On the Rate of Convergence for Critical Crossing Probabilities

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Abstract

For the site percolation model on the triangular lattice and certain generalizations for which Cardy's Formula has been established we acquire a power law estimate for the *rate* of convergence of the crossing probabilities to Cardy's Formula.

1 Introduction

Starting with the work [15] and continuing in: [6], [7], [18] [11], [4], [5], the validity of Cardy's formula [8] – which describes the limit of crossing probabilities for certain percolation models – and the subsequent consequence of an SLE_6 description for the associated limiting *explorer process* has been well established. The purpose of this work is to provide some preliminary quantitative estimates. Similar work along these lines has already appeared in [3] (also see [13]) in the context of the so–called loop erased random walk for both the observable and the process itself. Here, our attention will be confined to the percolation observable as embodied by Cardy's formula for crossing probabilities.

While in the case of the loop erased random walk, estimates on the observable can be reduced to certain Green's function estimates, in the case of percolation the observables are not so readily amenable. Instead of Green's functions, we shall have to consider the Cauchy integral representation of the complexified *crossing probability functions*, as first introduced in [15]. As demonstrated in [15] (see also [2] and [11]) these functions converge to conformal maps from the domain under consideration – where the percolation process takes place – to the equilateral triangle. Thus, a combination of some analyticity property and considerations of boundary value should, in principal, yield a rate of convergence.

However, the associated procedure requires a few domain deformations, each of which must be demonstrated to be "small", in a suitable sense. While such considerations are not important for *very regular* domains (which we will not quantify) in order to consider general domains, a more robust framework for quantification is called for. For this purpose, we shall introduce a procedure where all portions of the domain are explored via percolation crossing problems. This yields a multi–scale sequence of neighborhoods around each boundary point where the nature of the boundary irregularities determines the sequence of successive scales. Thus, ultimately, we are permitted to measure the distances between regions by counting the number of neighborhoods which separate them. This procedure is akin to the approach of Harris [12] in his study of the critical state at a time when detailed information about the nature of the state was unavailable.

Ultimately we establish a power law estimate (in mesh size) for the rate of convergence in any domain with boundary dimension less than two. (For a precise statement see the Main Theorem below.) As may or may not be clear to the reader at this point the hard quantifications must be done via percolation estimates – as is perhaps not surprising since we cannot easily utilize continuum estimates before having reached the continuum in the first place. The plausibility of a power law estimate then follows from the fact that most *a priori* percolation estimates are of this form.

Finally, we should mention, it has come to our attention (via reference no. 83 in [16] and also a seminar abstract) that similar results have been announced by Mendelson, Nachmias, Sheffield, Watson; however, at the time of this present paper, we know of no other definitive writings on this subject.

2 Preliminaries

2.1 The Models Under Consideration

We will be considering critical percolation models in the plane. However in contrast to the generality professed in [4], [5] – where, essentially, "all" that was required was a proof of Cardy's formula, here the mechanism of how Cardy's formula is established will come into play. Thus, we must restrict attention to the triangular site percolation problem considered in [15] and the generalization provided in [11]. These models can all be expressed in terms of random colorings (and sometimes double colorings) of hexagons. As is traditional, the competing colors are designated by blue and yellow.

We remind the reader that criticality implies that there are non–trivial (bounded away from 0 and 1) bounds on crossings of squares in both blue and yellow and that via the so–called Russo–Seymour–Welsh estimates, these generate scale–invariant bounds on crossings of longer rectangles.

2.2 The Observable

Consider a fixed domain $\Omega \subset \mathbb{C}$ that is a conformal rectangle with marked points (or prime ends) A, B, C and D which, as written, are in cyclic order. We let Ω_n denote the lattice approximation at scale $\varepsilon = n^{-1}$ to the domain Ω. The details of the construction of Ω_n – especially concerning boundary values and explorer processes – are somewhat tedious and have been described e.g., in [5] §3 & §4 and [4] §4.2. For present purposes, it is sufficient to know that Ω_n consists of the maximal union of lattice hexagons – of diameter $1/n$ – whose closures lie entirely inside Ω; we sometimes refer to this as the *canonical approximation*. We shall also have occasions later to use discrete interior approximating domains which are a subset of Ω_n. Moreover, boundary segments can be appropriately colored and lattice points A_n – D_n can be selected. We consider *percolation* problems in Ω_n.

The pertinent object to consider is a crossing probability: Performing percolation on Ω_n, we ask for the crossing probability – say in yellow – from (A_n, B_n) to (C_n, D_n). Below we list various facts, definitions and notations related to the observable that will be used throughout this work. In some of what follows, we temporarily neglect attention to the marked point A_n and regard Ω_n with the three remaining marked points as a conformal triangle.

- Let us recall the functions introduced in [15], here denoted by S_B, S_C, S_D where e.g., $S_B(z)$ with $z \in \Omega_n$ a lattice point, is the probability of a yellow crossing from (D_n, B_n) to (B_n, C_n) separating z from (C_n, D_n). Note that it is implicitly understood that the S_B, S_C, S_D–functions are defined on the discrete level; to avoid clutter, we suppress the n index for these functions. Moreover, we will denote the underlying *events* associated to these functions by $\mathbb{S}_B, \mathbb{S}_C, \mathbb{S}_D$, respectively.

- It is the case that the functions S_B, S_C, S_D are invariant under color switching: these models exhibit color switching symmetry (see [15] and [11]). While it is not essential to the arguments in this work, we sometimes may take liberties regarding whether we are considering a yellow or blue version of these functions.

- It is also easy to see that e.g., S_B has boundary value 0 on (C_n, D_n) and 1 at the point B_n. Moreover, the complexified function $S_n = S_B + \tau S_C + \tau^2 S_D$, with $\tau = e^{2\pi i/3}$, converges to the conformal map to the equilateral triangle with vertices at $1, \tau, \tau^2$, which we denote by $\mathbb{T}$. (See [15], [2], [5].)

- For finite n, we shall refer to the object $S_n(z)$ as the *Carleson–Cardy–Smirnov* function and sometimes abbreviated CCS–function.

- We will use $H_n : \Omega_n \to \mathbb{T}$ to denote the unique conformal map which sends (B_n, C_n, D_n) to $(1, \tau, \tau^2)$. Similarly, $H : \Omega \to \mathbb{T}$ is the corresponding conformal map of the continuum domain.

- With A_n reinstated, we will denote by $\mathscr{C}_n$ the crossing probability of the conformal rectangle Ω_n and $\mathscr{C}_\infty$ its limit in the domain Ω; i.e., Cardy's Formula in the limiting domain.

- Since $S_C(A_n) \equiv 0$,

$$S_n(A_n) = S_B(A_n) + \tau^2 S_D(A_n) = [S_B(A_n) - \frac{1}{2} S_D(A_n)] - i\frac{\sqrt{3}}{2} S_D(A_n).$$

  Now we recall (or observe) that $\mathscr{C}_n$ can be realized as $S_D(A_n)$ and so from the previous display, $\mathscr{C}_n = \frac{2}{\sqrt{3}} \cdot \mathrm{Im}[S_n(A_n)]$. Since it is already known that S_n converges to H (see [15], [2], [5]) it is also the case that $\mathscr{C}_\infty = \frac{2}{\sqrt{3}} \cdot \mathrm{Im}[H(A)]$. Therefore to establish a *rate* of convergence of $\mathscr{C}_n$ to $\mathscr{C}_\infty$, it is sufficient to show that there is some $\psi > 0$ such that

$$|S_n(A_n) - H(A)| \lesssim n^{-\psi};$$

- The functions S_n are not *discrete analytic* but they do have discrete analytic properties. (See [15], [2] and [11].) In particular, this is exhibited by the fact that the contour integral around some discrete contour Γ_n behaves like the length of Γ_n times n to some negative power. Also, the functions S_n are Hölder continuous with estimates which are uniform for large n. For details we refer the reader to Definition 4.1.

Our goal in this work is to acquire the following theorem on the rate of convergence of the finite volume crossing probability, $\mathscr{C}_n$, to its limiting value:

Main Theorem. *Let Ω be a domain and Ω_n its canonical discretization. Consider the site percolation model or the models introduced in [11] on the domain Ω_n. In the case of the latter we also impose the assumption that the boundary Minkowski dimension is less than 2 (in the former, this is not necessary). Let $\mathscr{C}_n$ be described as before. Then there exists some $\psi > 0$ such that $\mathscr{C}_n$ converges to its limit with the estimate*

$$|\mathscr{C}_n - \mathscr{C}_\infty| \lesssim n^{-\psi},$$

provided $n \geq n_0(\Omega)$ is sufficiently large and the symbol $\lesssim$ is described with precision in Notation 2.1 below.

Notation 2.1 *In the above and throughout this work, we will be describing asymptotic behaviors of various quantities as a function of small or large parameters (usually n in one form or another). The relation $X \lesssim Y$ relating two functions X and Y of large or small parameters (below denoted by M and m, respectively) means there exists a constant c which is of order unity independent of m and M such that for all M sufficiently large and/or m sufficiently small $X(m, M) \leq c \cdot Y(m, M)$.*

Remark 2.2. The restrictions on the boundary Minkowski dimension for the models in [11] is not explicitly important in this work and will only be implicitly assumed as it was needed in order to establish convergence to Cardy's Formula.

Remark 2.3. It would seem that complementary lower bounds of the sort presented in the Main Theorem are actually *not* possible. For example, in the triangular site model, the crossing probabilities for particular shapes are identically $\frac{1}{2}$ independently of n.

We end this preliminary section with some notations and conventions: (i) the notation $\mathrm{dist}(\cdot,\cdot)$ denotes the usual Euclidean distance while the notation $d_{sup}(\cdot,\cdot)$ denotes the sup–norm distance between curves; (ii) we will make use of both *macroscopic* and *microscopic* units, with the former corresponding to an $\varepsilon \to 0$ approximation and the latter corresponding to $n \to \infty$, thereby measuring distances relative to the size of a hexagon. Thus, $n = \varepsilon^{-1}$ – while analytical quantities are naturally in macroscopic units, it is at times convenient to use microscopic units when performing percolation constructions; (iii) we will use $a_1, a_2, \dots$ to number the powers of n appearing in the *statements* of lemmas, theorems, etc. Constants used in the course of a proof are considered temporary and duly forgotten after the Halmos box.

3 Proof of the Main Theorem

Our strategy for the proof of the Main Theorem is as follows: recall that H_n is the conformal map from Ω_n to $\mathbb{T}$ (the "standard" equilateral triangle) so that B_n, C_n, D_n map to the three corresponding vertices, where it is reiterated that $\mathscr{C}_n$ corresponds to a boundary value of S_n. Thus it is enough to uniformly estimate the difference between S_n and H_n and then the difference between H_n and H.

Foremost, the discrete domain may itself be a bit too rough so we will actually be working with an approximation to Ω_n which will be denoted by $\Omega_n^{\square}$ (see Proposition 3.2). Now, on $\Omega_n^{\square}$, we have the function $S_n^{\square}$ associated with the corresponding percolation problem *on this domain* and, similarly, the conformal map $H_n^{\square} : \Omega_n^{\square} \to \mathbb{T}$. Via careful consideration of *physical* (i.e., Euclidean) distances and distortion under the conformal map, we will be able to show that both $|S_n(A_n) - S_n^{\square}(A_n^{\square})|$ (for an appropriately chosen $A_n^{\square} \in \partial\Omega_n^{\square}$) and $|H(A) - H_n^{\square}(A_n^{\square})|$ are suitably small (see Theorem 3.3). Thus we are reduced to proving a power law estimate for the domain $\Omega_n^{\square}$.

Towards this goal, we introduce the *Cauchy–integral extension* of $S_n^{\square}$, which we denote by $F_n^{\square}$, so that

$$F_n^{\square}(z) := \frac{1}{2\pi i} \oint_{\partial\Omega_n^{\square}} \frac{S_n^{\square}(\zeta)}{\zeta - z}\, d\zeta.$$

Now by using the Hölder continuity properties and the *approximate* discrete analyticity properties of the S_n's, we can show that, barring the immediate vicinity of the boundary, the difference between $F_n^{\square}$ and $S_n^{\square}$ is power law small (see Lemma 3.5). It follows then that in an even smaller domain, $\Omega_n^{\blacklozenge}$, which can be realized as the inverse image of a uniformly shrunken version of $\mathbb{T}$, the function $F_n^{\square}$ is in fact *conformal* and thus it is uniformly close to $H_n^{\blacklozenge}$, which is *the* conformal map from $\Omega_n^{\blacklozenge}$ to $\mathbb{T}$ (see Lemma 3.9).

Now for $z \in \Omega_n^{\square}$ the dichotomy we have introduced is not atypical: on the one hand $F_n^{\square}(z)$ is manifestly analytic but does not necessarily embody the function $S_n^{\square}$ of current interest. On the other hand, $S_n^{\square}(z)$ has the desired boundary values – at least on $\partial\Omega_n^{\square}$ – but is, ostensibly, lacking in analyticity properties. Already the approximate discrete analyticity properties permits us to compare $F_n^{\square}$ to $S_n^{\square}$ in $\Omega_n^{\blacklozenge}$. In order to return to the domain $\Omega_n^{\square}$, we require some control on the "distance" between $\Omega_n^{\blacklozenge}$ and $\Omega_n^{\square}$ (not to mention a suitable choice of some point $A_n^{\blacklozenge} \in \partial\Omega_n^{\blacklozenge}$ as an approximation to A). It is indeed the case that *if* $\Omega_n^{\blacklozenge}$ is close to $\Omega_n^{\square}$ in the *physical* (i.e., Hausdorff) distance, then the proof can be quickly completed by using distortion estimates and/or Hölder continuity of the S function. However, such information translates into an estimate on the continuity properties of the *inverse* of $F_n^{\square}$, which is not *a priori* accessible (and, strictly speaking, not always true).

Further thought reveals that we in fact require the domain $\Omega_n^{\blacklozenge}$ to be close to $\Omega_n^{\square}$ in both the conformal sense and in the sense of "percolation" – which can be understood as being measured via local crossing probabilities. While with a *deliberate* choice of a point on the boundary corresponding to A we may be able to ensure that either one or the other of the two criteria can be satisfied, it is not readily demonstrable that *both* can be simultaneously satisfied without some additional detailed considerations; it is for this reason that we will introduce and utilize the notion of *Harris systems* (see Theorem 3.10) in order to quantify the distances between $\Omega_n^{\blacklozenge}$ and $\Omega_n^{\square}$.

The Harris systems are collections of concentric annular segments of various scales centered on points of $\partial\Omega_n^{\square}$ and heading towards some "central region" of $\Omega_n^{\square}$; they are constructed so that uniform estimates are available for both the traversing of each segment and the existence of an obstructing "circuit" (in dual colors). This leads to a natural choice of $A_n^{\blacklozenge}$: it is a point on $\partial\Omega_n^{\blacklozenge}$ which is in the Harris system of $A_n^{\square}$. Consequently, the distance between $A_n^{\square}$ and $A_n^{\blacklozenge}$ – and other such pairs as well – can be measured vis a counting of Harris segments (see Lemma 3.12).

Specifically, we will make use of another auxiliary point, $A_n^{\lozenge}$, which is also in the Harris system stationed at $A_n^{\square}$, chosen so that it is *inside* the domain $\Omega_n^{\blacklozenge}$. The task of providing an estimate for $|S_n^{\square}(A_n^{\lozenge}) - S_n^{\square}(A_n^{\square})|$ (and thus also $|F_n^{\square}(A_n^{\lozenge}) - S_n^{\square}(A_n^{\square})|$) is immediately accomplished by the existence of suitably many Harris segments surrounding both $A_n^{\square}$ and $A_n^{\lozenge}$ (see Proposition 3.15). Also, considering n to be *fixed*, the domain $\Omega_n^{\square}$ can be approximated at scales $N^{-1} \ll n^{-1}$ and the estimates derived from the Harris systems remain *uniform* in N as N tends to infinity (corresponding to "*continuum percolation*" on $\Omega_n^{\square}$) thus also immediately implying an estimate for $|H_n^{\square}(A_n^{\square}) - H_n^{\square}(A_n^{\lozenge})|$ (see Proposition 3.14).

At this point what remains to be established is an estimate relating the conformal map $H_n^{\blacklozenge}$, which is defined by percolation at scale n via $F_n^{\square}$, and $H_n^{\square}$, the "original" conformal map. It is here that we shall invoke a Marchenko theorem for the triangle $\mathbb{T}$ (see Lemma 3.16): indeed, again considering $\Omega_n^{\square}$ to be a fixed domain and performing percolation at scales $N^{-1} \ll n^{-1}$, we have by convergence to Cardy's Formula that $S_{n,N}^{\square}(s) \to H_n^{\square}(s)$ as $N \to \infty$, for all $s \in \partial\Omega_n^{\blacklozenge}$. The inherent *scale invariance* of the Harris systems permits us to establish that in fact $S_{n,N}^{\square}(s)$ is close to $\partial\mathbb{T}$, *uniformly* in N (see Lemma 3.18) and thus, $H_n^{\square}(\partial\Omega_n^{\blacklozenge})$ is close to $\partial\mathbb{T}$ (in fact in the *supremum norm*). Armed with this input, the relevant Marchenko theorem applied at the point $A_n^{\lozenge}$ immediately gives that $H_n^{\square}(A_n^{\lozenge}) - H_n^{\blacklozenge}(A_n^{\lozenge})$ is suitably small.

The technical components relating to the Cauchy–integral estimate and the construction of the Harris systems are relegated to Section 4 and Section 5, respectively. As for the rest, we will divide the proof of the main theorem into three subsections, corresponding to:

(i) the regularization of the boundary (introduction of $\Omega_n^{\square}$) and showing that crossing probabilities are close for the domains $\Omega_n^{\square}, \Omega_n$ & Ω;

(ii) the construction of the Cauchy–integral $F_n^{\square}$ and the construction of the domain $\Omega_n^{\blacklozenge}$;

(iii) the establishment of the remaining estimates needed to show that the domains $\Omega_n^{\blacklozenge}$ and $\Omega_n^{\square}$ are suitably close, by using the Harris systems of neighborhoods.

3.1 Regularization of Boundary Length

We now introduce the domain $\Omega_n^{\square} \subseteq \Omega_n$. The primary purpose of this domain is to reduce the boundary length of the domain that need be considered.

Definition 3.1. Let $1 > a_1 > 0$ and consider a square grid whose elements are squares of (approximately) microscopic size n^{a_1} and let $\Omega_n^{\square}$ denote the union of all (hexagons within the) squares of this grid that are entirely within the original domain Ω.

We have:

Proposition 3.2 *Let $\Omega \subseteq \mathbb{C}$ be a domain with boundary Minkowski dimension less than $1+\alpha'$ with $\alpha' \in [0,1]$, which we write as $M(\partial\Omega) < 1+\alpha$ for any $\alpha > \alpha'$. Then the domain $\Omega_n^{\square}$ satisfies $\Omega_n^{\square} \subseteq \Omega_n$ and*

$$|\partial\Omega_n^{\square}| \lesssim n^{\alpha(1-a_1)}.$$

Proof. Since $M(\partial\Omega) < 1+\alpha$ we have (for all n sufficiently large) that the number of boxes required to cover $\partial\Omega$ is essentially bounded from above by $(n^{1-a_1})^{1+\alpha}$ which is then multiplied by $\frac{1}{n^{(1-a_1)}}$, the size of the box (in macroscopic units). The fact that $\Omega_n^{\square} \subseteq \Omega_n$ is self–evident. $\square$

Next we will choose $A_n^{\square}, B_n^{\square}, C_n^{\square}, D_n^{\square} \in \partial\Omega_n^{\square}$ by some procedure to be outlined below and denote by $S_n^{\square}$ the corresponding CCS–function. Particularly, this can be done so that the crossing probabilities do not change much:

Theorem 3.3 *Let $\Omega_n^{\square} \subseteq \Omega_n$ with marked boundary points $(A_n, \dots, D_n)$ be as described, so particularly $\partial\Omega_n^{\square}$ is of distance at most n^{1-a_1} from $\partial\Omega_n$. Then there is an $A_n^{\square}$ as well as $B_n^{\square}$, $C_n^{\square}$ and $D_n^{\square}$ such that the corresponding $S_n^{\square}$ satisfies, for some $a_2 > 0$ and for all n sufficiently large,*

$$|S_n(A_n) - S_n^{\square}(A_n^{\square})| \lesssim n^{-a_2}$$

and, moreover,

$$|H(A) - H_n^{\square}(A_n^{\square})| \lesssim n^{-a_2}.$$

Remark 3.4. In the case that the separation between A_n and $\partial\Omega_n$ is the order of n^{a_1} – as is usually imagined – facets of Theorem 3.3 are essentially trivial. However, the reader is reminded that A_n could be deep inside a "fjord" and well separated from $\partial\Omega_n^{\square}$. In this language, the forthcoming arguments will demonstrate that, notwithstanding, an $A_n^{\square}$ may be chosen near the mouth of the fjord for which the above estimates hold.

