \rho_\infty(z_x)| + |\rho_\infty(z_y) - \rho_\infty(y)| \\
&\le \kappa[|x - z_x| + |y - z_y|] \le \kappa|x - y|;
\end{aligned} \qquad (4.16)$$

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the first inequality due to $\rho_\infty(z_x) = \rho_\infty(z_y) = 0$ and the last inequality because all four points lie in order on the same line. A similar argument can be used if, e.g., $\rho_\infty(x)$ is positive and $\rho_\infty(y)$ is zero.

All of this establishes enough regularity to unabashedly express Eq.(4.15) in Fourier modes:

$$0 = \sum_k \frac{|k|^2}{L^d} |\hat{\rho}_\infty(k)|^2 (1 + \frac{1}{2}\theta\hat{V}(k)). \tag{4.17}$$

By the defining property of $\theta^\sharp$ we have $1 + \frac{1}{2}\theta\hat{V}(k) > 0$ for all $k \neq 0$, thus Eq.(4.17) implies $\hat{\rho}_\infty(k) = 0$ for all $k \neq 0$, i.e. $\rho_\infty \equiv \rho_0$.

Now, we may use the monotonicity in time of $\mathcal{F}_\theta(\rho(t))$ and we finally have

$$\lim_{t\to\infty} \mathcal{F}_\theta(\rho(t)) = \lim_{n\to\infty} \mathcal{F}_\theta(\rho_n) = \mathcal{F}_\theta(\rho_\infty) = \mathcal{F}_\theta(\rho_0)$$

which is the stated claim. $\qquad\square$

By combining the above result with the uniform continuity in time, we can show the solution will become uniformly positive after a sufficiently large time.

Corollary 4.5. *Under the assumption of Lemma 4.4, we have*

$$\lim_{t\to\infty} \|\rho(\cdot,t) - \rho_0\|_{L^\infty(\mathbb{T}_L^d)} = 0,$$

hence there exists $T > 0$ depending on θ, $\|V\|_{C^2(\mathbb{T}_L^d)}$ and $\rho(\cdot,0)$, such that $\rho(x,t) > \rho_0/2$ for all $x \in \mathbb{T}_L^d, t > T$.

Proof. We prove the statement in the display. Supposing that this is not the case. Then there is a sequence of times, (τ_n) and points (y_n) – $y_n \in \mathbb{T}_L^d$ – and a $\delta > 0$ such that

$$|\rho(y_n, \tau_n) - \rho_0| > \delta.$$

Now, going to a further subsequence, we have $y_n \to y_\infty$ (with $y_\infty \in \mathbb{T}_L^d$ by compactness). But, along yet a further subsequence, not relabeled, we have, according to the arguments of Lemma 4.4 that $\rho(\cdot, \tau_n)$ is converging uniformly and the limit *must* be ρ_0. Thus

$$\lim_{n\to\infty} \rho_n(y_n, \tau_n) = \lim_{n\to\infty} [\rho_n(y_n, \tau_n) - \rho_n(y_\infty, \tau_n)] + \lim_{n\to\infty} \rho_n(y_\infty, \tau_n) = \rho_0$$

in contradiction with the preceding display. $\qquad\square$

Theorem 4.6. *Suppose $m = 2$ and $\theta < \theta^\sharp$, where $\theta^\sharp$ is as given in Eq.(4.4). Consider V that satisfies the assumptions (V1) and (V2). Let $\rho(x,t)$ be the weak solution to Eq.(1.1) on $[0,\infty) \times \mathbb{T}_L^d$, with non–negative initial data $\rho(x,0) \in L^\infty(\mathbb{T}_L^d)$ which integrates to 1. Then $\mathcal{F}_\theta(\rho(t))$ decays exponentially to $\mathcal{F}_\theta(\rho_0)$, where the rate depend on $\rho(x,0)$. Moreover, $\|\rho(\cdot,t) - \rho_0\|_{L^2(\mathbb{T}_L^d)} \to 0$ exponentially, i.e.*

$$0 \leq \mathcal{F}_\theta(\rho(t)) - \mathcal{F}_\theta(\rho_0) \leq C_1 \exp(-\frac{\rho_0 c'}{L^2}t),$$

and

$$\|\rho(t) - \rho_0\|_{L^2(\mathbb{T}_L^d)} \leq C_2 \exp(-\frac{\rho_0 c'}{L^2}t),$$

where c' and C_1 and C_2 depend on θ, V and $\rho(\cdot,0)$.