Proof of Theorem 3.3. For $\eta > 0$ and a subset $K \subset \overline{\Omega}$ we will denote by $N_\eta(K)$ the η–neighborhood of K intersected with Ω. Now let us first choose η sufficiently small so that

$$[[B, C, D] \cup N_{4\eta}(B) \cup N_{4\eta}(D) \cup N_{4\eta}(C)] \cap N_{4\eta}(A) = \emptyset,$$

where $[B, C, D]$ denotes the closed boundary segment containing the prime ends B, C, D.

Next we assume that $n > n_\circ$ where $n_\circ$ is large enough so that for all $n > n_\circ$, $A_n \in N_\eta(A)$, ..., $D_n \in N_\eta(D)$. Moreover, $\Omega_n^{\square} \cap N_\eta(A) \neq \emptyset$ and similarly for $\Omega_n^{\square} \cap N_\eta(B), \dots, \Omega_n^{\square} \cap N_\eta(D)$. Then, since

$$0 < \mathrm{dist}(([A, B] \setminus N_\eta(A)), ([D, A] \setminus N_\eta(A)))$$

it is assumed that for $n > n_\circ$, the above is very large compared with $n^{-(1-a_1)}$ and similarly for the other three marked points. Finally, consider the uniformization map $\varphi : \mathbb{D} \to \Omega$. Then taking $n_\circ$ larger if necessary, we assert that for all $n > n_\circ$, the distance between $\varphi^{-1}(N_\eta(A))$ and $[\varphi^{-1}(N_{4\eta}(A))]^{\mathrm{c}}$ satisfies

$$\mathrm{dist}[\varphi^{-1}(N_\eta(A)), [\varphi^{-1}(N_{4\eta}(A))]^{\mathrm{c}}] \gg n^{-\frac{1}{2}}. \tag{3.1}$$

We first state:

Claim. For $n > n_\circ$,

$$\mathrm{dist}(N_\eta(A), [B_n, C_n, D_n]) > 0.$$

Proof of Claim. We note that the pre–image of $\partial\Omega_n$ under uniformization has the following property: for n sufficiently large as specified above, consider the segment $\varphi^{-1}([A_n, B_n])$. Then starting at $\varphi^{-1}(A_n)$, once the segment enters $\varphi^{-1}(N_\eta(B_n))$, it must hit $\varphi^{-1}(B_n)$ before exiting $\varphi^{-1}(N_{4\eta}(B_n))$.

Indeed, supposing this were not true, then necessarily, there would be three or more crossings of the "annular region" $\varphi^{-1}(N_{4\eta}(B_n)) \setminus \varphi^{-1}(N_\eta(B_n))$. It is noted that all such crossings – indeed all of $\varphi^{-1}(\Omega_n)$ – lies within a distance of the order $n^{-1/2}$ of

$\partial\mathbb{D}$. This follows by standard distortion estimates (see e.g., [14], Corollary 3.19 together with Theorem 3.21) and the definition of canonical approximation: each point on $\partial\Omega_n$ is within distance $1/n$ of some point on $\partial\Omega$. It is further noted, by the final stipulation concerning $n_\circ$, that the separation scale of the above mentioned "annular region" is large compared with the distance $n^{-1/2}$.

Consider now a point on the "topmost" of these crossings which is well away – compared with $n^{-1/2}$ – from the lateral boundaries of the annular region and also the pre–image of its associated hexagon. Since this point is the pre–image of one on $\partial\Omega_n$, the hexagon in question must intersect $\partial\Omega$ and therefore its pre–image must intersect $\partial\mathbb{D}$. However, in order to intersect $\partial\mathbb{D}$, the pre–image of the hexagon in question must intersect *all* the lower crossings, since our distortion estimate does not permit this pre–image to leave (a lower portion of) the annular region. This necessarily implies it passes through the interior of Ω_n, which is impossible for a boundary hexagon.

The same argument also shows that once $\varphi^{-1}(\partial\Omega_n)$ exits $\varphi^{-1}(N_{4\eta}(B_n))$, it cannot re–enter $\varphi^{-1}(N_\eta(B_n))$ so therefore must be headed towards $\varphi^{-1}(C_n)$ and certainly cannot enter $\varphi^{-1}(N_\eta(A))$ since

$$\text{dist}(\varphi^{-1}(N_\eta(A)), \varphi^{-1}([B, C, D] \cup [N_{4\eta}(B) \cup N_{4\eta}(D) \cup N_{4\eta}(C)]) \gg n^{-1/2}$$

by assumption (by the choice of η, it is the case that $[B, C, D] \cup [N_{4\eta}(B) \cup N_{4\eta}(D) \cup N_{4\eta}(C)] \subseteq [N_{4\eta}(A)]^c$ from which the previous display follows from Equation (3.1)).

Altogether we then have that $\text{dist}(\varphi^{-1}(N_\eta(A)), \varphi^{-1}([B_n, C_n, D_n])) > 0$, and so the claim follows after applying φ.

$\square$

The above claim in fact implies that there exist points $A_n^p \in [A_n, B_n]$ and $A_n^g \in [A_n, D_n]$ such that

$$\text{dist}(A_n^p, A_n^g) < \frac{1}{n^{1-a_1}}$$

and

$$\text{dist}(A_n^p, \partial\Omega_n^\square),\ \text{dist}(A_n^g, \partial\Omega_n^\square) < \frac{1}{n^{1-a_1}}.$$

Indeed, consider squares of side length n^{a_1} intersecting $\partial\Omega_n$ which share an edge with $\partial\Omega_n^\square$ and have non–trivial intersection with $N_\eta(A)$, then since $\partial\Omega_n$ passes through such boxes, we can unambiguously label them as either an $[A_n, B_n]$, an $[A_n, D_n]$ box, or both, and by the claim there are no other possibilities. Therefore, a pair of such boxes of differing types must be neighbors or there is at least one single box of both types, so we indeed have points A_n^p, A_n^g as claimed. Finally, by the stipulation

$$\frac{1}{n^{1-a_1}} \ll \operatorname{dist}(([A,B] \setminus N_\eta(A)), ([D,A] \setminus N_\eta(A)))$$

it is clear that these points must lie in $N_\eta(A)$ since otherwise they would have a *single* label: either $[A_n, B_n]$ or $[A_n, D_n]$. Thus we choose $A_n^\square \in \partial\Omega_n^\square$ to be any representative point near the (A_n^p, A_n^g) juncture. Now consider the scale n^{a_3} with $1 > a_3 > a_1$. We may surround the points A_n^p, A_n^g and $A_n^\square$ with the order of $\log_2 n^{a_3 - a_1}$ disjoint concentric annuli each of which forms a conduit between $[A_n, D_n]$ and $[A_n, B_n]$. Let $\mathcal{A}$ denote the event that at least one of these annuli houses a blue circuit, then we have

$$\mathbb{P}(\mathcal{A}) \geq 1 - n^{-a_4} \tag{3.2}$$

for some $a_4 > 0$. Similar constructions may be enacted about the $B_n, B_n^\square; \ldots; D_n, D_n^\square$ pairs leading, ultimately, to the events $\mathcal{B}, \ldots, \mathcal{D}$ analogous to $\mathcal{A}$ with estimates on their probabilities as in Eq.(3.2). For future reference, we denote by $\mathcal{E}$ the event $\mathcal{A} \cap \cdots \cap \mathcal{D}$.

We are now in a position to verify that $|S_n(A_n) - S_n^\square(A_n^\square)|$ obeys the stated power law estimate. Indeed, the C–component of both functions vanish identically while the differences between the other two components amount to comparisons of crossing probabilities on the "topological" rectangles $[A_n, B_n, C_n, D_n]$ verses $[A_n^\square, B_n^\square, C_n^\square, D_n^\square]$. There are two crossing events contributing to the (complex) function $S_n(A_n)$ (and similarly for $S_n^\square(A_n^\square)$) but since the arguments are identical, it is sufficient to treat one such crossing event. Thus we denote by $\mathbb{K}_n$ the event of a crossing in Ω_n by a blue path between the $[A_n, D_n]$ and $[B_n, C_n]$ boundaries (the event contributing to $S_B(A_n)$) and similarly for the event $\mathbb{K}_n^\square$ for a blue path in $\Omega_n^\square$. It is sufficient to show that $|\mathbb{P}(\mathbb{K}_n^\square) - \mathbb{P}(\mathbb{K}_n)|$ has an estimate of the stated form.

The greater portion of the following is rather standard in the context of 2D percolation theory so we shall be succinct. Without loss of generality we may assume that $S_B^\square(A_n^\square) > S_B(A_n)$ since otherwise the S_D functions would satisfy this inequality and we may work with S_D instead. For the ease of exposition, let us envision $[A_n, B_n]$ and $[A_n^\square, B_n^\square]$ as the "bottom" boundaries and the D, C pairs as being on the "top".

Let Γ denote a crossing between $[A_n^\square, D_n^\square]$ and $[B_n^\square, C_n^\square]$ within $\Omega_n^\square$ and let $\Gamma_{\mathbb{K}_n^\square} \in \mathbb{K}_n^\square$ denote the event that Γ is the "lowest" (meaning $[A_n^\square, B_n^\square]$–most) crossing. These events form a disjoint partition so that $\mathbb{P}(\mathbb{K}_n^\square) = \sum_\Gamma \mathbb{P}(\mathbb{K}_n^\square \mid \Gamma_{\mathbb{K}_n^\square}) \cdot \mathbb{P}(\Gamma_{\mathbb{K}_n^\square})$. From previous discussions concerning A_n^p, A_n^g, we have that $\mathbb{P}(\mathcal{E}) \geq 1 - n^{-a_4}$, which we remind the reader, means that with stated probability these crossings do not go into these corners and hence there is "room" to construct a continuation.

Let $a_5 > a_1$ denote another constant which is less than unity (recall that in microscopic units $\text{dist}(\partial\Omega_n^\square, \partial\Omega_n) \leq n^{a_1}$). Then, to within tolerable error estimate (by the Russo–Seymour–Welsh estimates) it is sufficient to consider only the crossings Γ with right endpoint a distance in excess of n^{a_5} away from $C_n^\square$ and left endpoint similarly separated from $D_n^\square$.

Let Γ_D and Γ_C denote these left and right endpoints of Γ, respectively. Consider a sequence of intercalated annuli starting at the scale n^{a_1} – or, if necessary, in slight excess – and ending at scale n^{a_5} (where ostensibly they might run aground at $C_n^\square$) around Γ_C. A similar sequence should be considered on the left. Focusing on the right, it is clear that each such annulus provides a conduit between Γ and $\partial\Omega_n$ that runs through the $[B_n^\square, C_n^\square]$ boundary of $\Omega_n^\square$. Let $\bar{\gamma}_r$ denote an occupied blue circuit in one of these annuli and similarly for $\bar{\gamma}_\ell$ on the left.

The blue circuit $\bar{\gamma}_r$ must intersect Γ and, since e.g., Γ_C is at least n^{a_5} away from $A_n^\square, D_n^\square$, these circuits must end on the $[D_n^\square, A_n^\square]$ boundary so that the portion of the circuit above Γ forms a continuation to $\partial\Omega_n$; similar results hold for Γ_D and $\bar{\gamma}_\ell$ and the crossing continuation argument is complete. As discussed before, we may repeat the argument for the other crossing event contributing to the S–functions, so we now have that $|S_n(A_n) - S_n^\square(A_n^\square)| \leq n^{-a_6}$ for some $a_6 > 0$, concluding the first half of the theorem.

The second claim of this theorem, concerning the conformal maps $H_n(A_n)$ and $H_n^\square(A_n^\square)$ in fact follows readily from the *arguments* of the first portion. In particular, we claim that the estimate on the difference can be acquired by an identical sequence of steps by the realization of the fact that the S–function for a given percolative domain which is the canonical approximation to a conformal rectangle converges to the conformal map of said domain to $\mathbb{T}$ ([15] , [2], [5]).

Thus, while seemingly a bit peculiar, there is no reason why we may not consider Ω_n to be a fixed *continuum* domain and, e.g., for $N \geq n$, the domain $\Omega_{n,N}$ to be its canonical approximation for a percolation problem at scale N^{-1}. Similarly for $\Omega_{n,N}^\square$.

Of course here we underscore that e.g., $A_n^{\Box}, \dots D_n^{\Box}$ are regarded as fixed (continuum) marked points which have their own canonical approximates $A_{n,N}^{\Box}, \dots D_{n,N}^{\Box}$ and have no constructive relationship between them and the approximates $A_{n,N} \dots D_{n,N}$.

It is now claimed that uniformly in N, with $N \geq n$ and n sufficiently large the entirety of the previous argument can be transcribed *mutatis mutantis* for the percolation problems on $\Omega_{n,N}$ and $\Omega_{n,N}^{\Box}$. Indeed, once all points were located, the seminal ingredients all concerned (partial) circuits in (partial) annuli and/or rectangular crossings of uniformly bounded aspect ratios and dimensions not smaller than n^{-1}. All such events enjoy uniform bounds away from 0 or 1 (as appropriate) which do not depend on the scale and therefore apply to the percolation problems on $\Omega_{n,N}$ and $\Omega_{n,N}^{\Box}$ We thus may state without further ado that for all $N > n$ (and n sufficiently large)

$$|S_{n,N}(A_{n,N}) - S_{n,N}^{\Box}(A_{n,N}^{\Box})| \leq \frac{1}{n^{a_2}} \tag{3.3}$$

and so $|H_n(A_n) - H_n^{\Box}(A_n^{\Box})| \leq n^{-a_2}$ as well.

Finally, since the relationship between Ω_n and Ω is the same as that between $\Omega_n^{\Box}$ and Ω_n (both Ω_n, $\Omega_n^{\Box}$ are inner domains obtained by the union of shapes (squares or hexagons) of scale a power of n from Ω, Ω_n, respectively) the same continuum percolation argument as above gives the estimate that $|H_n(A_n) - H(A)| \leq n^{-a_2}$.

$\Box$

3.2 The Cauchy–Integral Extension

We will now consider the Cauchy–integral version of the function $S_n^{\Box}$. Ostensibly this is defined on the full $\Omega_n^{\Box}$ however as mentioned in the introduction to this section, its major rôle will be played on the subdomain $\Omega_n^{\blacklozenge}$ which will emerge shortly.

Lemma 3.5 *Let $\Omega_n^{\Box}$ and $S_n^{\Box}$ be as in Proposition 3.2 so that*

$$|\partial\Omega_n^{\Box}| \leq n^{\alpha(1-a_1)},$$

where $M(\partial\Omega) < 1 + \alpha$. For $z \in \Omega_n^{\Box}$ (with the latter regarded as a continuum object) let

$$F_n^{\Box}(z) = \frac{1}{2\pi i} \oint_{\partial\Omega_n^{\Box}} \frac{S_n^{\Box}(\zeta)}{\zeta - z} \, d\zeta. \tag{3.4}$$

Then for a_1 sufficiently close to 1 there exists some $\beta > 0$ such that for all $z \in \Omega_n^{\Box}$ (meaning lying on edges and sites of $\Omega_n^{\Box}$) so that dist$(z, \partial\Omega_n^{\Box}) > d_1$ for some $d_1 > 0$ a *power of* n^{-1},

$$\left|S_n^{\square}(z) - F_n^{\square}(z)\right| \lesssim n^{-\beta}.$$

The proof of this lemma is postponed until Section 4.2 and we remark that while $S_n^{\square}$ is only defined on vertices of hexagons *a priori*, it can be easily interpolated to be defined on all edges, as discussed in Section 4. We will now proceed to demonstrate that $F_n^{\square}$ is conformal in a subdomain of $\Omega_n^{\square}$. Let us first define a slightly smaller domain:

Definition 3.6. Let $\Omega_n^{\square}$ be as described. Let $d_1 > 0$ be some power of n^{-1}, as required by Lemma 3.5 and define, for temporary use,

$$\Omega_n^{\circledcirc} := \{z \in \Omega_n^{\square} : \mathrm{dist}(z, \partial\Omega_n^{\square}) \geq d_1\}.$$

We immediately have the following:

Proposition 3.7 *For n sufficiently large, there exists some $\beta > a_3 > 0$ (with β as in Lemma 3.5) such that*

$$d_{sup}(F_n^{\square}(\partial\Omega_n^{\circledcirc}), \partial\mathbb{T}) \lesssim n^{-a_3}.$$

Proof. Let us first re–emphasize that $S_n^{\square}$ maps $\partial\Omega_n^{\square}$ to $\partial\mathbb{T}$. This is in fact fairly well known (see e.g., [2] or in [5], Theorem 5.5) but a quick summary proceeds as follows: by construction $S_n^{\square}$ is continuous on $\partial\Omega_n^{\square}$ and e.g., takes the form $\lambda\tau + (1-\lambda)\tau^2$ on one of the boundary segments, where λ represents a crossing probability which increases monotonically – and continuously – from 0 to 1 as we progress along the relevant boundary piece. Similar statements hold for the other two boundary segments.

Now by Lemma 3.5, $F_n^{\square}(z)$ is at most the order $n^{-\beta}$ away from $S_n^{\square}(z)$ for any $z \in \partial\Omega_n^{\circledcirc}$, so the curve $F_n^{\square}(\partial\Omega_n^{\circledcirc})$ is in fact also that close to $S_n^{\square}(\partial\Omega_n^{\circledcirc})$ in the supremum norm. Finally, by the Hölder continuity of $S_n^{\square}$ up to $\partial\Omega_n^{\square}$ (see Proposition 4.3) and the fact that $\partial\Omega_n^{\circledcirc}$ is a distance which is an inverse power of n to $\partial\Omega_n^{\square}$, it follows that $S_n^{\square}(\partial\Omega_n^{\circledcirc})$ is also close to $\partial\mathbb{T}$ and the stated bound emerges. $\square$

Equipped with this proposition, we can now introduce the domain $\Omega_n^{\blacklozenge}$:

Definition 3.8. Let $a_4 > 0$ be such that $\beta > a_3 > a_4$ (with $a_3 > 0$ as in Proposition 3.7) and let us denote by

$$\mathbb{T}^{\blacklozenge} = (1 - n^{-a_4}) \cdot \mathbb{T}$$

the uniformly shrunken version of $\mathbb{T}$. Finally, let

$$\Omega_n^{\blacklozenge} := (F_n^{\square})^{-1}(\mathbb{T}^{\blacklozenge})$$

and denote by $(B_n^{\blacklozenge}, C_n^{\blacklozenge}, D_n^{\blacklozenge})$ the preimage of $(1 - n^{-a_4}) \cdot (1, \tau, \tau^2)$ under $F_n^{\square}$.

Lemma 3.9 *Let $F_n^{\square}$ and $\Omega_n^{\blacklozenge}$, etc., be as described. Then $F_n^{\square}$ is conformal in $\Omega_n^{\blacklozenge}$. Next let $H_n^{\blacklozenge} : \Omega_n^{\blacklozenge} \longrightarrow \mathbb{T}$ be the conformal map which maps $(B_n^{\blacklozenge}, C_n^{\blacklozenge}, D_n^{\blacklozenge})$ to $(1, \tau, \tau^2)$. Then for all $z \in \overline{\Omega_n^{\blacklozenge}}$,*

$$|F_n^{\square}(z) - H_n^{\blacklozenge}(z)| \lesssim n^{-a_4}.$$

Proof. Let $K_n := F_n^{\square}(\partial\Omega_n^{\circledcirc})$ and let us start with the following observation on the winding of K_n:

Claim. If $w \in \mathbb{T}^{\blacklozenge}$, then the winding of K_n around w is equal to one:

$$W(K_n, w) = \frac{1}{2\pi i} \int_{K_n} \frac{dz}{z - w} = 1.$$

Proof of Claim. The result is elementary and is, in essence, Rouché's Theorem so we shall be succinct and somewhat informal. Foremost, by continuity, the winding is constant for any $w \in \mathbb{T}^{\blacklozenge}$. (This is easily proved using the displayed formula and the facts that the winding is integer valued and that K_n is rectifiable.) Clearly, since $\partial\mathbb{T}$ and K_n are close in the supremum norm, it follows, by construction (for an argument see the end of the proof of Proposition 3.7 and use the fact that $F_n^{\square}$ only differs from $H_n^{\blacklozenge}$ by a small scale factor) that $\partial\mathbb{T}^{\blacklozenge}$ and K_n are also close in this norm.

Let $z_K(t)$ and $z_{\blacklozenge}(t)$, $0 \leq t \leq 1$ denote parameterizations of K_n and $\partial\mathbb{T}^{\blacklozenge}$ that are uniformly close moving counterclockwise. For $z_{\blacklozenge}$, this starts and ends on the positive real axis and we let $\theta_{\blacklozenge}(t)$ denote the evolving argument of $z_{\blacklozenge}(t)$ (with respect to the origin as usual): $0 \leq \theta_{\blacklozenge}(t) \leq 2\pi$. We similarly define $\theta_K(t)$: in this case, we stipulate that $|\theta_K(0)|$ is as *small* as possible – and thus approximately zero – but of course $\theta_K(t)$ evolves continuously with $z_K(t)$ and therefore ostensibly could lie anywhere in $(-\infty, \infty)$. But $|z_{\blacklozenge}(t)|$ and $|z_K(t)|$ are both of order unity (and in particular not small) and they are close to each other. So it follows that $|\theta_{\blacklozenge}(t) - \theta_K(t)|$ must be uniformly small, e.g., within some ϑ with $0 < \vartheta \ll \pi$ for all $t \in [0, 1]$. Now, since $\theta_{\blacklozenge}(1) - \theta_{\blacklozenge}(0) = 2\pi$, we have

$$|W(K_n, 0) - 1| = \left| \frac{\theta_K(1) - \theta_K(0)}{2\pi} - \frac{\theta_{\blacklozenge}(1) - \theta_{\blacklozenge}(0)}{2\pi} \right| \leq \frac{2\vartheta}{2\pi} \ll 1,$$

so we are forced to conclude that $W(K_n, 0) = 1$ by the integer–valued property. The preceding claim has been established. $\square$

The above implies that $F_n^{\square}$ is in fact 1-1 in $\Omega_n^{\blacklozenge}$: from Definition 3.8 we see that a_5 is chosen so that (for n sufficiently large) n^{-a_4} is large compared with n^{-a_3} (from the conclusion of Proposition 3.7) so that K_n (which is clearly a continuous and possibly self–intersecting curve) lies outside $\mathbb{T}^{\blacklozenge}$. Now fix some point $\xi \in \Omega_n^{\blacklozenge}$ and consider the function $h_\xi(z) := F_n^{\square}(z) - F_n^{\square}(\xi)$. Next parametrizing $\partial\Omega_n^{\square} := \gamma$ as $\gamma : [0,1] \to \mathbb{C}$, noting that $F_n^{\square}(\xi) \in \mathbb{T}^{\blacklozenge}$ and using the chain rule we have that

$$\begin{aligned}
1 = W(K_n, F_n^{\square}(\xi)) &= \frac{1}{2\pi i} \oint_{F_n^{\square}\circ\gamma} \frac{1}{\zeta - F_n^{\square}(\xi)}\, d\zeta \\
&= \frac{1}{2\pi i} \int_0^1 \frac{(F_n^{\square})'(\gamma(t))\gamma'(t)}{F_n^{\square}(\gamma(t)) - F_n^{\square}(\xi)}\, dt = \frac{1}{2\pi i} \oint_\gamma h'_\xi / h_\xi \, dz.
\end{aligned}$$

By the argument principle, the last quantity is equal to the number of zeros of h_ξ in the region enclosed by γ, i.e., in $\Omega_n^{\square}$. The desired one–to–one property is established.

We have now that $F_n^{\square} \mid_{\Omega_n^{\blacklozenge}}$ is analytic, maps $\Omega_n^{\blacklozenge}$ in a one–to–one fashion onto $\mathbb{T}^{\blacklozenge}$. Therefore $F_n^{\square} \mid_{\Omega_n^{\blacklozenge}}$ is *the* conformal map from $\Omega_n^{\blacklozenge}$ to $\mathbb{T}^{\blacklozenge}$ (mapping $B_n^{\blacklozenge}, C_n^{\blacklozenge}, D_n^{\blacklozenge}$ to $(1-n^{-a_4})\cdot(1, \tau, \tau^2)$, the corresponding vertices of $\mathbb{T}^{\blacklozenge}$). Thus by uniqueness of conformal maps we have that $H_n^{\blacklozenge} = \frac{1}{1-n^{-a_4}} \cdot \left(F_n^{\square} \mid_{\Omega_n^{\blacklozenge}}\right)$ and the stated estimate immediately follows. $\square$

3.3 Harris Systems

We will now introduce the Harris systems:

Theorem 3.10 (Harris Systems.) *Let $\Omega_n^{\square} \subseteq \Omega$ be as described with marked boundary points $A, B, C, D \in \partial\Omega$ and let z be an arbitrary point on $\partial\Omega_n$. Further, let 2Δ denote the supremum of the side–length of all squares (oriented with the lattice axes) contained in Ω, and let D_Δ denote a square of side Δ with the same center as a square for which the supremum is realized.*

Then there exists some $\Gamma > 0$ such that for all $n \geq n(\Omega)$ sufficiently large, the following holds: around each boundary point $z \in \partial\Omega_n^{\square}$ there is a nested sequence of at least $\Gamma \cdot \log n$ neighborhoods the boundaries of which are segments (lattice paths) separating z from D_Δ. We call this sequence of neighborhoods the Harris system *stationed at z.*

The regions between these cuts (inside $\Omega_n^{\square}$) are called Harris ring fragments (or just Harris rings).

Further, there exists some $0 < \vartheta < 1/2$ such that in each Harris ring, the probability of an occupied path (in blue or yellow) separating z from D_Δ is uniformly bounded from above and below by ϑ and $1 - \vartheta$, respectively.

Also, let J denote the the d_∞–distance (see Definition 5.1) between successive segments forming a (generic) relevant Harris ring and let $B > 0$ be such that the probability of a hard way crossing of a B by 1 rectangle (in both yellow and blue) is less than ϑ^2. The following properties hold:

1. *for $r \equiv r(\vartheta) > 0$ sufficiently large (particularly, $2^{-r}J < B$) the Harris rings can be tiled with boxes of scale $2^{-r} \cdot J$ and there is a main body of full boxes (unobstructed) which connect the segments forming the Harris rings;*
2. *successive segments Y, Y_Q satisfy*

$$2^{-r}J \leq \|Y\|_0 \leq \|Y\|_\infty \leq 4B \cdot J, \quad 2^{-r}J \leq \|Y_Q\|_0 \leq \|Y_Q\|_\infty \leq 4B \cdot J;$$

3. *let a be point in the Harris systems centered at $A_n^{\square}$ such that the number of Harris rings between a and D_Δ is of order $\log n$. Let $\mathcal{A}(a)$ denote the event of a blue (or yellow) circuit surrounding both a and $A_n^{\square}$ with endpoints on $[A_n^{\square}, B_n^{\square}]$ and $[D_n^{\square}, A_n^{\square}]$. Then there exists some constant $\lambda > 0$ such that*

$$\mathbb{P}(\mathcal{A}(a)) \geq 1 - n^{-\lambda};$$

   Similar estimates hold at the points $B_n^{\square}, C_n^{\square}, D_n^{\square}$ and hence the the estimate also holds for the intersected event, by FKG type inequalities (or just independence);

4. *finally, all estimates are uniform in lattice spacing in the sense of considering $\Omega_n^{\square}$ to be a fixed domain and performing percolation at scale N^{-1}.*

Proof. The constructions required for the establishment this theorem is the content of Section 5. That there exists at least of order $\log n$ such neighborhoods follows from the fact that each point on $\partial\Omega_n^{\square}$ is a distance at least Δ from D_Δ and so proceeding towards D_Δ in a "straight" tunnel and increasing the scale each time by the maximum allowed while fixing the aspect ratio already leads to of order $\log n$ such neighborhoods.

Finally, items 1 and 2 follow from Theorem 5.6 and Subsection 5.4, item 3 follows from the above together with Lemma 5.10 and item 4 is a direct consequence of the scale invariance of critical percolation. $\square$

Let us start with the quantification of the "distance" between the corresponding marked points of $\Omega_n^{\blacklozenge}$ and $\Omega_n^{\square}$:

Proposition 3.11 *$B_n^{\blacklozenge}$ is in the Harris system stationed at $B_n^{\square}$. Moreover, there exists some $\kappa > 0$ such that there are at least $\kappa \cdot \log n$ Harris rings from this Harris system which enclose $B_n^{\blacklozenge}$. Similar statements hold for $C_n^{\blacklozenge}, D_n^{\blacklozenge}$.*

Proof. The argument that $B_n^{\blacklozenge}$ is indeed in the Harris system stationed at $B_n^{\square}$ and the argument that there are many Harris rings enclosing $B_n^{\blacklozenge}$ are essentially the same.

First we have that by Lemma 3.5 and Definition 3.8 that e.g., $|S_B^{\square}(B_n^{\blacklozenge})| \gtrsim 1 - n^{-a_4} - n^{-\beta} \gtrsim 1 - n^{-a_4}$ (recall that $\beta > a_3 > a_4$). On the other hand, let us consider the "last" Harris ring separating $B_n^{\blacklozenge}$ from $B_n^{\square}$ which forms a conduit from $[D_n^{\square}, B_n^{\square}]$ and $[B_n^{\square}, C_n^{\square}]$, c.f., Theorem 3.10, item 3. We may enforce a long way crossing with probability ϑ (as in Theorem 3.10) and then via a box construction and a "large scale" crossing as in the proof of Lemma 3.12 the said crossing can be connected to $[C_n^{\square}, D_n^{\square}]$ in blue, i.e., there is some $V > 0$ such that the latter connection occurs with probability in excess of $n^{-\gamma V}$, if the number of Harris rings enclosing $B_n^{\blacklozenge}$ were *less* than $\gamma \cdot \log n$.

Since such a blue connection renders a yellow version of the event $\mathbb{S}_B^{\square}(B_n^{\blacklozenge})$ impossible, we conclude that there must be more than a_4/V Harris rings enclosing $B_n^{\blacklozenge}$, for n sufficiently large. Similar arguments yield the result also for $C_n^{\blacklozenge}, D_n^{\blacklozenge}$. $\square$

More generally, we have the following description of the distance between $\partial\Omega_n^{\square}$ and $\partial\Omega_n^{\blacklozenge}$:

Lemma 3.12 *Let $s \in \partial\Omega_n^{\blacklozenge}$ and $z \equiv z(s)$ the point on $\partial\Omega_n^{\square}$ which is closest to s (in the Euclidean distance). Then there exists some $\kappa > 0$ such that in the Harris system stationed at z, there are at least $\kappa \cdot \log n$ Harris rings that enclose s.*

Proof. Let us denote $\lambda := \text{dist}(s, z)$. First, logically speaking, we must rule out the possibility that s is outside the Harris system stationed at z altogether: if this were true, then it would imply that $\text{dist}(s, \partial\Omega_n^{\square}) = \lambda > \frac{1}{2}\Delta$ (since Harris circuits plug into

$\partial\Omega_n^{\square}$ the point s can only be outside the Harris system at z altogether if it is “beyond” the last Harris segment which parallels ∂D_{Δ}; see Theorem 3.10) which then readily implies that all of the S–functions are of order unity: indeed, in this case $S_B^{\square}(s), S_C^{\square}(s)$ and $S_D^{\square}(s)$ can all be bounded from below by large scale events of order unity (consider e.g., the crossing of a suitable annulus whose aspect ratio is order unity with s on the boundary of the inner square and the outer square touching $\partial\Omega_n^{\square}$ (from inside $\Omega_n^{\square}$) together with yet another couple of crossings from the inner square of this annulus to a larger rectangle which enclose all of $\Omega_n^{\square}$) which would place s well away from the boundary of $\Omega_n^{\blacklozenge}$ by Definition 3.8 and Lemma 3.5. Thus s is in a Harris ring of z.

If the separation – measured in number of Harris rings – between s and D_{Δ} is not so large, then we will show that $|S_n^{\square}(z)|$ is larger than a small inverse power of n. We will accomplish this by constructing configurations which lead to the occurrence of all three events corresponding to $S_B^{\square}, S_C^{\square}, S_D^{\square}$ with sufficiently large probability. To this end we will make detailed use of the Harris system.

Let J denote the separation distance of the Harris segments which form the ring containing s and let $r > 0$ be as given in Theorem 3.10. Now note that if the statement of the lemma were false, then there would be abundantly many Harris rings separating z from s. Consider the boxes of size $2^{-r}J$ which grid the ring containing s. Let us observe that there are three cases: 1) the *main* type, s is contained in a full box which is connected to the cluster which percolates through the ring (see Theorem 3.10, item 1); 2) the *partial* type, meaning that s is in a partial box, i.e., a box invaded by $\partial\Omega_n^{\square}$; 3) s is in a full box which is separated from the cluster of main types of percolating boxes by a partial box.

Let us rule out the possibility of 2) and 3). Case 2) is impossible since it implies that $\mathrm{dist}(s, z) = \mathrm{dist}(s, \partial\Omega_n^{\square}) \leq 2^{-r}J$ whereas s and z are separated by at least one ring of scale at least $B^{-1}J$ (see Theorem 3.10, item 2) which, by the choice of B is strictly larger (see Theorem 3.10, item 1): indeed, if s and z were in the same ring, then with probability in excess of (some constant times) $1 - n^{-\Gamma}$, with Γ as in Theorem 3.10, the occurrence or not of the events contributing to $S_B^{\square}, S_C^{\square}, S_D^{\square}$ would be the same for both s and z (c.f., the proof of Proposition 3.15 below) but then by Lemma 3.5 and Definition 3.8, it is the case that $|S_n^{\square}(z) - S_n^{\square}(s)| \gtrsim n^{-a_4} - n^{-\beta}$, which is a contradiction if a_4, β are appropriately chosen relative to Γ.

Similar reasoning shows that 3) is also not possible: indeed, since z is the closest point to s, z and s must lie along a straight line segment which lies in $\Omega_n^{\square}$ and this segment must pass through the partial box in question (i.e., the "bottleneck") which separates s from the percolating body of boxes. From previous considerations regarding $2^{-r}J$ (the scale of the boxes) versus $\text{dist}(s, z)$, it is clear that there is a point on $\partial\Omega_n^{\square}$ within this partial box which is closer to s than z, a contradiction.

Thus, we find s in the main percolating body of boxes and, similar considerations in fact places s in a box which is separated from $\partial\Omega_n^{\square}$ (specifically the portion of $\partial\Omega_n^{\square}$ forming the blue boundary of this ring containing s) by several layers of boxes.

We shall now proceed to construct, essentially by hand, any of the events $\mathbb{S}_B^{\square}(s)$, $\mathbb{S}_C^{\square}(s)$ or $\mathbb{S}_D^{\square}(s)$ corresponding to the functions $S_B^{\square}, S_C^{\square}, S_D^{\square}$, respectively, with "unacceptably large" probability.

It is understood that the constructions that follow utilize the main body of boxes percolating through a given Harris ring fragment, as detailed in Theorem 3.10, item 1. For convenience, we will base our construction on 3×1 bond events.

We remark, again, that arguments of this sort have appeared before, e.g., at least as far back as [1], so we will be succinct in our descriptions. The events are described as follows: let us assume, for ease of exposition, that three neighboring boxes form a horizontal 3×1 rectangle. The *bond event* – in yellow – would then consist of two disjoint left–right yellow crossing of the 3×1 rectangle together with two disjoint top–bottom yellow crossings in each of the outer two squares. It is seen that if a pair of such rectangles overlap on an end–square, and the bond event occurs for both of them, then, regardless of the orientations, there are two disjoint yellow paths which transmit from the beginning of one to the end of the other. I.e., these "bonds" have the same connectivity properties as the bonds of $\mathbb{Z}^2$ and provide us with double paths.

Starting with the square containing s we may suppose there is (or construct) a yellow ring in the eight boxes immediately surrounding and encircling this square. Via the bond events just described, we connect this encircling ring to the outward boundary of the Harris annulus to which s belongs. Each of these events – which are positively correlated – incurs a certain probabilistic cost. However, it is observed, with emphasis, that since the *relative* scales of the Harris ring and the bonds used in the construction are fixed independent of the actual scale, the cost may be bounded by a number independent of the *actual* scale.

Similarly, we may use the bonds to acquire a double path across the next (outward) ring and the two double paths may be connected to form a continuing double path by explicit use of a "patch" consisting of the smaller of the two bond types. Again, since the ratio of scales of (boxes of) successive Harris rings are uniformly bounded above and below, the probabilistic cost does not depend on actual scale. The procedure of double crossing via bond events and patches can be continued till the boundary of D_Δ is reached; thereupon, treating D_Δ and its vicinity as an annulus in its own right, the two paths can be connected to separate boundaries at an additional cost of order unity.

Now let us assume for the moment that $s \in [B_n^\blacklozenge, C_n^\blacklozenge]$, so that by Lemma 3.5 and Definition 3.8 it is the case that $S_D^\square(s) \leq C \cdot (n^{-a_4} + n^{-\beta})$ for some constant $C > 0$, so denoting by e^{-V} (for some $V > 0$) the uniform bound on the cost of one patch and one annular crossing via the double bonds, if $\kappa > 0$ is sufficiently small so that $\mathrm{e}^{-\kappa V \log n} = n^{-\kappa V} > C \cdot (n^{-a_4} + n^{-\beta})$ then it is evidently not possible that $s \in [B_n^\blacklozenge, C_n^\blacklozenge]$. By cyclically permuting the relevant B, C, D labels, the cases where $s \in [C_n^\blacklozenge, D_n^\blacklozenge], s \in [D_n^\blacklozenge, B_n^\blacklozenge]$ follow similarly. $\square$

The ensuing arguments will require an auxiliary point somewhat inside $\Omega_n^\blacklozenge$, which we will denote $A_n^\diamondsuit$:

Definition 3.13. Let $\Omega_n^\square, \Omega_n^\blacklozenge$, etc., be as described. Let $\eta > 0$ be a number to be specified in Proposition 3.14. Then we let $A_n^\diamondsuit$ be a point in the Harris ring which is separated from D_Δ by $\eta \cdot \log n$ Harris segments. Moreover, $A_n^\diamondsuit$ is in the center of a main type box of this ring. Here we are referring to boxes described in Theorem 3.10, item 1 and the meaning of main type is as in the proof of Lemma 3.12.

Proposition 3.14 *There exists some $\eta > 0$ such that if $A_n^\diamondsuit$ is as in Definition 3.13, then there exists some $\gamma > 0$ such that*

1. $|S_n^\square(A_n^\square) - S_n^\square(A_n^\diamondsuit)| \lesssim n^{-\gamma}$;
2. $|H_n^\square(A_n^\square) - H_n^\square(A_n^\diamondsuit) \lesssim n^{-\gamma}$;

In particular, with appropriate choice of γ, $A_n^\diamondsuit$ is strictly inside $\Omega_n^\blacklozenge$.

Proof. First let us establish item 1. It is claimed that for any configuration in which the event $\mathcal{A}(A_n^\diamondsuit)$ – of a blue circuit connecting $[D_n^\square, A_n^\square]$ to $[A_n^\square, B_n^\square]$ which surrounds both $A_n^{\square}$ and $A_n^{\diamondsuit}$ (as described in Theorem 3.10, item 3) – occurs, then the indicator function of the yellow version of $\mathbb{S}_n^{\square}(A_n^{\square})$ is *equal* to that of $\mathbb{S}_n^{\square}(A_n^{\diamondsuit})$. Indeed, for the $S_C^{\square}$–component, which always vanishes for $A_n^{\square}$, the requisite event in yellow is directly obstructed by the blue paths of $\mathcal{A}(A_n^{\diamondsuit})$. As for the rest, for either of the differences in the B or D components to be non–zero, there must be a long yellow path separating $A_n^{\square}$ from $A_n^{\diamondsuit}$ heading to a distant boundary, but this separating path is preempted by the blue event $\mathcal{A}(A_n^{\diamondsuit})$. We may thus conclude that

$$\mathbb{E}(|\mathbb{I}_{\mathbb{S}_n^{\square}(A_n^{\square})} - \mathbb{I}_{\mathbb{S}_n^{\square}(A_n^{\diamondsuit})}| \mid \mathcal{A}(A_n^{\diamondsuit})) = 0 \tag{3.5}$$

(where $\mathbb{I}_{(\bullet)}$ denotes the indicator) which together with Lemma 3.12 and Theorem 3.10, item 3 gives the result.

As for item 2, recalling the discussion near the end of the proof of Theorem 3.3, we may consider $\Omega_n^{\square}$ to be a fixed *continuum* domain and, e.g., for $N \geq n$, the domain $\Omega_{n,N}^{\square}$ to be its canonical approximation (together with appropriate approximations for the marked points $A_n^{\square}, B_n^{\square}$, etc.) for a percolation problem at scale N^{-1}. We will consider the corresponding CCS–functions $S_{n,N}^{\square}$ on the domains $\Omega_{n,N}^{\square}$.