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Proof. By Lemma 4.5, there exist some $T > 0$ depending on θ, V and $\rho(\cdot, 0)$, such that $\rho(x, t) > \rho_0/2$ for all $x \in \mathbb{T}_L^d, t > T$. Then for all $t_2 > t_1 > T$, Eq.(4.8) becomes

$$
\begin{aligned}
\mathcal{F}_\theta(\rho(\cdot, t_1)) - \mathcal{F}_\theta(\rho(\cdot, t_2)) &\geq \int_{t_1}^{t_2} \int_{\mathbb{T}_L^d} \frac{\rho_0}{2} |\nabla(2\rho + \theta\rho * V)|^2 dx dt \\
&= 2\rho_0 \int_{t_1}^{t_2} \frac{1}{L^d} \sum_k |k|^2 |\hat{\rho}(k)|^2 (1 + \frac{1}{2}\theta\hat{V}(k))^2 dt \\
&\geq \rho_0 c' \int_{t_1}^{t_2} \frac{1}{L^d} \sum_{k \neq 0} |\hat{\rho}(k)|^2 (1 + \frac{1}{2}\theta\hat{V}(k)) dt \\
&= \rho_0 c' \int_{t_1}^{t_2} (\mathcal{F}_\theta(\rho(\cdot, t)) - \mathcal{F}_\theta(\rho_0)) dt, \quad (4.18)
\end{aligned}
$$

where $c' = 2\min_{k\neq 0} |k^2|(1 + \frac{1}{2}\theta\hat{V}(k))$, which is positive when $\theta < \theta^\sharp$.

In the spirit of Remark 4.3 we may regard the above as a differential inequality for $g(t) := \mathcal{F}_\theta(\rho(\cdot, t)) - \mathcal{F}_\theta(\rho_0)$; the inequality reads

$$
-\frac{dg}{dt} \geq \rho_0 c' g(t).
$$

This immediately integrates to yield $g(t) \leq g(T)\exp\{-\rho_0 c'(t - T)\}$ for $t \geq T$. I.e.,

$$
\mathcal{F}(\rho(\cdot, t)) - \mathcal{F}(\rho_0) \leq C e^{-\rho_0 c' t}.
$$

Since $\mathcal{F}_\theta(\rho(\cdot, t)) - \mathcal{F}_\theta(\rho_0)$ is comparable to $\|\rho(t) - \rho_0\|_{L^2}$, we have $\|\rho(t) - \rho_0\|_{L^2} \to 0$ exponentially with the same rate. □

Remark 4.7. It is remarked that, via comparison to linearized theory, the above is essentially optimal. (The results differ by a factor of two which comes from the definition of $T =: T_{1/2}$. Using $T_\epsilon = \sup\{t > 0 \mid \|\rho(\cdot, t) - \rho_0\|_{L^\infty(\mathbb{T}_L^d)} > \epsilon\rho_0\}$, the *long* time asymptotic rates are actually in complete agreement.) Moreover, while for L of order unity, the result stands: c' – with or without an additional factor of two – might well be optimized at a wave number of order unity. However, as $L \to \infty$, it is clear that

$$
\min_{k\neq 0} |k|^2(1 + \frac{1}{2}\hat{V}(k)) \to (\frac{2\pi}{L})^2(1 + \frac{1}{2}\hat{V}(0)).
$$

So, in particular, for large L the rate scales as $L^{-(d+2)}$ – a result which may be an artifact of our normalization.

4.2 Some remarks on the supercritical case, when $m = 2$

When $\theta > \theta^\sharp$, we have $1 + \frac{1}{2}\theta\hat{V}(k_0) < 0$ for some $k_0 = \frac{2\pi}{L}\vec{n}_0$, where $\vec{n}_0 \in \mathbb{Z}^d$. In other words, at least one of the coefficients of the free energy Eq.(4.3) is negative. In the next proposition we show that in this case the constant solution ρ_0 is not linearly stable.