Let us now argue that the arguments for item 1 persist, uniformly, for all N sufficiently large. First, it is emphasized that all the results follow from the occurrence of paths in each Harris ring, which has probability uniformly bounded from below. We claim that this remains the case for percolation performed at scale N^{-1}. Indeed, while the scales of the Harris rings were constructed existentially to ensure uniform bounds on crossings at scale n^{-1}, it is recalled that these rings are gridded by boxes of scale 2^{-r} *relative* to the rings themselves (see Theorem 3.10, item 1). Thence, using uniform probability crossings in squares/rectangles, etc., the necessary crossings can be constructed by hand as in e.g., the proof of Lemma 3.12.

For the last statement, we invoke an argument similar to that in the proof of Lemma 3.12. Recapitulating the construction, we acquire a lower bound on the probability of occurrence of any of the events associated with the S–functions for $A_n^{\diamondsuit}$. Finally, since $S_n^{\square}$ is close to $F_n^{\square}$ by Lemma 3.5 the latter of which is used to *define* $\partial\Omega_n^{\blacklozenge}$, with appropriate choice of power of n, $A_n^{\diamondsuit}$ can be placed in the interior of $\Omega_n^{\blacklozenge}$. $\square$

Proposition 3.15 *There exists some $a_5 > 0$ such that*

$$|F_n^{\square}(A_n^{\diamondsuit}) - S_n^{\square}(A_n^{\square})| \leq n^{-a_5}.$$

Proof. This follows immediately from Proposition 3.14, item 1 and Lemma 3.5. $\square$

Finally, we will need a result concerning the conformal maps $H_n^{\blacklozenge}$ and $H_n^{\square}$. First we state a distortion estimate:

Lemma 3.16 *Let $\epsilon > 0$ and let $K \subseteq \mathbb{T}$ be a domain whose boundary is a Jordan curve such that the sup–norm distance between ∂K and $\partial \mathbb{T}$ is less than ϵ. We consider K to be a conformal triangle with some marked points K_B, K_C, K_D such that $|K_B - 1| < \epsilon$, $|K_C - \tau| < \epsilon$, $|K_D - \tau^2| < \epsilon$, and let g_K denote the conformal map from K to $\mathbb{T}$ mapping (K_B, K_C, K_D) to $(1, \tau, \tau^2)$. Then for $z \in K$ it is the case that*

$$|g_K(z) - z| \lesssim [\epsilon \cdot \log(1/\epsilon)]^{1/3}.$$

Proof. The result for the disk (without the power of 1/3) is a classical result going back to Marchenko (for a statement see [17], Section 3) and of course, we can transfer our hypotheses to the disk by applying a conformal map ϕ, which maps $\mathbb{T}$ to the unit disk such that $\phi(0) = 0$. The map ϕ does not increase the distances, because it is smooth up to the boundary everywhere but at 1, τ, and τ^2, where it behaves locally like ϵ^3, which in fact only *decreases* the distances.

We are almost in a position to directly apply Marchenko's Theorem except for a few caveats. First of all Marchenko's Theorem requires a certain geometric condition on the tortuosity of the boundary of K, which is manifestly satisfied under the assumption that ∂K and the boundary of the *triangle* are close in the *sup–norm* distance.

Secondly, Marchenko's Theorem is stated for some map f_K with $f_K(0) = 0$ and $f'_K(0) > 0$, and we have a possibly different normalization. Specifically, we have some map $G_K : \phi(K) \to \mathbb{D}$ so that $\phi^{-1} \circ G_K \circ \phi = g_K$, so it suffices to check that G_K has approximately the correct normalizations (indeed, the conformal self–map of the unit disc mapping a point a to the origin takes the form $e^{i\theta} \cdot \left(\frac{z-a}{1-\bar{a}z}\right)$).

Since $\phi(0) = 0$ and $1 + \tau + \tau^2 = 0$ it is the case that $\phi^{-1}((1-\epsilon) \cdot \phi(K_B + K_C + K_D))$ is close to 0 and also close to $w := \phi^{-1}((1-\epsilon) \cdot \phi(K_B)) + \phi^{-1}((1-\epsilon) \cdot \phi(K_C)) + \phi^{-1}((1-\epsilon) \cdot \phi(K_D))$; since it is also the case that $g_K(K_B) + g_K(K_C) + g_K(K_D)$ is close to 0, we have that $G_K(w)$ is close to 0. So we now have that $G_K(z)$ is close to some $e^{i\theta} z$ for some fixed θ. But since $\phi(K_B)$ is close to $\phi(1)$, and so $z_0 := \phi^{-1}((1-\epsilon)\phi(K_B))$ is close to both 1 and $e^{-i\theta} G_K(1)$, it follows that $|e^{i\theta} - 1| \lesssim \epsilon \cdot \log(1/\epsilon)$.

Finally, in transferring the result back to the triangle, the behavior near the vertices of the triangle requires us to replace the distances by their cube roots. $\square$

Remark 3.17. We remark that for our purposes, we can in fact avoid the fractional power: indeed, we shall only use this result at the point $A_n^{\diamondsuit}$, which we remind the reader is chosen to be in the Harris system stationed at $A_n^{\square}$ and by Lemma 5.10 we may assert that it is within a fixed small neighborhood of $A_n^{\square}$ and therefore outside fixed neighborhoods of the other marked points.

Lemma 3.18 *There exists some $a_6 > 0$ such that for all n sufficiently large,*

$$|H_n^{\blacklozenge}(A_n^{\diamondsuit}) - H_n^{\square}(A_n^{\diamondsuit})| \lesssim n^{-a_6}.$$

Proof. Denoting by G_n the conformal map mapping $H_n^{\square}(\Omega_n^{\blacklozenge})$ to $\mathbb{T}$ with $(H_n^{\square}(B_n^{\blacklozenge}), H_n^{\square}(C_n^{\blacklozenge}),$ $H_n^{\square}(D_n^{\blacklozenge}))$ mapping to $(1, \tau, \tau^2)$, we have by uniqueness of conformal maps that

$$H_n^{\blacklozenge} = G_n \circ H_n^{\square}.$$

The stated result will follow from Lemma 3.16, and in order to utilize this lemma, we need to verify that $(H_n^{\square}(B_n^{\blacklozenge}), H_n^{\square}(C_n^{\blacklozenge}), H_n^{\square}(D_n^{\blacklozenge}))$ is close to $(1, \tau, \tau^2)$ and to show that the sup–norm distance between $\partial[H_n^{\square}(\partial\Omega_n^{\blacklozenge})]$ and $\partial\mathbb{T}$ is less than $n^{-\gamma}$ for some $\gamma > 0$. The first statement is a direct consequence of Proposition 3.11: since $O(\log n)$ Harris rings surround both $B_n^{\blacklozenge}$ and $B_n^{\square}$, by an argument as in the proof of Proposition 3.14, their $S_n^{\square}$ values differ by an inverse power of n and the result follows since $S_n^{\square}(B_n^{\square}) \equiv 1$; similar arguments yield the result for $C_n^{\blacklozenge}, D_n^{\blacklozenge}$.

As for the second statement, first we have by Lemma 3.5 and Lemma 3.9 that the distance between $\partial[S_n^{\square}(\partial\Omega_n^{\blacklozenge})]$ and $\partial\mathbb{T}$ is less than (some constant times) $n^{-a_4} + n^{-\beta}$; we emphasize that here we in fact have closeness in the sup–norm since both lemmas yield pointwise estimates. Next, as near the end of the proof of Theorem 3.3, we may consider $\Omega_n^{\square}$ to be a fixed *continuum* domain and, e.g., for $N \geq n$, the domain $\Omega_{n,N}^{\square}$ to be its canonical approximation (together with appropriate approximations for the marked points $A_n^{\square}, B_n^{\square}$, etc.) for a percolation problem at scale N^{-1}. We will consider the corresponding CCS–functions $S_{n,N}^{\square}$ on the domains $\Omega_{n,N}^{\square}$.

We claim that there exists some $\gamma > 0$ such that uniformly in N for N sufficiently large, the sup–norm distance between $\partial[S_{n,N}^{\square}(\partial\Omega_n^{\blacklozenge})]$ and $\partial\mathbb{T}$ is less than $n^{-\gamma}$. Indeed, from Lemma 3.12, we know that for each point s on $\partial\Omega_n^{\blacklozenge}$, there are $\kappa \cdot \log n$ Harris rings stationed at $z(s)$ which separate it from the central region D_Δ. While by *fiat* $S^{\square}_{n,N=n}(\partial\Omega_n^{\blacklozenge})$ is close to $\partial\mathbb{T}$, we shall reprove this using the Harris systems since we require an estimate which is uniform in N. We start with the following observation concerning the central region D_Δ:

Claim. For n sufficiently large, with probability of order unity independent of n, there are monochrome percolative connections between D_Δ and any or all of the three boundary segments.

Proof of Claim. Consider the domain Ω with marked points B, C, D, viewed as a conformal triangle. It is recalled that D_Δ is roughly half the size of the largest circle which can be fit into Ω. Let us focus on two of the three marked points, say B and D. We now mark two boundary points on D_Δ and denote them by b and d and consider two disjoint curves which join B to b and D to d, thereby forming a conformal rectangle. Since the aspect ratio of said rectangle is fixed, it therefore follows, by convergence to Cardy's Formula, that for n sufficiently large, there is a uniform lower bound on the discrete realization of the desired connection. Similar arguments apply to the other two boundary segments. $\square$

Claim. Consider $s \in \partial\Omega_n^{\blacklozenge}$ and the Harris system stationed at $z(s) \in \partial\Omega_n^{\square}$, as in Lemma 3.12) which, without loss of generality, we assume to be in $[B_n^{\square}, D_n^{\square}]$. Then there exists some fixed constant $\Upsilon < \infty$ such that all but Υ of the Harris segments have at least one endpoint on $[B_n^{\square}, D_n^{\square}]$ *and* either accomplishes $\mathbb{S}_D$ or $\mathbb{S}_B$ for *both* s and $z(s)$ or have *both* endpoints on $[B_n^{\square}, D_n^{\square}]$. Similar statements hold if z belongs to the other boundary segments.

Proof of Claim. Let us first rule out the possibility that too many Harris segments have endpoints on $[B_n^{\square}, C_n^{\square}, D_n^{\square}]$. It is noted that each Harris segment of this type in fact separates all of $[B_n^{\square}, D_n^{\square}]$ from D_Δ. Thus, if there are say Υ such Harris segments, then the probability of a connection between D_Δ and $[B_n^{\square}, D_n^{\square}]$ would be less than $(1-\vartheta)^{\Upsilon}$, with $\vartheta > 0$ as in Theorem 3.10. It follows from the previous claim that Υ cannot scale with n.

Finally, if there are too many Harris segments with one endpoint on $[B_n^\square, D_n^\square]$, but accomplishes *neither* $\mathbb{S}_B^\square$ *nor* $\mathbb{S}_D^\square$, then necessarily the other endpoint must be on $[B_n^\square, C_n^\square]$ or $[C_n^\square, D_n^\square]$ in such a way that the Harris segment separates D_Δ from $[B_n^\square, C_n^\square]$ or $[C_n^\square, D_n^\square]$. The same reasoning as in the above paragraph then implies that this also cannot occur "too often". $\square$

We also note that there cannot be Harris segments of conflicting "corner types" since the Harris segments are topologically ordered and cannot intersect one another.

We can now acquire the needed conclusion that the Harris rings themselves force $S_{n,N}^\square(s)$ to be close to $\partial\mathbb{T}$. The essence of the argument can be captured by the (redundant) case $N = n$, so let us proceed. Consider then $s \in \partial\Omega_n^\blacklozenge$ and the Harris system stationed at $z(s) \in \partial\Omega_n^\square$ as above which, without loss of generality, we assume to be in $[B_n^\square, D_n^\square]$. Then we claim that $|S_n^\square(z(s)) - S_n^\square(s)| \lesssim n^{-\kappa}$. Indeed, from the previous claim, the possible landing locations for "most" of the Harris segments are very limited and in all cases (including the possibility of a "mixed" case) conditioned on the existence of paths in the appropriate color in the Harris segments, the indicator functions of all $S_n^\square$–events are the same for both s and $z(s)$.

Let us now argue that the above argument persists, uniformly, for all N sufficiently large. First, it is emphasized that all arguments follow from the occurrence of paths in each Harris ring, which has probability uniformly bounded from below. We claim that this remains the case for percolation performed at scale N^{-1}. Indeed, we reiterate, these rings are gridded by boxes of scale 2^{-r} *relative* to the rings themselves (see Theorem 3.10) and using uniform probability of crossings in squares/rectangles, etc., which is characteristic of critical 2D percolation problems, the necessary crossings can be constructed by hand as in e.g., the proof of Lemma 3.12.

Now by convergence to Cardy's Formula (or rather, the statement that the CCS–function converges uniformly on compact sets to the conformal map to $\mathbb{T}$) it is the case that $S_{n,N}^\square(s) \to H_n^\square(s)$. Uniformity in s follows from the fact that $\overline{\Omega_n^\blacklozenge} \subseteq \Omega_n^\square$ is a fixed (for n fixed) compact set, c.f., Section 5 in [5]. We conclude therefore that each point on $\partial\Omega_n^\blacklozenge$ maps to a point sufficiently close to $\partial\mathbb{T}$, and since $\partial[H_n^\square(\partial\Omega_n^\blacklozenge)]$ is a curve, it easily follows that the *Hausdorff distance* is small.

However, we require the stronger statement that the relevant objects are close in the *sup–norm*. We will now strengthen the above arguments to acquire this conclusion. Let us define the set of all points which are chosen as the $z(s)$ (the closest point to s) for some s in $\langle \partial\Omega_n^{\blacklozenge} \rangle_N$ (the approximation to $\partial\Omega_n^{\blacklozenge}$ at scale N^{-1}):

$$\mathcal{Z}_N := \{z \in \partial\Omega_{n,N}^{\square} \mid \exists s \in \langle \partial\Omega_n^{\blacklozenge} \rangle_N, z = z(s)\}.$$

Let us first observe that *a priori* $\mathcal{Z}_N$ is a discrete set of points on $\partial\mathbb{T}$ which we may consider to be a curve by linear interpolation. For simplicity let us consider the portion of $\partial\mathbb{T}$ corresponding to the $[C, D]$ boundary, i.e., the vertical segment connecting τ and τ^2. Let us focus attention on $S_{n,N}^{\square}([C_{n,N}^{\square}, D_{n,N}^{\square}] \cap \mathcal{Z}_N)$. By monotonicity of crossing probabilities, it is the case that these points are ordered along the vertical segment.

Now our contention is that there are no substantial gaps between successive points:

Claim. Let $\nu > 0$ be such that $n^{-\nu} \gg n^{-\kappa}$, where κ as above is such that $|S_{n,N}^{\square}(s) - S_{n,N}^{\square}(z(s))| \lesssim n^{-\kappa}$. Then for all $N > n$, it is the case that the maximum separation between successive points of $S_{n,N}^{\square}([C_{n,N}^{\square}, D_{n,N}^{\square}] \cap \mathcal{Z}_N)$ is less than $n^{-\nu}$.

Proof of Claim. Suppose there are two points $x_1, x_2 \in [C_{n,N}^{\square}, D_{n,N}^{\square}] \cap \mathcal{Z}_N$ say with $S_{n,N}^{\square}(x_1)$ below $S_{n,N}^{\square}(x_2)$ separated by a gap in excess of $n^{-\nu}$. Let us denote by $s_1, s_2 \in \langle \partial\Omega_n^{\blacklozenge} \rangle_N$ the points corresponding to x_1, x_2, respectively. Next consider the $\frac{1}{4} \cdot n^{-\nu}$ neighborhoods of $S_{n,N}^{\square}(s_1)$ and $S_{n,N}^{\square}(s_2)$ and consider the points "between" s_1 and s_2 (there must be points between s_1 and s_2 since $|S_{n,N}(s_1) - S_{n,N}(s_2)| \gtrsim n^{-\nu} - n^{-\kappa}$ so if they were neighbors, then their $S_{n,N}^{\square}$ values for n sufficiently large would be unacceptably large relative to the above inequality).

If these points all have $S_{n,N}^{\square}$–value which lie in the $\frac{1}{4} \cdot n^{-\nu}$ neighborhoods described above, then there would be a neighboring pair whose $S_{n,N}^{\square}$ values are separated by $\frac{1}{2} \cdot n^{-\nu}$, which would again be unacceptably large. We conclude therefore that there exists some point between s_1 and s_2 with $S_{n,N}^{\square}$ value outside these neighborhoods and therefore a point in $\mathcal{Z}_N$ whose $S_{n,N}^{\square}$ value lies between those of x_1 and x_2. This is a contradiction. $\square$

Finally, let us describe the parametrization. Let us denote by U_N the number of points in $\mathcal{Z}_N$ then we may parametrize say the vertical portion of $\partial\mathbb{T}$ by having, for $t = j$, the curve on the j^{th} site of $\mathcal{Z}_N$, linearly interpolating for the non–integer times. Similarly, we parametrize the corresponding portion of $S_{n,N}^{\square}(\langle \partial\Omega_n^{\blacklozenge} \rangle_N$, so that pairs of points at integer times correspond to their $s, z(s)$ pair. The above claim then implies that with this parametrization, the two curves are within $n^{-\nu}$ at all times. We have verified that $S^{\square}_{n,N}(\langle \partial\Omega^{\blacklozenge}_n \rangle_N)$ is sup–norm close to $\partial\mathbb{T}$, uniformly in N.

The stated result now follows from Lemma 3.16. $\square$

Proof of the Main Theorem. The required power law estimate for the rate of convergence of crossing probabilities now follows by concatenating the various theorems, propositions and lemmas we have established. Let us temporarily use the notation $A \sim B$ to mean that A and B differ by an inverse power of n.

Starting with $S_n(A_n)$, we have that $S_n(A_n) \sim S^{\square}_n(A^{\square}_n)$ by Theorem 3.3; $S^{\square}_n(A^{\square}_n) \sim S^{\square}_n(A^{\diamondsuit}_n)$ by Proposition 3.14, item 1; $S^{\square}_n(A^{\diamondsuit}_n) \sim F^{\square}_n(A^{\diamondsuit}_n)$ by Lemma 3.5; $F^{\square}_n(A^{\diamondsuit}_n) \sim H^{\blacklozenge}_n(A^{\diamondsuit}_n)$ by Lemma 3.9; $H^{\blacklozenge}_n(A^{\diamondsuit}_n) \sim H^{\square}_n(A^{\diamondsuit}_n)$ by Lemma 3.18; $H^{\square}_n(A^{\diamondsuit}_n) \sim H^{\square}_n(A^{\square}_n)$ by Proposition 3.14, item 2; finally, $H^{\square}_n(A^{\square}_n) \sim H(A)$ by Theorem 3.3.

$\square$

4 σ–Holomorphicity

The main goal in this section is to establish the so–called Cauchy integral estimates which is one of the more technical aspects required for the proof of Lemma 3.5. We will address such issues in somewhat more generality than strictly necessary by extracting the two properties of functions of the type $S_n(z)$ which are of relevance: i) Hölder continuity and ii) that their discrete (closed) contour integrals are asymptotically zero as the lattice spacing tends to zero. As for the latter, it should be remarked that the details of how our particular $S_n(z)$ exhibits its cancelations on the *microscopic scale* can be directly employed to provide the Cauchy–integral estimates.

4.1 (σ, ρ)–Holomorphicity

As a starting point – and also to fix notation – let us review the concept of a discrete holomorphic function on a hexagonal lattice. Let $\mathbb{H}_\varepsilon$ denote the hexagonal lattice at scale ε, i.e., the length of the sides of each hexagon is ε, so we envision $\varepsilon = n^{-1}$, where the hexagons are oriented horizontally (i.e., two of the sides are parallel to the x–axis). For now, let Λ_ε denote any collection of hexagons and $Q : \Lambda_\varepsilon \to \mathbb{C}$ a function on the vertices of Λ_ε. For each pair of adjacent vertices in Λ_ε let us linearly interpolate Q on the edges (so that in particular, Q as a function on *edges* when integrated with respect to arc length yields the average of the values of Q at the two endpoints):

> Then we say that Q is *discrete holomorphic* on Λ_ε if for any hexagon $h_\varepsilon \in \Lambda_\varepsilon$ with vertices $(v_1, \dots v_6)$ – in counterclockwise order with v_1 the leftmost of the lowest two – the following holds:
>
> $$0 = \left(\frac{Q(v_1) + Q(v_2)}{2} + \cdots + e^{i\frac{5}{3}\pi} \cdot \frac{Q(v_6) + Q(v_1)}{2}\right) = \varepsilon^{-1} \cdot \oint_{\partial h_\varepsilon} Q\ dz.$$

That is, the usual discrete contour integral (by this or any equivalent) definition vanishes. By way of contrast, we have the following mild generalization pertaining to *sequences* of functions.

Definition 4.1. Let $\Lambda \subset \mathbb{C}$ be a simply connected domain and denote by Λ_ε the (interior) discretized domain given as $\Lambda_\varepsilon := \bigcup_{h_\varepsilon \subset \Lambda} h_\varepsilon$ and let $(Q_\varepsilon : \Lambda_\varepsilon \to \mathbb{C})$ be a sequence of functions defined on the vertices of Λ_ε. Here ε is tending to zero and, without much loss, may be taken as a discrete sequence. we say that the sequence (Q_ε) is *σ–holomorphic* if there exist constants $0 < \sigma, \rho \leq 1$ such that for all ε sufficiently small:

(i) Q_ε is Hölder continuous (down to the scale ε) and up to $\partial\Lambda_\varepsilon$, in the sense that there exists some $\psi > 0$ (envisioned to be small) such that 1) Q_ε is Hölder continuous in the usual sense for $z_\varepsilon, w_\varepsilon \in \Lambda_\varepsilon \setminus N_\psi(\partial\Lambda)$: if $|z_\varepsilon - w_\varepsilon| < \psi$, then $|Q_\varepsilon(z_\varepsilon) - Q_\varepsilon(w_\varepsilon)| \lesssim \left(\frac{|z_\varepsilon - w_\varepsilon|}{\psi}\right)^\sigma$ and 2) if $z_\varepsilon \in N_\psi(\partial\Lambda)$, then there exists some $w_\varepsilon^\star \in \partial\Lambda_\varepsilon$ such that $|Q_\varepsilon(z_\varepsilon) - Q_\varepsilon(w_\varepsilon^\star)| \lesssim \left(\frac{|z_\varepsilon - w_\varepsilon^\star|}{\psi}\right)^\sigma$.

(ii) for any simply closed lattice contour Γ_ε,

$$|\oint_{\Gamma_\varepsilon} Q\ dz| = |\sum_{h_\varepsilon \in \Lambda'_\varepsilon} \oint_{\partial h_\varepsilon} Q\ dz| \lesssim |\Gamma_\varepsilon| \cdot \varepsilon^\rho, \tag{4.1}$$

with $\Lambda'_\varepsilon, |\Gamma_\varepsilon|$ denoting the region enclosed by Γ_ε and the Euclidean length of Γ_ε, respectively.

Remark 4.2. (i) Obviously any sequence of discrete holomorphic functions *which also satisfy the Hölder continuity condition* are σ–holomorphic.

(ii) There are of order $|\Gamma_\varepsilon|/\varepsilon$ terms in a discrete contour integration but each term is multiplied by ε and so in cases where $|\Gamma_\varepsilon| = O(1)$ (a contour of fixed finite length)

$|\Gamma_\varepsilon|$ need not be explicitly present on the right hand side of Equation (4.1). We have introduced a more general definition as we shall have occasion to consider contours whose lengths scale with ε (specifically they are discrete approximations to contours that are not rectifiable).

(iii) From the assumption of Hölder continuity alone, we already have that $|\oint_{\partial h_\varepsilon} Q\, dz| \lesssim \varepsilon^{1+\sigma}$, but on a moment's reflection, it is clear that this is quite far from what is necessary to provide adequate estimates for the integral around contours of *larger scales* that are amenable to the $\varepsilon \to 0$ limit.

We will now gather the necessary ingredients to establish that the (complexified) CCS–functions are (σ, ρ)–holomorphic. The arguments here are certainly not new: various ideas and statements needed are almost already completely contained in [15], [11] and [5].

Proposition 4.3 *Let Λ denote a conformal triangle with marked points (or prime ends) B, C, D and let Λ_ε denote an interior approximation (see Definition 3.1 of [5]) of Λ with $B_\varepsilon, C_\varepsilon, D_\varepsilon$ the associated boundary points. Let $S_\varepsilon(z)$ denote the complex crossing function defined on Λ_ε. Then for all ε sufficiently small, the functions $(S_\varepsilon : \Lambda_\varepsilon \to \mathbb{C})$ are (σ, ρ)–holomorphic for some $\sigma, \rho > 0$.*

Proof. We will first establish, using some conformal mapping ideas, that S_ε enjoys Hölder continuity up to the boundary; since arguments like this already appear in [4] and [5], we will be brief. Let us start with a pointwise statement:

Claim. Suppose we have a point A on the $[D, B]$ boundary, then we claim that there is some $\Delta^\star \equiv \Delta^\star(A)$ (with $1 \gg \Delta^\star \gg \varepsilon$) and a connected set $N_{\Delta^\star} \subset \Lambda_\varepsilon$, also contained in the $\Delta^\star$ neighborhood of A_ε and connected to A_ε, such that the following holds: there exists some $\sigma > 0$ such that for any $z \in N_{\Delta^\star}$,

$$|S_\varepsilon(z) - S_\varepsilon(A_\varepsilon)| \lesssim \left[\frac{|z - A_\varepsilon|}{\Delta^\star}\right]^\sigma.$$

Proof of Claim. Let $z \in \Lambda_\varepsilon$ and consider the $S_\varepsilon(z)$ to be described by blue paths. Then it is clear that if there is a yellow path starting on $[D_\varepsilon, A_\varepsilon]$ and ending on $[A_\varepsilon, B_\varepsilon]$ which encircles z then events contributing to $S_\varepsilon(z)$ and $S_\varepsilon(A_\varepsilon)$ occur together and there is no contribution to $|S_\varepsilon(z) - S_\varepsilon(A_\varepsilon)|$. The power $(|z - A_\varepsilon|/\Delta^\star)^\sigma$ corresponds to having the order of $|\log(|z - A_\varepsilon|/\Delta^\star)|$ annuli (or coherent portions thereof) connecting the two parts of the $[D_\varepsilon, B_\varepsilon]$ boundary with an independent chance of such a yellow circuit in each segment with uniformly bounded probability. Thus the principal task is to construct the reference scale $\Delta^\star$ in a manner which is uniform in ε. While the entire issue is trivial when $|A - A_\varepsilon|$, $|B - B_\varepsilon|$ etc., are small compared to the distance between various relevant "points" on Λ, we remind the reader that under certain circumstances, the separation between these points and their approximates may be spuriously large. Thus we turn to uniformization.

To this end, let $\varphi : \mathbb{D} \to \Lambda$ denote the uniformization map. Let X'_A denote a crosscut neighborhood of $\varphi^{-1}(A)$ which does not contain any of the inverse images of the marked points $\varphi^{-1}(B), \dots$ nor, for ε small, the inverse images of their approximates $\varphi^{-1}(B_\varepsilon), \dots$ but which *does* (for ε small) contain $\varphi^{-1}(A_\varepsilon)$. Next we set $X_A := X'_A \cap \varphi^{-1}(\Lambda_\varepsilon)$ so that

$$\varphi(X_A) = \varphi(X'_A \cap \varphi^{-1}(\Lambda_\varepsilon)).$$

Note that $(\varphi_\varepsilon^{-1} \circ \varphi)(X_A)$ is itself a crosscut neighborhood of the image of A_ε since Λ_ε is an *interior* approximation; here φ_ε denotes the uniformization map associated with Λ_ε.

Next let $r_\Pi = r_\Pi(A_\varepsilon)$ be standing notation for the square centered at A_ε of side Π. Then, for $\Delta^\star$ sufficiently small, it is the case that $\varphi^{-1}(r_{\Delta^\star}) \subset X_A$ and it is worth observing that $\varphi_\varepsilon^{-1}(\partial(r_\Pi \cap \varphi(X_\delta)))$ is a crosscut containing $\varphi_\varepsilon^{-1}(A_\varepsilon)$ for all $\Pi \leq \Delta^\star$.

But now, it follows that there is a nested sequence of (partial) annuli, down to scale $|z - A_\varepsilon|$, contained inside $r_{\Delta^\star}$, within each of which there is a connected monochrome chain with uniform and independent probability separating z from A_ε. $\square$

From the claim we have that corresponding to each boundary point of Λ, we have a neighborhood $\Delta^\star(z)$ in which we have Hölder continuity and it is certainly the case that $\partial\Lambda \subseteq \bigcup_{z\in\partial\Lambda} N_{\Delta^\star(z)}$, so by compactness there exist $z^{(1)}, \dots, z^{(k)}$ such that $\partial\Lambda \subseteq \bigcup_{\ell=1}^k N_{\Delta^\star(z^{(\ell)})}$. Adding a few $N_{\Delta(z)}$'s if necessary so that all neighborhoods have non–trivial overlap, this implies the existence of some $\psi > 0$ such that $N_\psi(\partial\Lambda) \subseteq \bigcup_{\ell=1}^k N_{\Delta^\star(z^{(\ell)})}$ (here $N_\psi(\partial\Lambda)$ denotes the Euclidean ψ–neighborhood of $\partial\Lambda$). In particular, $\psi \leq \Delta^\star(z^{(\ell)}), \ell = 1, \dots, k$, so if $\varepsilon \ll \psi$, and $z_\varepsilon \in N_\psi(\partial\Lambda)$, then $z_\varepsilon \in N_{\Delta^\star(z^{(\ell)})}$ for some ℓ and so $|S_\varepsilon(z_\varepsilon) - S_\varepsilon(z_\varepsilon^{(\ell)})| \lesssim \left(\frac{|z_\varepsilon - z_\varepsilon^{(\ell)}|}{\psi}\right)^\sigma$. For $z_\varepsilon, w_\varepsilon \in \Lambda_\varepsilon \setminus N_\psi(\partial\Lambda)$, $|z_\varepsilon - w_\varepsilon| < \psi$, there are clearly of the order $\log(|z_\varepsilon - w_\varepsilon|/\psi)$ annuli surrounding both z_ε from w_ε and we obtain $|z_\varepsilon - w_\varepsilon| \lesssim \left(\frac{|z_\varepsilon - w_\varepsilon|}{\psi}\right)^\sigma$.

Finally, the statement concerning the behavior of discrete contour integrals of S_ε can be directly found in [15] for the triangular lattice (also c.f., discussion in [2]) and in [11], §4.3, for the extended models. $\square$

4.2 Cauchy Integral Estimate

We will start by establishing a multiplication lemma for an actual holomorphic function with a nearly–holomorphic function:

Lemma 4.4 *Let Q_ε be part of a (σ, ρ)–holomorphic sequence as described in Definition 4.1. above. Let $\varepsilon > 0$ and suppose Γ_ε is a discrete closed contour consisting of edges of hexagons at scale ε. Let $q(z)$ be a holomorphic function on Λ restricted to Λ_ε (regarded as a subset of $\mathbb{C}$). Next let $1 \gg D \gg \varepsilon$ (both considered small). Then for all $\varepsilon \geq 0$ sufficiently small*

$$|\oint_{\Gamma_\varepsilon} q \cdot Q_\varepsilon \; dz| \lesssim (\|q\|_\infty \cdot \frac{\varepsilon^\rho}{D} + \|q\|_{C^1} \cdot D^\sigma) \cdot (|\mathrm{Int}(\Gamma_\varepsilon)| + |\Gamma_\varepsilon| \cdot D).$$

Proof. Consider a square–like grid of scale D and let R_k denote the k^{th} such square which has non–empty intersection with Λ_ε. Next we let

$$\gamma_k := \partial(R_k \cap \mathrm{Int}(\Gamma_\varepsilon)).$$

Note that γ_k is not necessarily a single closed contour, but each γ_k is a collection of closed contours. It is observed that if F is a function, then $\oint_{\Gamma_\varepsilon} F \; dz = \sum_k \oint_{\gamma_k} F \; dz$, where by abuse of notation, as mentioned above, each term on the righthand side may represent the sum of several contour integrals. Next let us register an estimate within a single region bounded a γ_k, the utility of which will be apparent momentarily:

Claim. Let $z_k \in R_k$ (if R_k intersects $\partial\Lambda_\varepsilon$ then choose z_k in accordance with item (i) of the definition of σ–holomorphicity so that Hölder continuity of Q can be assumed). Then

$$\oint_{\gamma_k} q \cdot Q \; dz = q(z_k) \cdot \oint_{\gamma_k} Q \; dz + \mathcal{E}_k, \tag{4.2}$$

where

$$|\mathcal{E}_k| \lesssim |\gamma_k| \cdot \|q\|_{C^1} \cdot D^{1+\sigma}$$

and to avoid clutter, we omit the ε subscript on the Q's.

Proof of Claim. Let us write

$$Q(z) = Q(z_k) + \delta Q(z).$$

Similarly, let us write

$$q(z) = q(z_k) + \delta q(z).$$

We then have that

$$\oint_{\gamma_k} q \cdot Q \; dz - q(z_k) \cdot \oint_{\gamma_k} Q \; dz = \oint_{\gamma_k} \delta Q \cdot \delta q \; dz + Q(z_k) \cdot \oint_{\gamma_k} \delta q \; dz.$$

The second term on the right hand side vanishes identically by analyticity of q whereas the integrand of the first term, by the assumed Hölder continuity of Q and analyticity of q, can be estimated via $\lesssim \|q\|_{C^1} \cdot D \cdot D^\sigma$ and the claim follows. □

Therefore we may write

$$\oint_{\Gamma_\varepsilon} q \cdot Q \; dz = \sum_k \oint_{\gamma_k} q \cdot Q \; dz := \sum_k q(z_k) \cdot \oint_{\gamma_k} Q \; dz + \sum_k \mathcal{E}_k,$$

where z_k is a representative point in the region $R_k \cap \mathrm{Int}(\Gamma_\varepsilon)$. We divide the error on the righthand side into two terms, corresponding to *interior* boxes – which do not intersect Γ_ε, and boundary boxes – the complementary set.

Let us first estimate the interior boxes. Here, from the claim we have that the integral over each such box incurs an error of $\|q\|_{C^1} \cdot D^{2+\sigma}$ since here $|\gamma_k| \lesssim D$. There are of order $|\mathrm{Int}(\Gamma_\varepsilon)| \cdot D^{-2}$ interior boxes so we arrive at the estimate $\|q\|_{C^1} \cdot D^\sigma \cdot |\mathrm{Int}(\Gamma_\varepsilon)|$. On the other hand, for boundary boxes, the contribution to the errors from the boundary boxes will certainly contain the original contour length $|\Gamma_\varepsilon|$. To this we must add $\lesssim D \times$ [the number of boundary boxes] corresponding to the "new" boundary of the boxes themselves that we might have introduced by considering the boxes in the first place. This is estimated as follows:

Claim. Let $M(\Gamma_\varepsilon, D)$ denote the number of boundary boxes – i.e., the number of boxes on the grid visited by Γ_ε. Then $M \lesssim |\Gamma_\varepsilon|/D$.

Proof of claim. Since arguments of this sort have appeared in the literature (e.g., [4], [9], [10]) many times, we shall be succinct: We divide the grid into 9 disjoint sublattices each of which indicated by its position on a 3×3 square. Let $M_1, \ldots, M_9$ denote the number of boxes of each type that are visited by Γ_ε. We may assume without loss of generality that $\forall j,\ M_1 \geq M_j$. Let us consider the coarse grained version of Γ_ε as a sequence of boxes on the first lattice (visited by Γ_ε); revisits of a given box are not recorded until/unless a different element of the sublattice has been visited in–between. Since the distance between each visited box is more than D it follows that corresponding to each visited box the curve Γ_ε must "expend" at least D of its length, i.e., $|\Gamma_\varepsilon| \cdot D \geq M_1 \geq (1/9) \cdot M$ and the claim follows. $\square$

It is specifically observed that the additional boundary length incurred is at most comparable to the original boundary length. In any case altogether we acquire an estimate of the order $|\Gamma_\varepsilon| \cdot \|q\|_{C^1} \cdot D^{1+\sigma}$. We have established

$$|\sum_k \mathcal{E}_k| \lesssim \|q\|_{C^1} \cdot D^\sigma \cdot (|\mathrm{Int}(\Gamma_\varepsilon)| + |\Gamma_\varepsilon| \cdot D).$$

Finally, by item ii) of (σ, ρ)–holomorphicity,

$$\sum_k |q(z_k) \cdot \oint_{\gamma_k} Q \; dz| \lesssim \|q\|_\infty \cdot \varepsilon^\rho \cdot (|\mathrm{Int}(\Gamma_\varepsilon)| \cdot D^{-1} + |\Gamma_\varepsilon|).$$

This follows from the decomposition similar to the estimation of the $\mathcal{E}_k$ terms with the first term corresponding to interior boxes and the second to boundary boxes. The lemma been established. $\square$

We can now immediately control the Cauchy integral of a (σ, ρ)–holomorphic function uniformly away from the boundary:

Corollary 4.5 *Let Q_ε be part of a (σ, ρ)–holomorphic sequence as described in Definition 4.1 above. Let $G_\varepsilon(z)$ be given as the Cauchy–integral of Q_ε – as in Eq.(4.4) – over some (discrete Jordan) contour Γ_ε. Let z denote any lattice point in $\mathrm{Int}(\Gamma_\varepsilon)$ such that*

$$\mathrm{dist}(z, \Gamma_\varepsilon) \geq d_1$$

for some $d_1 > 0$ and let $D \gg \varepsilon$ (both considered small). Then for all $\varepsilon > 0$ sufficiently small, and any $d_2 < d_1$,

$$
\begin{aligned}
|G_\varepsilon(z) - Q_\varepsilon(z)| &= |\frac{1}{2\pi i}\oint_{\Gamma_\varepsilon}(Q_\varepsilon(\zeta) - Q_\varepsilon(z))\cdot\frac{1}{\zeta - z}\ d\zeta\ | \\
&\lesssim \left(\frac{\varepsilon^\rho}{d_2 D} + \frac{D^\sigma}{d_2^2}\right)\cdot(|\mathrm{Int}(\Gamma_\varepsilon)| + |\Gamma_\varepsilon|\cdot D) + \left(\frac{d_2}{d_1}\right)^\sigma
\end{aligned}
\tag{4.3}
$$

Proof. This is the adaptation of standard arguments from the elementary theory of analytic functions to the present circumstances. Let γ_{d_2} denote an approximately circular contour that is of radius d_2 and which is centered at the point z. Let Γ'_ε denote the contour Γ_ε together with γ_{d_2} – traversed backwards – and a back and forth traverse connecting the two. We have, by Lemma 4.4, that

$$
|\frac{1}{2\pi i}\oint_{\Gamma'_\varepsilon}(Q_\varepsilon(\zeta))\cdot\frac{1}{\zeta - z}\ d\zeta\ | \lesssim \left(\frac{\varepsilon^\rho}{d_2 D} + \frac{D^\sigma}{d_2^2}\right)\cdot(|\mathrm{Int}(\Gamma_\varepsilon)| + |\Gamma_\varepsilon|\cdot D)
$$

where, in the language of this lemma, we have used $\|q\|_\infty \lesssim d_2^{-1}$ and $\|q\|_{C^1} \lesssim d_2^{-2}$. Thus we write

$$
G_\varepsilon(z) = \frac{1}{2\pi i}\oint_{\gamma_{d_2}}\frac{Q_\varepsilon(\zeta)}{\zeta - z}d\zeta + \mathcal{E}_2
$$

we have that $|\mathcal{E}_2|$ is bounded by the right hand side of the penultimate display. So, subtracting $Q_\varepsilon(z)$ in the form

$$
Q_\varepsilon(z) = \frac{1}{2\pi i}\oint_{\gamma_{d_2}}\frac{Q_\varepsilon(z)}{\zeta - z}d\zeta
$$

we have that

$$
|G_\varepsilon(z) - Q_\varepsilon(z)| \lesssim |\mathcal{E}_2| + \frac{1}{2\pi}\oint_{\gamma_{d_2}}\frac{|Q_\varepsilon(z) - Q_\varepsilon(\zeta)|}{\zeta - z}d\zeta
$$

and the stated result follows immediately from the Hölder continuity of Q_ε. □

By inputing information on $|\partial\Omega_n^\square|$, the required Cauchy–integral estimate now follows:

Proof of Lemma 3.5. We first recall the statement of the lemma:

Let $\Omega_n^\square$ and $S_n^\square$ be as in Proposition 3.2 so that

$$
|\partial\Omega_n^\square| \leq n^{\alpha(1-a_1)},
$$

where $M(\partial\Omega) = 1 + \alpha$. For $z \in \Omega_n^\square$ (with the latter regarded as a continuum object) let

$$
F_n^\square(z) = \frac{1}{2\pi i}\oint_{\partial\Omega_n^\square}\frac{S_n^\square(\zeta)}{\zeta - z}\ d\zeta. \tag{4.4}
$$

Then for a_1 sufficiently close to 1 there exists $0 < \beta < \sigma, \rho$ such that for all $z \in \Omega_n^\square$ so that dist$(z, \partial\Omega_n^\square) > d_1$ for some $d_1 > 0$ (a sublinear power of n)

$$\left|S_n^\square(z) - F_n^\square(z)\right| \lesssim n^{-\beta}.$$

By Proposition 4.3, we have that the functions $S_n^\square(z)$ (with $\varepsilon = n^{-1}$) have the (σ, ρ)–holomorphic property. In addition, we shall also have to keep track of a few other powers of ε, which we now enumerate:

i) let us define $b_1 > 0$ so that in macroscopic units we have

$$|\partial\Omega_N^\square| \leq \varepsilon^{-\alpha(1-a_1)} := \varepsilon^{-\alpha b_1};$$

ii) let us define

$$d_2 := \varepsilon^s,$$

for some $s > 0$ to be specified later;

iii) finally, we define

$$D := \varepsilon^t,$$

where the role of D will be the same as in the proof of Lemma 4.4 (it is the size of a renormalized block).