Proposition 4.8. *Suppose $m = 2$ and $\theta < \theta^\sharp$, where $\theta^\sharp$ is as given in Eq.(4.4). Consider an interaction V that satisfies the assumptions (V1) and (V2). Then the constant solution ρ_0 is not a local minimizer of the free energy Eq.(4.3) in $\mathscr{P}$.*

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Proof. We choose $k_0 = \frac{2\pi}{L}\vec{n}_0$ such that $1 + \frac{1}{2}\theta\hat{V}(k_0) < 0$, where $\vec{n}_0 \in \mathbb{Z}^d$. We add a small pertubation $\epsilon\eta$ to the constant solution ρ_0, where

$$\eta := \cos(\frac{2\pi n_0 \cdot x}{L}).$$

Then

$$\mathcal{F}_\theta(\rho_0 + \epsilon\eta) = \mathcal{F}_\theta(\rho_0) + L^d\epsilon^2(1 + \frac{1}{2}\theta\hat{V}(k_0)),$$

which is strictly less than $\mathcal{F}_\theta(\rho_0)$ by the defining property of k_0. $\qquad\square$

Remark 4.9. In fact, using the same perturbation term in the proof, we would know that when $\theta > \theta^\sharp$, any strictly positive function is not a local minimizer of the free energy Eq.(4.3).

In the supercritical case, while Eq.(4.3) immediately implies that ρ_0 is not a local minimizer of $\mathcal{F}_\theta$ in $\mathscr{P}$, it gives us little information about what is the global minimizer. The difficulty comes from the restriction $\rho(x) \geq 0$ for all x, which evidently plays an important role in the supercritical case, since any minimizer should touch zero somewhere due to Remark 4.9. After Fourier transform, the non-negativity of ρ actually gives us infinite numbers of restrictions, which causes the difficulty.

5   Exponential decay for $1 < m < 2$ and weak interaction

In this section, we continue our study of Eq.(1.1) with $m \in (1, 2)$ and here we will assume that θ is "small". Unfortunately, θ will *not* be uniformly small in volume. In particular, we shall require $\theta L^{d(2-m)}$ to be a small number of order unity and, under these conditions we shall acquire all the results of the previous section. We claim that without additional (physics based) assumptions – in particular H–stability of the interaction – the above condition is essentially optimal. Specifically, our cornerstone result of a unique stationary state does not hold for non–H–stable interactions when $\theta L^{d(2-m)}$ is a sufficiently *large* number of order unity. However, from an æsthetic perspective, this uniqueness result is the sole instance where $\theta L^{d(2-m)}$ must be considered small. In the aftermath of Proposition 5.1 and its corollary, we will only require θ itself to be a small quantity.

We start with a priliminary result (which is, actually, just a quantitative version of the argument used in Lemma 4.4 in the vicinity of Eq.(4.16)).

Proposition 5.1. *Consider an interaction V that satisfies the assumptions **(V1)** and **(V2)**. Let*

$$\varepsilon_0 := \theta L^{d(2-m)}$$

be a sufficiently small number of order unity. Let ρ denote any solution to the Kirkwood–Monroe equations which here read, whenever $\rho > 0$,

$$\nabla\rho^{m-1} = -\varepsilon_0\frac{m-1}{m}\rho * \nabla V$$

and let

$$R := \|\rho\|_{L^\infty(\mathbb{T}_L^d)}.$$

Then if ε_0 is a small number of order unity then R is also a small number of order unity (if L is large). In particular,

$$R \leq \kappa_4 \max\{[\varepsilon_0]^{\frac{d}{d(m-1)+1}}, L^{-d}\}$$

with κ_4 a constant of order unity.

Proof. From the mean–field equations,

$$|\nabla \rho^{m-1}| \leq \frac{m-1}{m}\varepsilon_0 \int_{\mathbb{T}_L^d} |\nabla V(x-y)|\rho(y)dy \leq \frac{m-1}{m}\|V\|_{C_1}\varepsilon_0 =: \kappa_1\varepsilon_0.$$

Let x_0 mark the spot where ρ achieves R. Then, for all x,

$$\rho^{m-1}(x) \geq R^{m-1} - \kappa_1\varepsilon_0|x-x_0|.$$

Thus, if r is the length scale of the region about x_0 where ρ^{m-1} exceeds, a.e., $\frac{1}{2}R^{m-1}$ we have

$$r \geq \frac{R^{m-1}}{2\kappa_1\varepsilon_0}$$

provided the right hand side does not exceed L. Otherwise, obviously, $r = L$. Since ρ integrates to unity we have, assuming $r < L$,

$$1 = \int_{\mathbb{T}_L^d} \rho dx \geq \kappa_2 r^d R \geq \kappa_2 \frac{R^{d(m-1)+1}}{(2\kappa_1\varepsilon_0)^d} =: \frac{1}{\kappa_3^d}\frac{1}{\varepsilon_0^d}R^{d(m-1)+1}$$

(with κ_2 a geometric constant of order unity) and otherwise we acquire the mundane bound. After a small step, the stated bound is obtained with an appropriate definition of κ_4. □

With the above in hand, we can establish that ρ_0 is the unique stationary solution. We start with

Corollary 5.2. *Under the conditions stated in Proposition 5.1, if ε_0 is sufficiently small – but of order unity independent of L – the unique solution to the mean–field equations is $\rho = \rho_0$.*

Proof. From the mean–field equation, we may write

$$0 = \int_{\mathbb{T}_L^d} \nabla\rho \cdot \nabla(\rho^{m-1} + \varepsilon_0\frac{m-1}{m}\rho * V)dx.$$

By recapitulating the Lipchitz continuity that was featured in the vicinity of Eq.(4.16) we have full justification to manipulate classically under the integral. Letting R_{ε_0} denote the upper bound on the L^∞ norm of ρ that was featured in Proposition 5.1. Then, pointwise a.e. on the support of ρ,

$$\nabla\rho \cdot \nabla\rho^{m-1} = \frac{m-1}{\rho^{2-m}}|\nabla\rho|^2 \geq \frac{m-1}{R_{\varepsilon_0}^{2-m}}|\nabla\rho|^2$$

since, we remind the reader, $2 - m > 0$. In other words,

$$0 \geq \int_{\mathbb{T}_L^d} \frac{1}{R_{\varepsilon_0}^{2-m}}|\nabla\rho|^2 + \frac{\varepsilon_0}{m}\nabla\rho \cdot \nabla(\rho * V)dx.$$

We can again go to Fourier modes and the above reads

$$0 \geq \sum_{k\neq 0} k^2|\hat{\rho}(k)|^2[\frac{1}{R_{\varepsilon_0}^{2-m}} + \frac{\varepsilon_0}{m}\hat{V}(k)].$$

For ε_0 sufficiently small (but of order unity independent of L) the coefficient of $|\hat{\rho}(k)|^2$ is positive for all terms so the later must vanish identically. The desired result is proved. □

Based on the fact that ρ_0 is the unique stationary solution, in the next lemma we prove that $\rho(\cdot, t)$ will converge to ρ_0 uniformly, (but not with a quantitative estimate on the *rate.*)

Lemma 5.3. *Suppose the conclusions in Corollary 5.2 are satisfied. Let $\rho(x,t)$ be the weak solution to Eq.(1.1) on $[0,\infty) \times \mathbb{T}_L^d$, with non-negative initial data $\rho(x,0) \in L^\infty(\mathbb{T}_L^d)$ which integrates to 1. Then $\sup_x |\rho(\cdot, t) - \rho_0| \to 0$ as $t \to \infty$.*

Proof. This is more or less identical to the proof of Corollary 4.5 based on Lemma 4.4 so we shall be succinct. Assuming the result false, we could find a sequence of times $t_n \to \infty$ and points $x_n \to x_\infty \in \mathbb{T}_L^d$ such that $\rho(\cdot, t_n)$ converges uniformly and yet $|\rho(x_n, \tau_n) - \rho_0| > \delta$. So, denoting by $\rho_\infty(\cdot)$ the uniform limit, we would have $|\rho_\infty(x_\infty) - \rho_0| > \delta$.