Plugging into Corollary 4.5, we obtain that

$$\begin{aligned}
\left|S_n^\square(z) - F_n^\square(z)\right| &\lesssim \left(\frac{\varepsilon^\rho}{d_2 D} + \frac{D^\sigma}{d_2^2}\right) \cdot (|\mathrm{Int}(\partial\Omega_n^\square)| + |\partial\Omega_n^\square| \cdot D) + \left(\frac{d_2}{d_1}\right)^\sigma \\
&\lesssim \left(\varepsilon^{\rho-(s+t)} + \varepsilon^{t\sigma-2s}\right) \cdot (1 + \varepsilon^{-\alpha b_1 + t}) + \frac{\varepsilon^{s\sigma}}{d_1^\sigma} \\
&= \varepsilon^{\rho-(s+t)} + \varepsilon^{\rho-s-\alpha b_1} + \varepsilon^{t\sigma-2s} + \varepsilon^{(1+\sigma)t-\alpha b_1-2s} + \frac{\varepsilon^{s\sigma}}{d_1^\sigma}.
\end{aligned}$$

With σ fixed, the parameters $s, t > 0$ and d_1 can be chosen so that all terms in the above are positive powers of ε: set $t = \lambda\sigma$, where $\lambda \in (0, 1)$ so that $\sigma > \frac{1-\lambda}{\lambda}$. This choice of t implies that $(1+\sigma)t > \sigma > t$. Now let $s > 0$ and $b_1 > 0$ be sufficiently small so that $2s < t\sigma$ and $\alpha b_1 < t$ so altogether we have the last two terms are positive powers of ε. Next take t and then s and b_1 even smaller if necessary, we can also ensure $\rho > s + t$ and $\rho > s + \alpha b_1$. Finally, d_1 can be chosen to be some power of ε so that $\varepsilon^s \ll d_1$.

$\square$

5 Harris Systems

5.1 Introductory remarks

For many purposes, the pertinent notion of distance – or separation – is Euclidean; in the context of critical percolation, what is more often relevant is the *logarithmic* notion of distance: how many scales separate two points. These matters are relatively simple deep in the interior of a domain or in the presence of smooth boundaries. However, for points in the vicinity of rough boundaries, circumstances may become complicated. For certain continuum problems, including, in some sense, the limiting behavior of critical percolation, there is a natural notion for a system of increasing neighborhoods about a boundary point: the preimages under uniformization of the logarithmic sequence of cross cuts centered about the preimage of the boundary point in question. This device was employed implicitly and explicitly at several points in the works [4], [5]. In the present context, we cannot so easily access the limiting behavior we are approaching. Moreover, in order to construct such a neighborhood sequence at the discrete level, it will be necessary to work directly with Ω_n itself.

We will construct a neighborhood system for each point in $\partial\Omega_n$ by inductively exploring the entire domain via a sequence of crossing questions. Our construction demonstrates (as is *a posteriori* clear from the convergence of S_n to a conformal map) that various domain irregularities e.g., nested tunnels, which map to a small region under uniformization are, in a well–quantified way, also unimportant as far as percolation is concerned.

5.2 Preliminary Considerations

Let $\Omega \subset \mathbb{C}$ be a simply connected domain with $\text{diam}(\Omega) < \infty$ and let 2Δ denote the supremum of the radius of all circles which are contained in Ω. Further, let D_Δ denote a circle of side Δ with the same center as a circle for which the supremum is realized. We will denote by Ω_m some interior discretization of Ω, as before. For $\omega \in \partial\Omega_m$ we will define a sequence of segments the boundaries of which are paths beginning and ending on $\partial\Omega_m$. As a rule, these segments separate ω from D_Δ. The dimensions of these segments will be determined by percolation crossing probabilities analogous to the system of annuli (of which these are fragments) investigated by Harris in [12]. We will call the resultant objects *Harris rings.*

We will start with some preliminary considerations. Let $S_0(\omega)$ denote the smallest square centered at some $\omega \in \partial\Omega_m$ whose boundary is tangent to ∂D_Δ. We set $R_0(\omega) := S_0(\omega) \cap \Omega_m$; the successive topological rectangles $R_1(\omega), \dots, R_k(\omega), \dots$ will be constructed via a non–trivial inductive procedure: 1) there will be deformations of the shape of the annular segments; 2) the sizes of the smaller squares (location of the next boundary) will be determined by percolation crossing probabilities; 3) the basic shape will not always be a square centered at ω.

We denote an annular ring fragment by e.g., $A_m := [S_{m-1} \setminus S_m] \cap \Omega_m$. We think of the segment A_m as having four boundary segments, forming a topological rectangle. Part of our inductive procedure involves a coloring – i.e., a determination – of portions of the inner and outer boundaries of A_m as yellow, but the remaining boundaries, considered blue, will be portions of $\partial\Omega_m$. Indeed, note that the boundary $\partial\Omega_m$ cuts through such a ring and thus the portion of interest (i.e., lying in Ω_m) *is* a topological rectangle and it is clear that there are dual crossing problems of say a yellow crossing plugging into parts of the ring from the original annulus, which we may consider to be the yellow boundary, *and* a blue crossing joining the said blue boundaries.

Key in the definition is that for some $0 < \vartheta < 1/2$, it will be the case that the probability of a yellow crossing between the yellow segments of the boundaries *and* the probability of a blue crossing between the blue segments of the boundaries are both in excess of ϑ (and therefore less than $1 - \vartheta$).

We will describe what is fully required in successive stages of increasing complexity, but before we begin, let us dispense with some lattice details. While the definitions and conventions which follow are certainly not immediately necessary, we have elected to display them first since on the one hand such details are ultimately inessential but on the other hand may serve to foreshadow what is to come.

Definition 5.1. The moral behind these definitions is that all lattice details should be resolved in as organic a way as possible via the definition of the percolation model of interest. The models of interest for us are hexagonal based: each model provides some *smallest independent unit* (abbreviated SIU) in the sense that such a unit (a subset of the lattice) can be stochastically configured independently and any smaller subset is correlated with some neighbor; in the case of hexagonal tiling the smallest such unit is simply a hexagon whereas for the generalized models in [11] the smallest unit can be either a single hexagon or a flower (which consist of 7 hexagons). All notions of neighborhood, self–avoiding, etc., then should be thought of in terms of the intrinsic definition of connectivity. (We warn the reader, however, that in the case of the model introduced in [11], path transmissions may take place over fractions of hexagons/flowers.)

1. unless otherwise specified $x \rightsquigarrow y$ (with $x, y \in \overline{\Omega}_m$) means a monochrome percolation connection *inside* Ω_m from x to y;
2. we shall often use descriptions like horizontal and vertical, and this should be understood to mean the closest lattice approximation to either a horizontal or vertical segment (e.g., assuming hexagons are oriented so that there are two vertical edges parallel to the y–axis, a horizontal segment of hexagons would just be a consecutive string of such hexagons whereas a vertical segment of hexagons would "zigzag");
3. thus we envision the plane as having been coordinatized by SIU, and if $x = (x_1, x_2), y = (y_1, y_2) \in \Omega_m$ are SIU, we will make use of the distance
$$d_\infty(x, y) = \max\{|x_1 - x_2|, |y_1 - y_2|\},$$
understood to mean e.g., if $y_1 = y_2$, then $|x_1 - x_2|$ is the number of hexagons/flowers lying between x_1 and x_2 in the horizontal directions;
4. if Γ is a lattice segment consisting of only (approximations of) horizontal and vertical subsegments, then $\|\Gamma\|_\infty$ denotes the *maximum* of the total horizontal length and the total vertical length;
5. if Γ is a lattice segment consisting of only (approximations of) horizontal and vertical subsegments, then $\|\Gamma\|_0$ denotes the *minimum* of the total horizontal length and the total vertical length;
6. if Γ is a lattice segment, the lattice k–neighborhood of Γ, denoted $N_k(\Gamma)$, consists of all hexagons that can be reached from Γ by a lattice path of length $\leq k$;
7. if Γ is a lattice segment with endpoints on two points of $\partial\Omega_m$ which divides $\partial\Omega_m$ into two connected components $\mathscr{C}_\Gamma(z_1)$ and $\mathscr{C}_\Gamma(z_2)$ containing points z_1 and z_2,

respectively, then a *successor* of Γ is $\delta N_k(\Gamma)$ – by which we mean the "boundary" of the k^{th} lattice k–neighborhood of Γ which in the case of hexagons should be a connected path consisting of edges of hexagons associated with the "boundary", intersected with either $\mathscr{C}_\Gamma(z_1)$ or $\mathscr{C}_\Gamma(z_2)$, so that a successor necessarily also connects two points of $\partial\Omega_m$;

8. if Γ is a lattice segment as in the previous item, then *sliding* Γ in e.g., $\mathscr{C}_\Gamma(z_1)$ means considering successive successors of Γ in $\mathscr{C}_\Gamma(z_1)$: $\delta N_1(\Gamma)\cap\mathscr{C}_\Gamma(z_1), \delta N_2(\Gamma)\cap \mathscr{C}_\Gamma(z_1), \delta N_3(\Gamma)\cap\mathscr{C}_\Gamma(z_1)$... where it is tacitly assumed that these neighborhoods do not run into z_1;
9. we say that e.g., a box is *contiguous* with some lattice segment if it is the case that (the lattice approximation to) the boundary of the box overlaps a portion of the lattice segment;
10. we say that e.g., a box is *flush against* some horizontal segment if the bottom or top boundary of the box overlaps an end portion of the lattice segment: e.g., the right endpoint of the segment coincides with the right end point of the bottom boundary of the box.

Let $\omega \in \partial\Omega_m$ and note that since $\partial\Omega_m \cap \partial D_\Delta = \emptyset$ whereas ∂D_Δ has non–trivial intersection with $\partial S_0(\omega)$, we can declare the first yellow segment of the boundary, denoted Y_0, to simply be the connected component of $\partial S_0(\omega)\cap\partial D_\Delta$ in Ω_m.

One of the properties we will require of our Harris regularization scheme is that ω can be connected to D_Δ via a sequence of boxes whose size do not increase or decrease too fast; the boxes themselves will be comparable in size to that of the Harris segment within which they reside. Dually, ω can be "sealed off" from D_Δ by the independent events of separating chains which have an approximately uniform probability in each segment. Thus we envision an orientation to our constructions leading from D_Δ to ω. (Indeed, it is this orientation which permits us to choose the appropriate components to be colored yellow at various stages of the construction.) Moreover, from these considerations, it emerges that only the first $O(\log n)$ of these segments are relevant for the percolation problem at hand. If Ω_m has a smooth boundary this would be, in fact, all of them; under general circumstances, the configurations in the region beyond the first $O(\log n)$ segments has negligible impact on the percolation problem at hand.

5.3 Preliminary Constructions

It turns out that in order to acquire the necessary quantitative control on the domain, we in general have need for three types of constructions, which we will call the S–construction, the Q–construction and the R–construction. We will describe them in order as they require more and more detailed control on successive Harris segments.

The S–Construction. The starting point is the S–construction, which involves concentric squares centered at ω. Consider a *successor* square $S \equiv S(\omega) \subset S_0(\omega)$ which is concentric with $S_0(\omega)$ and consider all possible self–avoiding paths

$$\mathscr{P} : \omega \rightsquigarrow \partial S \rightsquigarrow Y_0$$

such that i) $\mathscr{P} \subseteq \Omega_m$, ii) $\mathscr{P} \cap \partial\Omega_m = \{\omega\}$ and iii) once $\mathscr{P}$ leaves S it never re–enters S, so that the second portion of the path takes place entirely in $S_0 \setminus S$. We then define the yellow segment of ∂S, denoted Y_S, to be the set of all exit points of all such paths $\mathscr{P}$ on ∂S. We will now establish some topological properties of these yellow segments Y_S.

First we claim that Y_S is well–defined:

Claim. All of Y_S belong to a single connected component in $\overline{\Omega}_m$.

Proof of Claim. First it is noted that $\partial\Omega_m$ may in general divide ∂S into many components, and it is clear that $Y(S)$ fills any such component which it has non–trivial intersection with. Suppose then that there are two such components and containing points $z_1, z_2 \in \partial S$, respectively. Consider paths $\mathscr{P}_1, \mathscr{P}_2$ associated with z_1 and z_2. Clearly, $\mathscr{P}_1 \cup \mathscr{P}_2$ together with the relevant portion of Y_0 form a loop inside $\overline{\Omega}_m$ and because of property iii) of $\mathscr{P}_1, \mathscr{P}_2$, inside this loop lies the entire portion of ∂S which connects z_1 and z_2; but since z_1, z_2 are in different components (relative to $\partial\Omega_m$) there exist points inside the loop which are part of $\partial\Omega_m$. This necessarily implies that some portion of the loop intersected $\partial\Omega_m$ which is impossible since Y_0 $\mathscr{P}_1$ and $\mathscr{P}_2$ were all, purportedly, separate from the boundary. $\square$

Next we have the following "partial ordering" property:

Claim. Suppose $Y_{S'}$ is a successor to Y_S, then Y_S separates $Y_{S'}$ from Y_0 in the sense that every path $\mathscr{P} : Y_{S'} \rightsquigarrow Y_0$ inside Ω_m which does not intersect $\partial\Omega_m$ must pass through Y_S.

Proof of Claim. It is clear that we have *some* path $\Gamma : Y_{S'} \rightsquigarrow Y_0$ which intersects Y_S, e.g., the latter portion of a path associated with Y_S. Now if the separation statement in this claim is not true, then there exists a path $\tilde{\Gamma} : Y_{S'} \rightsquigarrow Y_0$ which is disjoint from Y_S. But then the loop formed by $\Gamma \cup \tilde{\Gamma}$ and the relevant portions of $Y_{S'}$ and Y_0 must enclose some point in $\partial\Omega_m$; this is impossible since Ω_m is simply connected. $\square$

Given S, a successor square to S_0, and the corresponding yellow boundary Y_S, the blue boundary is defined to be the portions of $\partial\Omega_m$ connecting the endpoints of Y_0 and Y_S. The *topological rectangle* formed by the blue boundaries and Y_0, Y_S will be denoted by $R_S \subset [S_0(\omega) \setminus S(\omega)] \cap \Omega_m$ – where it is noted that the inclusion can be strict. Finally, the size of the successor square will be selected in the following way: we require the resulting topological rectangle R to be so that both the yellow and blue crossing problems satisfy definitive bounds. That this can be accomplished is the subject of the following lemma:

Lemma 5.2 (Sliding Scales.) *Let $S_0(\omega)$ be the square centered at some $\omega \in \partial\Omega_m$ as described above and consider successor squares contained in $S_0(\omega)$, which are generically denoted S. Then the size of S can be adjusted so that the yellow crossing probability satisfies*

$$\vartheta \leq \mathbb{P}(Y_0 \rightsquigarrow Y_S) \leq 1 - \vartheta,$$

where $0 < \vartheta < 1/2$ is a definitive constant only depending on details of the percolation model and is such that $\mathbb{P}(Y_0 \rightsquigarrow \omega) < \vartheta$.

Proof. Consider the procedure of sliding S inwards starting from S_0 itself one step at a time, as described in Definition 5.1. It is clear by the separation claim concerning S that the crossing probabilities between the relevant yellow segments monotonically decrease. Furthermore, we may bound such a crossing probability from above by the crossing of a full annulus: the topological rectangle R of relevance is contained in the full annulus $S_0 \setminus S$ hence a crossing of the full annulus certainly implies a crossing of R . Thus we see that these probabilities are bounded above by that of a one–arm event which in turn can be bounded by a power of the aspect ratios. Let us enumerate the successive sliding trials of S by $S^{(1)}, S^{(2)}, \ldots,$ etc. It is clear that for some ℓ,

$\mathbb{P}(Y_0 \rightsquigarrow Y_{S(\ell)}) > 1 - \vartheta$ while $\mathbb{P}(Y_0 \rightsquigarrow Y_{S(\ell+1)}) \leq 1 - \vartheta$. Thus it is sufficient to show that $\mathbb{P}(Y_0 \rightsquigarrow Y_{S(\ell+1)}) \geq \vartheta$. By the separation claim above (see also the introductory paragraph of Definition 5.1) if there is a requisite path at the ℓ^{th} level, then it is only necessary to "attach" one more unit of yellow to achieve the desired connection up through the $(\ell+1)^{\text{st}}$ level. In the independent model this occurs with probability 1/2, while for the generalizations in [11], this occurs with some probability $r > 0$. Therefore, if

$$\vartheta < \frac{r}{1+r},$$

then we are guaranteed that $\mathbb{P}(Y_0 \rightsquigarrow Y_{S(\ell+1)}) \geq \vartheta$. The stated result has been proved.

□

The Q–Construction. While we may envision a regularization where we inductively perform the S–construction, yielding $Y_1, Y_2, \dots$, it turns out that this is not sufficient to capture certain irregularities which may be present in the domain Ω – nor to achieve our purpose. This problem manifests itself on two levels: the successive yellow regions may be vastly different in length, as can be caused by a narrow tunnel suddenly leading to a wide region; on a more subtle level, there are cases where the S–construction yields consecutive yellow regions which are of comparable size but the "effective" yellow region where the crossing would actually take place is in fact much smaller, which is again indicative of "pinching" of $\partial\Omega_m$. In any case, the problem here is that, in essence, the process is proceeding much too quickly. The cure is then to reduce the relevant scales in order to slow the growth of the evolving neighborhood sequence. Geometrically this requires a re–centering and re–sizing of the basic shape we use to construct the crossing rectangle.

To a *first approximation*, a successor square is not valid for us if it is the case that the yellow segment Y_{k+1} is too large or too small relative to the separation between Y_k and Y_{k+1}, in which case we will instead consider annuli grown around some "effective region" determined by *sub*segments of Y_k and Y_{k+1}. The notion of "effective regions" is made precise in the following lemma:

Lemma 5.3 (Effective Regions.) *Suppose we have successive yellow regions Y_k, Y_{k+1} in the S–construction, which are parts of the boundary of squares $S_k(\omega), S_{k+1}(\omega)$, where the size of $S_{k+1}(\omega)$ is such that the conclusion of Lemma 5.2 is satisfied. Denote* *the separation distance between S_k and S_{k+1} by J_k. Then there exists some B with $1 < B < \infty$ and some subsegments $Y_k^{(e^+)} \subset Y_k, Y_{k+1}^{(e^-)} \subset Y_{k+1}$ with*

$$\|Y_k^{(e^+)}\|_\infty, \|Y_{k+1}^{(e^-)}\|_\infty \leq 3B \cdot J_k$$

such that all relevant crossing events are essentially determined within the rectangle formed by the effective regions; in particular:

$$\mathbb{P}(Y_k^{(e^+)} \rightsquigarrow Y_{k+1}^{(e^-)}) \geq \vartheta - \vartheta^4,$$

with ϑ as in Lemma 5.2.

Proof. Let us choose the constant B to be such that the hard way crossing of a B by 1 rectangle in either blue or yellow is less than ϑ^2. Without loss of generality, let us envision the segments Y_k and Y_{k+1} to be, by and large, horizontal, with Y_k lying above Y_{k+1}. By construction of Y_k, Y_{k+1}, it is the case that the boundary of Ω_m connects pairs of endpoints of Y_k to Y_{k+1}. Consider the path $\mathscr{P}_\ell$ which is the portion of $\partial\Omega_m$ starting with the left end point of Y_k proceeding to the left end point of Y_{k+1}; similarly consider the path $\mathscr{P}_r$ starting with the right end point of Y_{k+1} and proceeding to the right end point of Y_k. (It must be the case that $\mathscr{P}_\ell \cap \mathscr{P}_r = \emptyset$ since otherwise no yellow crossing would be possible in the region enclosed by Y_k and Y_{k+1}. Indeed, $\mathscr{P}_\ell$ and $\mathscr{P}_r$ are considered to be the left and right boundaries of the topological rectangle $R_k = (S_k \setminus S_{k+1}) \cap \Omega_m)$. Let us now define Γ_ℓ to be the straight vertical segment joining Y_k and Y_{k+1} which intersects the rightmost point of $\mathscr{P}_\ell$ inside the region bounded by Y_k and Y_{k+1} and similarly define Γ_r for $\mathscr{P}_r$.