Hence, since ρ_∞ is continuous, it is definitively *not* equal to ρ_0. However, any subsequential limit must satisfy the mean–field equation and by Corollary 5.2 this is uniquely ρ_0 in contradiction with the preceding. This completes the proof. $\qquad\square$

In the next lemma, we show that once ρ and ρ_0 becomes comparable, $\mathcal{F}_\theta(\rho) - \mathcal{F}_\theta(\rho_0)$ also becomes comparable with $L^{d(2-m)}\|\rho - \rho_0\|_{L^2(\mathbb{T}_L^d)}$. Indeed, as alluded to earlier, this will be proved under the weaker assumption that θ – not $\theta L^{d(m-2)}$ – is small. We start with:

Lemma 5.4. *Suppose that $\theta > 0$ is sufficiently small (but of order unity independent of L). Let ρ be such that $\|\rho - \rho_0\|_{\mathbb{T}_L^d} < \frac{1}{2}\rho_0$. Then we have*

$$\alpha L^{d(2-m)}\|\rho - \rho_0\|_{L^2(\mathbb{T}_L^d)}^2 \le \mathcal{F}_\theta(\rho) - \mathcal{F}_\theta(\rho_0) \le \beta L^{d(2-m)}\|\rho - \rho_0\|_{L^2(\mathbb{T}_L^d)}^2 \qquad (5.1)$$

for some $\alpha, \beta > 0$ of order unity.

Proof. First, by any number of methods we have

$$\int_{\mathbb{T}_L^d \times \mathbb{T}_L^d} \rho(x)\rho(y)V(x-y)dxdy \ge -K_V\|\rho - \rho_0\|_{L^2(\mathbb{T}_L^d)}^2;$$

e.g., we may take, using the Fourier decomposition, $K_V = [\theta^\sharp]^{-1}$. Similarly for a corresponding *upper* bound with a positive constant. Let us turn to the entropic–like terms.

Writing $\rho = \rho_0(1 + \eta)$, our assumption implies that $|\eta| \le \frac{1}{2}$. From this it is easy to verify that, pointwise,

$$(1 + \eta)^m \ge 1 + m\eta + \frac{m(m-1)}{2}\left(\frac{2}{3}\right)^{2-m}\eta^2 := 1 + m\eta + a\eta^2,$$

and for the other direction we have

$$(1 + \eta)^m \le 1 + m\eta + \frac{m(m-1)}{2}(3)^{2-m}\eta^2 := 1 + m\eta + b\eta^2.$$

Thence $\rho^m - \rho_0^m = \rho_0^m[(1 + \eta)^m - 1] = \rho_0^m[(1 + \eta)^m - 1 - m\eta + m\eta] \ge \rho_0^m[m\eta + a\eta^2]$. So

$$\int_{\mathbb{T}_L^d} (\rho^m - \rho_0^m)dx \ge a\rho_0^m\|\eta\|_{L^2(\mathbb{T}_L^d)}^2 = aL^{d(2-m)}\|\rho - \rho_0\|_{L^2(\mathbb{T}_L^d)}^2,$$

and similarly we have

$$\int_{\mathbb{T}_L^d} (\rho^m - \rho_0^m)dx \le bL^{d(2-m)}\|\rho - \rho_0\|_{L^2(\mathbb{T}_L^d)}^2.$$

Combining this with the bounds on the energy term, the stated claim has been established. $\qquad\square$

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Finally, in the next theorem, we prove that the free energy decays exponentially to its minimum value.

Theorem 5.5. *Suppose the conclusions acquired in Corollary 5.2 are satisfied and suppose that θ is a sufficiently small number which is of the order of unity. Let $\rho(x,t)$ be the weak solution to Eq.(1.1) on $[0,\infty) \times \mathbb{T}_L^d$, with non-negative initial data $\rho(x,0) \in L^\infty(\mathbb{T}_L^d)$ which integrates to 1. Then $\mathcal{F}_\theta(\rho(t))$ decays exponentially to $\mathcal{F}_\theta(\rho_0)$. More precisely,*

$$\mathcal{F}(\rho(\cdot,t)) - \mathcal{F}(\rho_0) \le C_1 e^{-\rho_0^{m-1} c' t} \tag{5.2}$$

for various constants c' and C_1. Similarly for the L^2–norm of $(\rho - \rho_0)$ with a different prefactor.

Proof. According to Lemma 5.3, there exist some $T > 0$ depending on θ, L, V and $\rho(\cdot,0)$, such that $|\rho(x,t) - \rho_0| < \frac{1}{2}\rho_0$ for all $x \in \mathbb{T}_L^d, t > T$. Then for all $t_2 > t_1 > T$, we manipulate the integrand on the right hand side of Eq.(4.8) – the lower bound on $\mathcal{F}_\theta(\rho(\cdot,t_1)) - \mathcal{F}_\theta(\rho(\cdot,t_2))$:

$$\int_{\mathbb{T}_L^d} \rho |\nabla \frac{m}{m-1}\rho^{m-1} + \theta L^{d(2-m)}\nabla\rho * V|^2 dx \ge$$

$$\int_{\mathbb{T}_L^d} \rho \left[\frac{1}{2} |\nabla \frac{m}{m-1}\rho^{m-1}|^2 - |\theta L^{d(2-m)}\nabla\rho * V|^2 \right] dx$$

$$= \int_{\mathbb{T}_L^d} \left[\frac{1}{2}m^2\rho^{2m-3}|\nabla\rho|^2 - \rho\theta^2 L^{2d(2-m)}|\nabla(\rho * V)|^2 \right] dx$$

$$\ge \int_{\mathbb{T}_L^d} \left[g\rho_0^{2m-3}|\nabla\rho|^2 - \frac{3}{2}\rho_0\theta^2 L^{2d(2-m)}|\nabla(\rho * V)|^2 \right] dx \tag{5.3}$$

where the value of g – which is always of order unity – depends on whether $2m - 3$ is positive or not. Note that all terms are proportional to $\rho_0^{2m-3} = \rho_0^{m-1}L^{d(2-m)}$.

Going to Fourier modes, the final (spatial) integral in the above string becomes

$$\rho_0^{m-1}L^{d(2-m)} \cdot \frac{1}{L^d} \sum_k k^2 |\hat{\rho}(k)|^2 [g - \frac{3}{2}\theta^2|\hat{V}(k)|^2]$$

where, for sufficiently small θ, we may assert that the summand is positive.

We thus have

$$\mathcal{F}_\theta(\rho(\cdot,t_1)) - \mathcal{F}_\theta(\rho(\cdot,t_2)) \ge$$

$$\rho_0^{m-1}\beta c' \int_{t_1}^{t_2} \int_{\mathbb{T}_L^d} L^{d(2-m)}(\rho - \rho_0)^2 dx dt \ge \rho_0^{m-1}c' \int_{t_1}^{t_2} [\mathcal{F}_\theta(\rho(\cdot,t)) - \mathcal{F}_\theta(\rho_0)] dt \tag{5.4}$$

where in the above, β is the constant from Lemma 5.4 which has been conveniently absorbed into the definition of c':

$$c'\beta := \min_{k \ne 0}[k^2(g - \frac{3}{2}\theta^2|\hat{V}(k)|^2)]$$

and in the final step we have used Lemma 5.4.