We now observe that the horizontal distance between Γ_ℓ and Γ_r cannot exceed $B \cdot J_k$: first if Γ_ℓ were to the left of Γ_r, then the relevant yellow crossing probability in R_k would be bounded from below by the easy way crossing of the $B \cdot J_k \times J_k$ rectangle bounded by $\Gamma_\ell, \Gamma_r, Y_k, Y_{k+1}$ which by choice of B would exceed $1 - \vartheta^2$, but by Lemma 5.2, R_k is constructed so that this yellow crossing probability is at most $1 - \vartheta$. On the other hand, if Γ_ℓ were to the right of Γ_r, then any yellow crossing must traverse a horizontal distance at least $B \cdot J_k$, which by choice of B is less than ϑ^2, again contradicting the choice of ϑ as in Lemma 5.2.

Next we will extend the horizontal region defined by Γ_ℓ and Γ_r by an additional $B \cdot J_k$ on each side. (We do this if space is available; otherwise Y_k, Y_{k+1} are already comparable to J_k.) We denote the bounding vertical segments by γ_ℓ, γ_r, the resulting yellow regions by $Y_k^{(e^+)}, Y_{k+1}^{(e^-)}$ and call the regions bounded within the *effective region.* We now argue that the yellow crossing actually occurs inside the effective region, with high "conditional" probability.

Consider the event $\mathscr{D} := \{Y_k \rightsquigarrow Y_{k+1}\} \setminus \{Y_k^{(e^+)} \rightsquigarrow Y_{k+1}^{(e^-)}\}$. We claim that the event $\mathscr{D}$ implies the existence of both a blue and a yellow crossing of aspect ratio at least B. First consider the case where Γ_ℓ is to the left of Γ_r. Here non–existence of a yellow crossing in the effective region implies a blue crossing between some point on $\mathscr{P}_\ell$ and γ_r, whose probability is bounded by the hard–way crossing of a $B \cdot J_k \times J_k$ rectangle; finally a yellow crossing in the original R_k conditioned on such a blue crossing must traverse horizontal distance at least $B \cdot J_k$. Similarly, if Γ_ℓ is to the right of Γ_r, then the occurrence of $\mathscr{D}$ implies a similarly long blue crossing and an even longer yellow crossing (traversing horizontal distance at least $2B \cdot J_k$). By choice of B, we obtain that $\mathbb{P}(\mathscr{D}) \leq \vartheta^4$, and the final claimed result follows.

□

Let us now describe the re–centering and re–scaling procedure (going forwards): suppose that up to step k the S–construction has been employed successivefully, i.e., 1) the successive yellow segments are all within $3B$ times the separation distance; 2) the crossing probabilities in the relevant rectangle satisfies the conclusion of Lemma 5.2. Now suppose that Y_{k+1} is the first yellow segment such that $|Y_{k+1}| > 3B \cdot J_k$. Let us again envision that Y_k and Y_{k+1} are primarily horizontal with Y_k above Y_{k+1} and consider the subsegment $Y_{k+1}^{(e^-)} \subseteq Y_{k+1}, \gamma_\ell, \gamma_r$ as in the proof of Lemma 5.3; notice that since the horizontal distance between $\mathscr{P}_\ell$ and $\mathscr{P}_r$ (portions of $\partial\Omega_m$ forming the left and right boundaries of R_k) cannot exceed $B \cdot J_k$, γ_ℓ and γ_r must intersect $\partial\Omega_m$ before they reach Y_k, i.e., $|\gamma_\ell|, |\gamma_r| \leq B \cdot J_k$. Finally, let us consider the topological rectangle formed by $Y_k, Y_{k+1}^{(e^-)}, \gamma_\ell, \gamma_r$ together with relevant portions of $\partial\Omega_m$ and define it to be R_{k+1}, with yellow segment

$$\overline{Y}_{k+1} := Y_{k+1}^{(e^-)} \cup \tau_\ell \cup \tau_r,$$

where $\tau_\ell \subseteq \gamma_\ell$ is the portion of the γ_ℓ connected to $Y_{k+1}^{(e^-)}$ before it hits $\partial\Omega_m$ and similarly for γ_r. First note that the probability of a yellow connection between Y_k and γ_ℓ or γ_r is bounded by the probability of a long way crossing of a J_k by $B \cdot J_k$ rectangle which by choice of B (again see the proof of Lemma 5.3) is bounded above by ϑ^2. Thus from the construction of Y_{k+1} and Lemma 5.2 we have

$$\mathbb{P}(Y_k \rightsquigarrow \overline{Y}_{k+1}) \leq \mathbb{P}(Y_k \rightsquigarrow Y_{k+1}^{(e^-)}) + \vartheta^2 \leq \mathbb{P}(Y_k \rightsquigarrow Y_{k+1}) + \vartheta^2 \leq 1 - \vartheta + \vartheta^2.$$

On the other hand, the conclusion of Lemma 5.3 gives that

$$\mathbb{P}(Y_k \rightsquigarrow \overline{Y}_{k+1}) \geq \mathbb{P}(Y_k^{(e^+)} \rightsquigarrow Y_{k+1}^{(e^-)}) \geq \vartheta - \vartheta^4.$$

Altogether we have that

$$\vartheta - \vartheta^4 \leq \mathbb{P}(Y_k \rightsquigarrow \overline{Y}_{k+1}) \leq (1 - \vartheta) - \theta^2.$$

We will consider $\overline{Y}_{k+1}$ to be the $(k+1)^{\text{st}}$ yellow segment. In order to define Y_{k+2}, we consider successors T_{k+2} of $\overline{Y}_{k+1}$ in the direction of ω (we refer to Definition 5.1 for the meaning of the direction associated to a successor of such a segment). More precisely, as in the case of the S–construction, $\overline{Y}_{k+1}$ divides $\overline{\Omega}_m$ into two components, $\mathscr{C}_{\overline{Y}_{k+1}}(\omega), \mathscr{C}_{\overline{Y}_{k+1}}(D_\Delta)$, the connected component of ω and Y_0, respectively; we then say that a successor T_{k+2} is in the direction of ω if it is contained in $\mathscr{C}_{\overline{Y}_{k+1}}(\omega)$. In particular, this implies that any lattice walk from ω to $\overline{Y}_{k+1}$ must pass through T_{k+2}. Let us then consider all possible lattice walks $\mathscr{P} : \omega \rightsquigarrow \overline{Y}_{k+1}$ such that once it enters $\mathscr{C}_{\overline{Y}_{k+1}}(D_\Delta)$ it never leaves; the set of points on T_{k+2} reached this way then constitutes Y_{k+2} (see the analogous Claims in the description of the S–construction).

The relevant blue boundary as before is defined to be the portions of $\partial\Omega_m$ connecting the endpoints of $\overline{Y}_{k+1}$ and T_{k+2}. The topological rectangle formed by the blue boundaries and $\overline{Y}_{k+1}, Y_{k+2}$ will be denoted R_{k+2}, where as before, we adjust how far into $\mathscr{C}_{\overline{Y}_{k+1}}(\omega)$ we slide Y_S so that the crossing probability in R_S satisfy the conclusion of Lemma 5.2 (it is easily verified that the proposition is readily modified to accommodate topological rectangles formed this way, since the Y_S's are slid toward ω one step at a time, as discussed in Definition 5.1).

Next we must address the question of effective regions in this more complicated geometry. In this case, the separation distance J_{k+1} will be measured in the sup–norm distance: $J_{k+1} = d_\infty(Y_{k+1}, Y_{k+2})$ and we compare $\|\overline{Y}_{k+1}\|_\infty$ and $\|Y_{k+2}\|_\infty$ to J_{k+1}. We again employ the quantity B (which the reader will recall is defined to be such that the hard way crossing of a B by 1 rectangle in either blue or yellow is less than ϑ^2). The analogues of the segments Γ_ℓ and Γ_r are clear: there are two disjoint portions $\mathscr{P}_\ell, \mathscr{P}_r$ of $\partial\Omega_m$ which join endpoints of Y_{k+1} to Y_{k+2} and we define Γ_ℓ, Γ_r to be either horizontal *or* vertical segments joining $\overline{Y}_{k+1}$ to Y_{k+2} which intersect the "furthest points" of $\mathscr{P}_\ell$ and $\mathscr{P}_r$ in the region enclosed by $\overline{Y}_{k+1}$ and Y_{k+2}. We can proceed to extend Γ_ℓ, Γ_r in either the horizontal or vertical direction (depending on the original orientation and in cases where only the vertical or horizontal length – but not both – exceed $3B \cdot J_{k+1}$, we can extend one of them by $2B \cdot J_{k+1}$ and then not extend the other) and define the effective region accordingly. The rest of the argument proceeds as in the proof of Lemma 5.3.

In general, we will also have to consider the case where $|Y_k| > 3B \cdot J_k$, in which case we may (under conditions to be described later) need to re–center, re–scale and proceed *backwards* towards D_Δ. Let us record here that the topological separation statements from the S–construction as well as the conclusions of Lemma 5.2 and Lemma 5.3 can be adapted to the cases where the Y_k's are more general lattice segments:

Lemma 5.4 *Let $\omega \in \Omega_m$ be fixed. Let Y denote a connected lattice segment connecting two points of $\partial\Omega_m$, neither of which is ω and such that $Y \cap D_\Delta = \emptyset$ and $\mathbb{P}(Y \rightsquigarrow \omega) < \vartheta$ with ϑ as in Lemma 5.2. The segment Y divides Ω_m into two connected components, $\mathscr{C}_Y(\omega)$ and $\mathscr{C}_Y(D_\Delta)$, the connected components of ω and D_Δ, respectively. Then*

i) successors of Y in $\mathscr{C}_Y(\omega)$ and $\mathscr{C}_Y(D_\Delta)$ are ordered in the sense that if e.g., Y_Q is a successor of Y then any path $\mathscr{P} : \omega \rightsquigarrow Y$ which does not intersect $\partial\Omega_m$ must pass through Y_Q and similarly for successors in $\mathscr{C}_Y(D_\Delta)$;

ii) the analogue of Lemma 5.2 remains valid: successively sliding Y in $C_Y(\omega)$, it is possible to adjust the location of the successor Y_Q so that the yellow crossing probability satisfies

$$\vartheta \leq \mathbb{P}(Y \rightsquigarrow Y_Q) \leq 1 - \vartheta;$$

iii) the analogue of Lemma 5.3 remains valid: suppose Y_Q is a successor of Y and let $J := d_\infty(Y, Y_Q)$. Then there exists some $1 < B < \infty$ and some subsegments $Y^{(e)} \subset Y, Y_Q^{(e)} \subset Y_Q$ with

$$\|Y^{(e)}\|_\infty, \|Y_Q^{(e)}\|_\infty \leq 3B \cdot J$$

such that all relevant crossing events are determined within the rectangle $\overline{R}$ with boundaries Y, Y_Q, the relevant portions of $\partial\Omega_m$, the analogues of τ_r, τ_ℓ (for Y) and $\tau_{Q_r}, \tau_{Q,\ell}$

(for Y_Q) as described above: defining $\overline{Y} := Y^{(e)} \cup [\tau_r \cup \tau_\ell]$ and $\overline{Y}_Q := Y_Q^{(e)} \cup [\tau_{Q,\ell} \cup \tau_{Q,r}]$ to be the relevant yellow segments, we have

$$\vartheta - \vartheta^4 \leq \mathbb{P}(\overline{Y} \rightsquigarrow \overline{Y}_Q) \leq (1 - \vartheta) + \vartheta^2.$$

We have similar results if Y_Q is a successor of Y in $\mathscr{C}_Y(D_\Delta)$ if Y is such that $\mathbb{P}(Y \rightsquigarrow D_\Delta) > 1 - \vartheta$.

Proof. It is clear from the paragraphs preceding the statement of the lemma that these statements, borrowed from Lemma 5.3, continue to hold for general polygonal segments. Particularly note that the procedures are symmetric in Y and Y_Q. □

The R–Construction. Our objectives will eventually be achieved by establishing the existence of monochrome percolation connections between e.g., (the vicinity of) ω and the central region D_Δ. This will be accomplished by showing that there is a sequence of *contiguous* (see Definition 5.1) boxes leading from ω to D_Δ such that each successive box has comparable scale.

Suppose now that we are in the setting of Lemma 5.4, iii): $\overline{Y}, \overline{Y}_Q$ are legitimate yellow segments after taking into account the effective region and renormalizing so that $\|\overline{Y}\|_\infty \leq 3B \cdot J, \|\overline{Y}_Q\|_\infty \leq 3B \cdot J$ with $J = d_\infty(\overline{Y}, \overline{Y}_Q)$. Let us tile the region enclosed by $\overline{Y}, \overline{Y}_Q$ (which we will denote by $\overline{R}$) by boxes of size $2^{-r} \cdot J$ with some $r > 0$, then:

Claim. There exists some fixed $r \equiv r(\vartheta) > 0$ independent of the particulars of the region such that there is a connection between Y and Y_Q via boxes of size $2^{-r} \cdot J$. Here by connection we mean that successive boxes share a side and no box is intersected by $\partial\Omega_m$, as discussed in Definition 5.1.

Proof of Claim. Suppose no such connection exists for some particular r. Then necessarily there exists a box R_* of size $2^{-r} \cdot J$ inside $\overline{R}$ such that $\mathscr{P}_\ell \cap R_* \neq \emptyset$ *and* $\mathscr{P}_r \cap R_* \neq \emptyset$ (here again $\mathscr{P}_\ell$ and $\mathscr{P}_r$ denote the two pieces of $\partial\Omega_m$ ("left" and "right") forming the blue boundary of the topological rectangle $\overline{R}$ formed by $\overline{Y}, \overline{Y}_Q$ and $\partial\Omega_m$).

Consider the $r - 1$ annuli around R_* of doubling sizes: $2^{-r}J \times [2, 4, \dots, 2^r]$. In each such annulus, by weak scale invariance of critical percolation, a blue circuit independently exists with probability at least some $\lambda \equiv \lambda(\vartheta) > 0$ and each such blue circuit, intersected with Ω_m, will connect $\mathscr{P}_r$ to $\mathscr{P}_\ell$ thereby preventing the possibility of a yellow connection between $\overline{Y}$ and $\overline{Y}_Q$. Thus,

$$\mathbb{P}(\overline{Y} \rightsquigarrow \overline{Y}_Q) \leq (1-\lambda)^r < \vartheta - \vartheta^4,$$

if r is chosen sufficiently large depending on ϑ, contradicting Lemma 5.4, iii). □

Remark 5.5. Up to adding or deleting one layer of boxes (which at most introduces some fixed constants into our estimates) we may – and will – assume that the renormalized boxes are *flush* against $\overline{Y}, \overline{Y}_Q$ (see Definition 5.1).

Next consider the layer of boxes of scale $2^{-r}J$ contiguous with $\overline{Y}_Q$. By the previous claim it is the case that there exists at least one such box which can be connected to $\overline{Y}$ via boxes which are unobstructed by $\partial\Omega_m$; let $\overline{Y}_Q^{(b)} \subseteq \overline{Y}_Q$ denote the portion of $\overline{Y}_Q$ consisting of sides of such boxes (intersected with $\overline{Y}_Q$) which can be connected back to $\overline{Y}$. Let us first observe:

Claim. $\overline{Y}_Q^{(b)}$ is connected.

Proof of Claim. Let $\mathscr{C}$ denote one connected component of $\overline{Y}_Q$ which can be connected to $\overline{Y}$ via boxes. First suppose that $\mathscr{C}$ does not share an endpoint with $\overline{Y}_Q$: in this case on *both sides* of $\mathscr{C}$ the boundary $\partial\Omega_m$ comes within $2^{-r}J$ of $\overline{Y}_Q$, but then no box of $2^{-r}J$ can connected $\mathscr{C}$ to $\overline{Y}$, a contradiction.

It follows then that $\mathscr{C}$ shares at least one endpoint with $\overline{Y}_Q$, say the "left" endpoint (i.e., an endpoint of $\mathscr{P}_\ell$) so that $\partial\Omega_m$ comes within $2^{-r}J$ of $\overline{Y}_Q$ immediately to the "right" of the "left" endpoint of $\mathscr{C}$: if this is accomplished by $\mathscr{P}_\ell$, then $\mathscr{C}$ itself cannot be connected to $\overline{Y}$ by boxes, whereas if this is accomplished by $\mathscr{P}_r$, then certainly no portion of $\overline{Y}_Q$ to the right of $\mathscr{C}$ can be connected to $\overline{Y}$ via boxes (of scale $2^{-r}J$). We are forced to conclude that there can only be one such component and so $\overline{Y}_Q^{(b)}$ is connected. □

Our next claim is that far as a yellow connection to $\overline{Y}$ is concerned, it is sufficient to consider the subsegment $\overline{Y}_Q^{(b)}$, up to a very small change in the crossing probability:

Claim. Let $\overline{Y}_Q^{(b)}$ be as above, then

$$\mathbb{P}(\overline{Y} \rightsquigarrow \overline{Y}_Q^{(b)}) \geq \vartheta - 2\vartheta^4.$$

Proof of Claim. If $\overline{Y}_Q^{(b)} = \overline{Y}_Q$ then there is nothing prove. Otherwise, as in the proof of the previous claim, let us assume that $\overline{Y}_Q^{(b)}$ shares its left endpoint with that of $\overline{Y}_Q$ and $\mathscr{P}_r$ intersects the box, again denoted by R_*, of scale $2^{-r}J$ immediately to the right (endpoint) of $\overline{Y}_Q^{(b)}$. Consider again the $r-1$ annuli around R_* formed about R_*. Then, just as in the proof of the penultimate claim, by the standard ("Russo–Seymour–Welsh") estimates we see that the existence of a blue circuit (independently) in each of these annuli has probability in excess of some $\lambda \equiv \lambda(\vartheta) > 0$. Any one of these, together with $\mathscr{P}_r$, certainly seals off the region to the right of $\overline{Y}_Q^{(b)}$ from $\overline{Y}$ and so

$$\mathbb{P}(\overline{Y} \rightsquigarrow [\overline{Y}_Q \setminus \overline{Y}_Q^{(b)}]) \le (1-\lambda)^r.$$

Choosing, if necessary, the quantity r to be even larger than previously required, we can ensure that $(1-\lambda)^r \le \vartheta^4$. $\square$

Finally, we replace the yellow segment $\overline{Y}_Q$ by $\overline{Y}_Q^{(b)}$ together with portions of ∂R_* so that it joins up with $\mathscr{P}_r$; we will denote this portion of ∂R_* by ζ_Q and note that clearly $\|\zeta_Q\|_\infty \lesssim 2^{-r}J$ and $\mathbb{P}(\overline{Y} \rightsquigarrow \zeta_Q) \le \vartheta^4$, by the same arguments as in the proof of the previous claim.

We summarize the result of all our *preliminary* constructions in this subsection in the following theorem:

Theorem 5.6 *Let $\omega \in \partial\Omega_m$ be fixed. Let Y denote a connected lattice segment connecting two points of $\partial\Omega_m$, neither of which is ω and such that $Y \cap D_\Delta = \emptyset$ and $\mathbb{P}(Y \rightsquigarrow \omega) < \vartheta$, with $0 < \vartheta < 1/2$ as in Lemmas 5.2, 5.4. Successively sliding Y in $\mathscr{C}_Y(\omega)$, suppose that the successor Y_Q is such that the yellow crossing probability satisfies*

$$\vartheta \le \mathbb{P}(Y \rightsquigarrow Y_Q) \le 1-\vartheta,$$

with ϑ as given in Lemmas 5.2, 5.4, ii). Next let $B > 0$ be such that the probability of a hard way crossing of a B by 1 rectangle (in both yellow and blue) is less than ϑ^2 and $\overline{Y} := Y^{(e)} \cup [\tau_r \cup \tau_\ell], \overline{Y}_Q := Y_Q^{(e)} \cup [\tau_{Q,\ell} \cup \tau_{Q,r}]$ be as in Lemma 5.4, iii) and $J := d_\infty(Y, Y_Q)$.