Note that Eq.(5.4) has the same form as Eq.(4.18) therefore we can again treat it as a differential inequality as in the proof of Theorem 4.6. We obtain that

$$\mathcal{F}(\rho(\cdot,t)) - \mathcal{F}(\rho_0) \le C_1 e^{-\rho_0^{m-1} c' t}.$$

A further application of Lemma 5.4 implies a similar result for the L^2–norm of $(\rho - \rho_0)$ and the proof is finished. $\square$

Remark 5.6. Here as in the case $m = 2$, when L is large, $c' \propto L^{-2}$ and we have the large L scaling of the rate proportional to $L^{-(2+d(m-1))}$ in agreement with a perturbative analysis. However in this case, our arguments do not provide agreement with the constant of proportionality. We also note that by Theorem 2.4 we have that $\rho(\cdot, t)$ is uniformly Hölder continuous in space and time for all $t \geq T$, where the Hölder coefficient and exponent depends on θ, L and V. Thus we can bound, the L^∞–norm of $\rho - \rho_0$ by some power of its L^2–norm. Hence the exponential convergence of $\|\rho - \rho_0\|_{L^2(\mathbb{T}_L^d)}$ implies the exponential convergence of $\|\rho - \rho_0\|_{L^\infty(\mathbb{T}_L^d)}$. However a bound along these lines is "even more" non–optimal since the two norms should, presumably, differ by a factor of L^d.

6   Appendix: Proof of Lemma 2.5

Proof of Lemma 2.5. We will do the comparison between ρ^- and w first; the comparison between ρ^+ and w can be done in the same way.

First note that w also satisfies Eq.(2.9) with $\Phi \equiv 0$, therefore the inequality Eq.(2.25) also hold for w, namely

$$w - \rho^- \geq -C_1 a, \tag{6.1}$$

where C_1 depends on m, d, M_1, M_2.

We define $f := w - \rho^-$, and our goal is to obtain an upper bound for f. More precisely, we want to show there exists some constant C and β depending on m, d, M_1, M_2, such that $f(x, t) \leq Ca^\beta$ in $\tilde{\Omega} \times [1, 2]$.

Our strategy is as following. First, we claim that

$$g(T) := \sup_{y \in \tilde{\Omega}} \int_{B(y,1) \cap \tilde{\Omega}} f(x, T) dx < C_0 a \text{ for all } T \in [0, 2], \tag{6.2}$$

where C_0 depends on m, d, M_1, M_2. We will prove this claim momentarily. Once we have the claim, we know the space integral of $f(x, t)$ in any unit ball is of order a, for $0 < t < 2$. To get $f(x, t) \leq O(a^\beta)$ for $t \in [1, 2]$, it suffices to show f is Hölder continuous in space with exponent and constant that are uniform in time for all $t \in [1, 2]$, which is indeed true, since Theorem 11.2 of [DGV] guarantees this uniform Hölder continuity of ρ^- and w for $t \in [1, 2]$.

Now it suffices to prove our claim. It is proved by writing both equations in weak form, choosing an appropriate test function and applying the Gronwall inequality. By writing both Eq.(2.19) and Eq.(2.26) in weak form and subtracting the two equations, we arrive at

$$\underbrace{\int_{\tilde{\Omega}} f(x, T)\varphi(x)dx}_{I_1} = \underbrace{\int_{\tilde{\Omega}} f(x, 0)\varphi(x)dx}_{I_2} + \underbrace{\int_0^T \int_{\tilde{\Omega}} \left(w^m - \rho^- |\rho^-|^{m^- - 1}\right)\Delta\varphi(x) + Ma\varphi(x) \, dx \, dt}_{I_3}, \tag{6.3}$$

where $\varphi \in C_0^\infty(\tilde{\Omega})$ is a test function chosen as follows. For a fixed $T > 0$, there exists $z \in \tilde{\Omega}$, such that the maximum of $\int_{B(y,1) \cap \tilde{\Omega}} f(x, T) dx$ is achieved at z. We then define

$$\varphi(x) := \mu * h^z(x),$$

where μ is a standard mollifier supported in $B(0, \frac{1}{10})$, and

$$h^z(x) := \begin{cases} 1 - |x - z|^2/2 & \text{for } |x - z| \leq 1 \\ (|x - z| - 2)^2/2 & \text{for } 1 < |x - z| \leq 2 \\ 0 & \text{for } |x - z| > 2 \end{cases} \tag{6.4}$$

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For such φ, we have $0 < \varphi < 1$, inside the ball $B(z,1)$ and $\int_{\tilde{\Omega}} \varphi dx < |B(z,3)| < 6^d$.

To estimate I_1, note that $\varphi(x) \geq 1/3$ in $B(z,1)$, and $f(x,T) + C_1 a \geq 0$ in $\tilde{\Omega}$, which implies

$$
\begin{aligned}
I_1 &= \int_{\tilde{\Omega}} (f(x,T) + C_1 a)\varphi(x)dx - \int_{\tilde{\Omega}} C_1 a \varphi(x)dx \\
&\geq \frac{1}{3} \int_{B(z,1)\cap\tilde{\Omega}} (f(x,T) + C_1 a)dx - 6^d C_1 a \\
&\geq \frac{g(T)}{3} - 6^d C_1 a.
\end{aligned}
$$

For I_2, since $f(x,0) = (\frac{m}{m^-})^{m^-} \tilde{\rho}(x,0)^{1-ca} - \tilde{\rho}(x,0)$, we would obtain $f(x,0) < C_2 a$, where C_2 depends on $m, \|\tilde{\rho}(\cdot,0)\|_\infty$ and c, (hence depends on m, d, M_1, M_2), which yields

$$
I_2 \leq C_2 a \int_{\tilde{\Omega}} \varphi(x)dx \leq 6^d C_2 a.
$$

Now we start to estimate I_3. Due to the definition of m^- in Eq. (2.15), we have $m^- - m \leq 2(m-1)ca$. Also, we can derive some *a priori* bound of $\rho^-(x,t)$ and $w(x,t)$ for $t \in [1,2]$, which depend on m, d, M_1, M_2. Then we have

$$
\left| w^m - \rho^- |\rho^-|^{m^- - 1} \right| \leq C_3 |w - \rho^-| + C_4 a \quad \text{in } \tilde{\Omega} \times [0,2],
$$

where C_3, C_4 depends on m, d, M_1, M_2. Together with the fact that $|\Delta\varphi|$ is bounded, in particular by d, in $B(z,3)$ and vanishes outside of $B(z,3)$, we obtain the following bound for I_3:

$$
\begin{aligned}
I_3 &\leq \int_{\tilde{\Omega}} (C_3|f| + C_4 a)|\Delta\varphi| + Ma\varphi dx \\
&\leq dC_3 \int_{B(z,3)\cap\tilde{\Omega}} |f(x,t)|dx + 6^d(dC_4 + M)a \\
&\leq dC_3 \sum_{i=1}^{c_d} \int_{B(z+x_i,1)\cap\tilde{\Omega}} |f(x,t)|dx + 6^d(dC_4 + M)a, \quad (6.5)
\end{aligned}
$$

where in the last inequality we denote by c_d the number such that $B(0,3)$ can be covered by c_d numbers of balls of radius 1, centered at $x_1, \ldots x_{c_d}$ respectively. Note that c_d is a constant only depending on d.

Finally, we wish to control $\int_{B(z+x_i,1)\cap\tilde{\Omega}} |f|dx$. Note that $f \geq -C_1 a$ implies $|f| \leq f + 2C_1 a$, which yields

$$
\begin{aligned}
\int_{B(z+x_i,1)\cap\tilde{\Omega}} |f(x,t)|dx &\leq \int_{B(z+x_i,1)\cap\tilde{\Omega}} fdx + 2^d 2C_1 a \\
&\leq g(t) + 2^{d+1}C_1 a
\end{aligned}
$$

Plugging it into Eq.(6.5), we obtain

$$
I_3 \leq dC_3 c_d g(t) + (dC_3 c_d 2^{d+1} C_1 + 6^d dC_4 + 6^d M)a
$$

By putting estimates of I_1, I_2, I_3 together, we have

$$
g(T) \leq C_5 \int_0^T g(t)dt + C_6 a \quad \text{for } T \in [0,2]
$$

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where C_5, C_6 only depend on m, d, M_1, M_2. And for initial data, we have $g(0) \leq |B(0,1)| \sup_x f(x,0) \leq 2^d C_2 a$. By Gronwall inequality, we have $g(T) \leq C_0 a$ for all $T \in [0,2]$, where C_0 only depends on m, d, M_1, M_2, hence our claim Eq.(6.2) is proved. $\square$

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