Then for $r \equiv r(\vartheta) > 0$ sufficiently large (particularly, $2^{-r} \ll B^{-1}$) there are further connected subsegments $\overline{Y}^{(b)} \subseteq \overline{Y}$, $\overline{Y}_Q^{(b)} \subseteq \overline{Y}_Q$ and small segments ζ, ζ_Q connected to

$\overline{Y}^{(b)}, \overline{Y}_Q^{(b)}$, respectively, whose lengths are of order $2^{-r}J$, such that

$$\widehat{Y} := \overline{Y}^{(b)} \cup \zeta, \quad \widehat{Y}_Q := \overline{Y}_Q^{(b)} \cup \zeta_Q$$

are lattice segments which connect two points of $\partial\Omega_m$ and $\mathbb{P}(Y \rightsquigarrow \zeta_Q) \leq \vartheta^4, \mathbb{P}(Y_Q \rightsquigarrow \zeta) \leq \vartheta^4$ so that

i) $\vartheta - 3\vartheta^4 \leq \mathbb{P}(\widehat{Y} \rightsquigarrow \widehat{Y}_Q) \leq (1-\vartheta) + \vartheta^2 + 2\vartheta^4$;

ii) all of $\overline{Y}_Q^{(b)}$ can be connected to (all of) $\overline{Y}^{(b)}$ via boxes of size $2^{-r}J$ completely unobstructed by $\partial\Omega_m$;

iii) it is the case that

$$2^{-r}J \leq \|\widehat{Y}\|_0 \leq \|\widehat{Y}\|_\infty \leq 4B \cdot J, \quad 2^{-r}J \leq \|\widehat{Y}_Q\|_0 \leq \|\widehat{Y}_Q\|_\infty \leq 4B \cdot J.$$

Similar statements hold if Y_Q is a successor of Y in $\mathscr{C}_Y(D_\Delta)$ provided Y is such that $\mathbb{P}(Y \rightsquigarrow D_\Delta) < \vartheta$.

Proof. The statements hold for both $\widehat{Y}_Q$ *and* $\widehat{Y}$ since the constructions and claims are clearly symmetric in Y and Y_Q so for item i) we simply add in the error incurred when applying the claims also to Y; item ii) follows immediately from the previous claims. Finally, as for item iii) the upper bounds follow immediately from Lemma 5.4, iii) since clearly $\|\widehat{Y}\|_\infty \leq \|\overline{Y}\|_\infty + \|\zeta\|_\infty \leq 4B \cdot J$ (since $\|\zeta\|_\infty \lesssim 2^{-r}J$ which is certainly less than BJ) and similarly for $\|\widehat{Y}_Q\|_\infty$ and the lower bounds are direct consequences of item ii). □

Remark 5.7. Recall that the goal is to ensure box connections all the way from $Y_0 \subseteq \partial D_\Delta$ to the vicinity of ω. This boils down to the requirement that the conclusions of Theorem 5.6 hold for all successive yellow segments and since both constructions could possibly change both a lattice segment Y and its successor Y_Q, the final inductive procedure will have to entail 1) modifying two successive yellow segments in one step and 2) "backwards" constructions. These additions will be featured in the forthcoming section.

5.4 The Full Construction

Let $Y_0 := \partial S_0(\omega) \cap \partial D_\Delta$ be as described before. If $\mathbb{P}(Y_k \rightsquigarrow \omega) \geq \vartheta$ for any k, then stop. Otherwise the base case is the construction (by successively performing the

S, Q, R–constructions) of the yellow segments $\widehat{Y}_0$ and $\widehat{Y}_1$ (both of which are lattice segments connecting two points of $\partial\Omega_m$ and Y_1 is a successor of Y_0 in $\mathscr{C}_{Y_0}(\omega)$) so that the conclusions of Theorem 5.6 (items i), ii), iii)) are satisfied.

We will inductively construct

$$P_0 \equiv \widehat{Y}_0, \; P_1 \equiv \widehat{Y}_1, \; \dots, \; P_{k-1} \equiv \widehat{Y}_{k-1}, \; T_k \equiv \widehat{Y}_k$$

so that

1) the conclusions of Theorem 5.6 hold for P_{k-1} and T_k;

2) the conclusions of Theorem 5.6, i), iii) hold for successive segments $P_\ell, P_{\ell+1}$, $0 \leq \ell < k-1$ and the conclusion of Theorem 5.6, ii) holds with the possibility of connections via boxes of size $\frac{1}{2} \cdot 2^{-r} J_\ell$.

(The second item is on account of a small detail which will become clear when we perform the inductive step.) Here the letter P denotes what is considered a *permanent* yellow segment whereas T denotes *temporary* and subject to re–construction as described in the Q–construction and R–construction sections in the previous subsection. As before we will denote by J_ℓ the separation between successive yellow regions: $J_\ell = d_\infty(P_\ell, P_{\ell+1})$.

We next define T_{k+1} to be a successor of T_k in $\mathscr{C}_{T_k}(\omega)$ constructed so that the conclusion of Lemma 5.4 ii) on crossing probabilities (upper and lower bound by $\vartheta, 1-\vartheta$) is satisfied. We note immediately that this definition of T_{k+1} implies that Theorem 5.6, item i) for T_k and T_{k+1} is automatically satisfied. If it is the case that *all* the conclusions of Theorem 5.6 are satisfied for Y_k and Y_{k+1} then we set $P_k \equiv T_k, P_{k+1} \equiv T_{k+1}$ and continue.

Otherwise, if the cause for failure is only the violation of Theorem 5.6, item ii) (i.e., the availability of box connections from T_k to T_{k+1}) then we may carry out the modifications detailed in the R–construction section for T_{k+1} to yield $\widehat{T}_{k+1}$ and set $P_k \equiv T_k, P_{k+1} \equiv \widehat{T}_{k+1}$ and continue. (Indeed, by the inductive hypothesis, we already have the appropriate box connections up to T_k.)

We are left with the case that T_k, T_{k+1} violate Theorem 5.6, item iii). Here we will have to re–center and re–scale (as described in the S–construction section) and possibly have to proceed backwards towards P_{k-1} (i.e., towards Y_0). More precisely, by the choice of r (so that $2^{-r} < B$) it cannot be the case that the lower bounds are violated so we must have either $\|T_{k+1}\|_\infty > 3B \cdot J_k$ or $\|T_k\|_\infty > 3B \cdot J_k$ (or both). Here a backwards construction towards P_{k-1} may be required (especially now if $\|T_k\|_\infty > 3B \cdot J_k$) since the effective region $T_k^{(e^+)}$ may be much smaller than T_k (and hence J_{k-1}) implying that more yellow segments are needed between T_{k+1} and P_{k-1} so that successive yellow segments have lengths which do not exceed their separation by too much.

In any case, consider the topological rectangle $\overline{R}_k$ formed by the effective regions $T_k^{(e^+)}, T_{k+1}^{(e^-)}$ as in Lemma 5.4, iii)) and the corresponding $\widehat{T}_k^{(e^+)}, \widehat{T}_{k+1}^{(e^-)}$ after performing the R–construction, so that the conclusions of Theorem 5.6 hold for $\widehat{T}_k^{(e^+)}, \widehat{T}_{k+1}^{(e^-)}$. If it is the case that the conclusions of Theorem 5.6 also hold for $P_{k-1}, \widehat{T}_k^{(e^+)}$ then we set $P_k \equiv \widehat{T}_k^{(e^+)}, P_{k+1} \equiv \widehat{T}_{k+1}^{(e^-)}$ and continue.

Otherwise, it must be the case that $\|\widehat{T}_k^{(e^+)}\|_0 < 2^{-r} J_{k-1}$ so that there cannot be a legitimate box connection from P_{k-1} to $\widehat{T}_k^{(e^+)}$. Let us now perform the backwards construction. First set $\widehat{Q}_1 := \widehat{T}_k^{(e^+)}$ and construct sliding Harris rings starting at this scale going towards P_{k-1}: consider successive successors of Q_1 in $\mathscr{C}_{Q_1}(D_\Delta)$ and perform the S, Q, R–constructions to yield $\widehat{Q}_3, \dots, \widehat{Q}_{\ell+2}$ so that the conclusions of Theorem 5.6 hold for successive $\widehat{Q}$'s (here successors have *larger* index) where the index ℓ is defined by the fact that $\widehat{Q}_{\ell+2}$ lies in the region enclosed by P_{k-1} and P_{k-2}. We claim that ℓ is well–defined and *finite*:

Claim. It is the case that $\|\widehat{Q}_l\|_0 \geq 2^{-r} J_{k-1}$ for $2 \leq l \leq \ell + 1$ and therefore $\ell < \infty$.

Proof of Claim. It is sufficient to observe that $\widehat{Q}_2, \dots, \widehat{Q}_{\ell+2}$ all lie strictly in the region determined by $P_{k-1}, T_k^{(e^+)} \subseteq T_k$ and relevant portions of $\partial\Omega_m$ and so if the stated bounds were violated, then $\partial\Omega_m$ comes closer than $2^{-r} J_{k-1}$ to itself, contradicting the inductive hypothesis that there is a box connection with boxes of scale $2^{-r} J_{k-1}$ between P_{k-1} and T_k.

Finally letting $J_l^{(Q)} := d_\infty(\widehat{Q}_l, \widehat{Q}_{l+1})$, we have by Theorem 5.6, iii) that $J_l^{(Q)} \geq \frac{\|Q_l\|_0}{4B} \geq \frac{2^{-r}}{4B} \cdot J_{k-1}$, for $1 \leq l \leq \ell$, which directly implies that $\ell - 1 \leq 4B \cdot 2^r < \infty$. $\square$

Next we observe that this is sufficient for us to return to level $k-2$ and hence the induction is complete:

Claim. $\widehat{Q}_{\ell+2}$ can be connected to P_{k-2} by boxes of scale $\frac{1}{2} \cdot 2^{-r} J_{k-2}$ without being obstructed by $\partial\Omega_m$.

Proof of Claim. First note that by the same argument as in the previous claim, since $\widehat{Q}_{\ell+1}$ lies in the region determined by P_{k-1}, P_{k-2}, it is the case that $\|\widehat{Q}_{\ell+2}\|_0 \geq 2^{-r} J_{k-2}$. By the inductive hypothesis we already have boxes of scale $2^{-r} J_{k-1}$ connecting back to P_{k-2} so depending on the location of $\widehat{Q}_{\ell+2}$ cutting down the scale of some of these boxes by half if necessary, we have the required connection.

$\square$

Finally we re–index so that $P_k \equiv \widehat{Q}_\ell, P_{k+1} \equiv \widehat{Q}_{\ell-1}, \ldots, P_{k+\ell} \equiv \widehat{Q}_1, T_{k+\ell+1} \equiv \widehat{T}_{k+1}^{(e^-)}$. The induction can now be continued towards ω, starting with $P_{k+\ell}, T_{k+\ell+1}$, provided that $\mathbb{P}(T_{k+\ell+1} \rightsquigarrow \omega) < \vartheta$ – otherwise we stop.

Remark 5.8. Let us record for later purposes that we may have gone two Q's and one rectangle too far in our backwards construction. Indeed, consider the crossing probability in the topological rectangle with yellow boundaries formed by $\widehat{Q}_\ell$ and P_{k-1}: by *fiat*, the blue crossing probability would be too large and the yellow too small.

Thus, as far as circuit events are concerned, we may bound the crossing probability from above by crossing of this rectangle (and then the next circuit event would be a crossing in the region formed by P_{k-1}, P_{k-2}). On the other hand, as far as crossing events (say from the vicinity of ω to D_Δ) are concerned, we will require crossings in the rectangles formed by $\widehat{Q}_\ell, \widehat{Q}_{\ell+1}$, then the rectangle formed by $\widehat{Q}_{\ell+1}, \widehat{Q}_{\ell+2}$ and finally the rectangle formed by $\widehat{Q}_{\ell+2}, P_{k-2}$.

Finally, note that this discrepancy cannot occur too many times since for each "backwards" (towards D_Δ) construction there is at least one "forward" (towards ω) construction: if the induction stops at some P_L, then there are $L-1$ rings around ω, and we may bound the probability of ω being sealed into $\partial\Omega_m$ by say constructing blue circuits in *at least* $(L-1)/2$ of these rings.

5.5 Consequences and Refinements

We will now establish some consequences and properties of the the procedure described in the previous subsections. First let us define some terminology:

Definition 5.9. Let $\Omega \subset \mathbb{C}$ be a bounded, simply connected domain and Ω_m some interior discretization of Ω. For $\omega \in \Omega_m$, consider the inductive construction as described, yielding P_1, P_2, etc., until the crossing probability criterion between ω and the last P_ℓ is less than ϑ indicating that we have approximately reached the unit scale.

We will refer to the the topological rectangles formed by successive P_ℓ's as *Harris rings* and the amalgamated system of these segments around ω the *Harris system stationed at* ω.

For our purposes we will also need to show that for n sufficiently large, for the marked point corresponding to A, the relevant Harris segments have endpoints lying in the anticipated boundary regions:

Lemma 5.10 *Let $\Omega \subseteq \mathbb{C}$ be a bounded simply connected domain with marked boundary prime ends $A, B, C, D \in \partial\Omega$ (in counterclockwise order) and suppose Ω_m is an interior approximation to Ω with $A_m, B_m, C_m, D_m \in \partial\Omega_m$ approximating A, B, C, D. Consider the hexagonal tiling problem studied in [15] or the flower models introduced in [11] (in which case we assume the Minkowski dimension of $\partial\Omega$ is less than 2) and the Harris system stationed at A_m. Then there is a number v_A such that for all m sufficiently large, all but v_A of the Harris segments are conduits from $[D_m, A_m]$ to $[A_m, B_m]$. More precisely, under uniformization, there exists some $\eta > 0$ such that all but $v_A = v_A(\eta)$ of these segments begin and end in the η–neighborhood of the pre–image of A.*

Proof. Let $\varphi : \mathbb{D} \to \Omega$ be the uniformization map with $\varphi(0) = z_0$ for some $z_0 \in D_\Delta$ and let $\zeta_A, \zeta_B, \zeta_C, \zeta_D$ denote the pre–images of A, B, C, D, respectively. Let $\eta > 0$ denote any number smaller than e.g., half the distance separating any of these pre–images. Let $N_\eta(\zeta_A)$ denote the η–neighborhood of A and let $\{r_d, r_b\}$ denote the pair $N_\eta(\zeta_A) \cap \partial\mathbb{D}$ with r_d in between ζ_A and ζ_D and r_b between ζ_A and ζ_B. Similarly, about the point ζ_C we have $N_\eta(\zeta_A) \cap \partial\mathbb{D} := \{s_d, s_b\}$.

We denote by $\mathscr{G}_d$ the continuum crossing probability from $[\zeta_A, r_d]$ to $[s_d, \zeta_C]$ (with $(\mathbb{D}; \zeta_A, \zeta_C, s_d, r_d)$ regarded as a conformal rectangle) and similarly $\mathscr{G}_b$ for the continuum crossing probability from $[r_b, \zeta_A]$ to $[\zeta_C, s_b]$. It is manifestly clear that these are non–zero since all relevant cross ratios are finite.

Now consider Ω as a conformal polygon with (corresponding) marked points (or prime ends) $A, R_b, B, S_b, \dots R_d$ and Ω_m with marked boundary points $A_m, \dots R_{b_m}$ the relevant discrete approximation. It is emphasized, perhaps unnecessarily, that this is just Ω_m with A, B, C, D and with four additional boundary points marked and added in. It follows by conformal invariance and convergence to Cardy's formula that the probability of a crossing in Ω_m from $[R_{b_m}, A_m]$ to $[C_m, S_{b_m}]$ converges to $\mathscr{G}_b$ and similarly for the crossings from $[A_m, R_{d_m}]$ to $[S_{d_m}, C_m]$.

We shall need an additional construct, denoted by Φ_m which is best described as the intersection of three events: (i) a yellow connection between ∂D_Δ and $[R_{d_m}, S_{d_m}]$, (ii) a similar connection between ∂D_Δ and $[R_{b_m}, S_{b_m}]$ and (iii) a yellow circuit in $\Omega_m \setminus D_\Delta$. It is observed that the intersection of these three events certainly implies a crossing between $[R_{d_m}, S_{d_m}]$ and $[R_{b_m}, S_{b_m}]$.

It is noted that item (iii) has probability uniformly bounded from below since D_Δ is contained in a circle twice its size. As for the other two, we must return to the continuum problem in $\mathbb{D}$. Let $\mathcal{E} \subset \mathbb{D}$ denote the preimage of D_Δ under uniformization with corresponding evenly spaced boundary points p_1, p_2, p_3 and p_4. Let us pick an adjacent pair of points – conveniently assumed to be p_1 and p_2 – which may be envisioned as approximately facing the $[s_d, r_d]$ segment of $\partial\mathbb{D}$. We now connect r_d and p_1 with a smooth curve in $\mathbb{D}$ and similarly for s_d and p_2. It is seen that these two lines along with the $[s_d, r_d]$ portion of $\partial\mathbb{D}$ and the $[p_1, p_2]$ portion of $\mathcal{E}$ are the boundaries of a conformal rectangle. We let $\mathscr{L}_d$ denote the continuum crossing probability from $[p_1, p_2]$ to $[s_d, r_d]$ within the specified rectangle.

We perform a similar construct involving p_3, p_4, s_b and r_b and denote by $\mathscr{L}_b$ the corresponding continuum crossing probability. Thus, as was the case above, in the corresponding subsets of Ω_m, it is the case that as $m \to \infty$, the probability of observing yellow crossings of the type corresponding to the aforementioned crossings in (i) and (ii) tend to $\mathscr{L}_d$ and $\mathscr{L}_b$, respectively. (While of no essential consequence, we might mention that at the discrete level, the relevant portions of ∂D_Δ may be *defined* to coincide with the inner approximations of the subdomains we have just considered.)

Let us call $\mathbb{G}_m$ the intersection of all these events: Φ_m and the pair of $[R_{b_m}, R_{d_m}] \rightsquigarrow [S_{b_m}, S_{d_m}]$ crossings (corresponding to $\mathscr{G}_b$ and $\mathscr{G}_b$). Then we have, uniformly in m for m sufficiently large,

$$\mathbb{P}(\mathbb{G}_m) \geq \sigma$$

for some $\sigma = \sigma(\eta) > 0$.

We next make the following claim:

Claim. Consider the event that there is blue path beginning and ending on $\partial\Omega_m$ that seperates A_m from D_Δ. Then, if the event $\mathbb{G}_m$ also occurs, it must be the case that (modulo orientation) the path begins on $[R_{b_m}, A_m]$ and ends on $[A_m, R_{d_m}]$.

Proof of Claim. To avoid clutter, we will temporarily dispense with all m–subscripts. Note that since we have divided the boundary into six segments, there are $\frac{1}{2} \cdot 6 \cdot 7 = 21$ cases to consider and, therefore, twenty to eliminate. Let us enumerate the cases:

∘ A crossing from $[C, A]$ to $[R_b, S_b]$ or from $[A, C]$ to $[S_d, R_d]$ (5 cases): each possibility is prevented by (at least) one of the yellow crossings between the segments in $[R_d, R_b]$ and $[S_b, S_d]$.

∘ Corner cases, e.g., $[C, S_d]$ to $[S_d, R_d]$ (4 cases): recalling that the blue path must separate D_Δ and A, these are obstructed by the yellow circuit about D_Δ which is connected to the *opposite* $R \cdot S$ boundary, which in this example corresponds to $[R_b, S_b]$. (We note that these circuits are constructed precisely to prevent the possibility of connections "sneaking" through D_Δ.)

∘ An $[R_d, R_b]$ segment connected to a $[S_b, S_d]$ segment (4 cases): these are prevented by the yellow crossing from $[S_d, R_d]$ to $[R_b, S_b]$.

∘ Diagonal (same to same) paths, e.g., $[C, S_d]$ to $[C, S_d]$ (6 cases): recalling the separation clause, these are obstructed by the connection of the circuit around D_Δ and its connection to whichever – or both – $R \cdot S$ segment which is *not* where the blue path begins and ends. In this example this corresponds to both $[S_d, R_d]$ and $[R_b, S_b]$.

∘ Finally, $[S_d, C]$ to $[C, S_b]$: this is the same as the previous case.

The claim is proved. □

With the above in hand, the rest of the proof of this lemma is immediate. Let v'_A denote the number of Harris segments in the system stationed at A_m which do *not* begin on $[R_{b_m}, A]$ and end on $[A, R_{d_m}]$. (I.e., the twenty cases treated above.) Letting $\mathbb{B}_m$ denote the event of a blue circuit of the type described in the claim, we have

$$1 - \sigma \geq 1 - \mathbb{P}(\mathbb{G}_m) \geq \mathbb{P}(\mathbb{B}_m) \geq 1 - (1 - \vartheta)^{v'_A}$$

which necessarily implies v'_A is bounded above (independently of m) by the ratio $\log \sigma / \log(1 - \vartheta)$. Clearly, $v'_A \geq v_A$ as in the statement of the lemma so the result has been established.

□

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