Lecture #7

CR Equations: Full result.

Writing: $f(z) = u(x,y) + iv(x,y)$ and $z_0 = x_0 + iy_0$, we now know that if f has a complex derivative @ $z = z_0$, it must be the case that

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) \quad \& \quad \frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0).$$

Now want to show that this is in fact sufficient.
That is to say if the above are satisfied (at $z = z_0$) then the complex derivative:

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists. Note: once such a limit is established (to exist), must equal $u_x + iv_x$ -- or $v_y - iu_y$ or ...

This will be "worst" analysis of the course. Will not be part of exams but *will* require you understand this -- at least momentarily.
Will not prove this under most general possible hypothesis. Enough for us that both u and v have two continuous derivatives.

Main idea -- not so difficult -- is way function behaves when you vary argument by small amount. (Here is where having the two (bounded) derivatives helps.) Suppose

$$q = q(x,y)$$

any function (with two derivatives). Then can write

$$q(x_0 + \Delta x, y_0 + \Delta y) = q(x_0, y_0) + [\Delta x]q_x(x_0, y_0) + [\Delta y]q_y(x_0, y_0) + \ldots$$

Well, what is "..."? Claim: Actually equals

$$[\Delta x]^2\alpha(\Delta x, \Delta y) + [\Delta y]^2\beta(\Delta x, \Delta y) + [\Delta x\Delta y]\gamma(\Delta x, \Delta y)$$

where α, β and γ are bounded functions (that depend on x_0 and y_0). In fact,

$$\lim_{(\Delta x, \Delta y)\to 0}\alpha(\Delta x, \Delta y) = \frac{1}{2}q_{xx}(x_0, y_0)$$

$$\lim_{(\Delta x, \Delta y)\to 0}\beta(\Delta x, \Delta y) = \frac{1}{2}q_{yy}(x_0, y_0)$$

and

$$\lim_{(\Delta x, \Delta y)\to 0}\gamma(\Delta x, \Delta y) = q_{xy}(x_0, y_0)$$

but this not really important; just the "bounded" part.

So, now do this for $f(z_0 + \Delta z) = u(x_0 + \Delta x, y_0 + \Delta y) + iv(x_0 + \Delta x, y_0 + \Delta y)$:

Well: $$u(x_0 + \Delta x, y_0 + \Delta y) = u(x_0, y_0) + [\Delta x]u_x(x_0, y_0) + [\Delta y]u_y(x_0, y_0) + {} + [\Delta x]^2 A(\Delta x, \Delta y) + [\Delta x]^2 B(\Delta x, \Delta y) + C(\Delta x, \Delta y).$$

with $A \to u_{xx}(x_0, y_0)$ as $(\Delta x, \Delta y) \to 0$, etc. Similar for v.

First claim: $$\lim_{(\Delta x, \Delta y)\to 0} \frac{[\Delta x]^2 A(\Delta x, \Delta y)}{\Delta z} = 0.$$

Proof: Since A tends to a definite limit -- or is anyway has absolute value bounded, enough to show

$$\lim_{(\Delta x, \Delta y)\to 0} \frac{[\Delta x]^2}{\Delta z} = 0.$$ "Somehow obvious"

And look. Enough to show $$\lim_{(\Delta x, \Delta y)\to 0} \left|\frac{[\Delta x]^2}{\Delta z}\right| = 0$$

but $$\left|\frac{[\Delta x]^2}{\Delta z}\right| = \left|\frac{[\Delta x]^2}{\sqrt{[\Delta x]^2 + [\Delta y]^2}}\right| \leq \frac{[\Delta x]^2 + [\Delta y]^2}{\sqrt{[\Delta x]^2 + [\Delta y]^2}} = \sqrt{[\Delta x]^2 + [\Delta y]^2} \to 0.$$

Exactly the same argument shows $\displaystyle\lim_{(\Delta x,\,\Delta y)\to 0} \frac{[\Delta y]^2\, B(\Delta x, \Delta y)}{\Delta z} = 0.$

Now, last little business, show: $\displaystyle\lim_{(\Delta x,\,\Delta y)\to 0} \frac{[\Delta x \Delta y]C(\Delta x, \Delta y)}{\Delta z} = 0,$ but just amounts to showing

$$\lim_{(\Delta x,\,\Delta y)\to 0} \frac{[\Delta x \Delta y]}{\Delta z} = 0.$$

Not hard, use $2\Delta x\Delta y \le [\Delta x]^2 + [\Delta y]^2$ -- complete the square -- and same follows.

Alright, now down to business. Expand our u and v:

$$u(x_0 + \Delta x, y_0 + \Delta y) + iv(x_0 + \Delta x, y_0 + \Delta y) - [u(x_0, y_0) + iv(x_0, y_0)]$$

$$= [\Delta x]u_x(x_0, y_0) + [\Delta y]u_y(x_0, y_0) + i[\Delta x]v_x(x_0, y_0) + i[\Delta y]v_y(x_0, y_0) + [\Delta x]^2\mathbb{A} + [\Delta y]^2\mathbb{B} + [\Delta x\Delta y]^2\mathbb{D}.$$

Here $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{D}$ are the (complex) left–overs; tend to a definitive limit and, anyway are bounded.

So now divide by Δz and attempt to take the limit as $\Delta z \to 0$:

$$f'(z_0) \;\; ¿=? \;\; \lim_{(\Delta x,\, \Delta y)\to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

should this limit exist

$$\lim_{(\Delta x,\, \Delta y)\to 0} \frac{[\Delta x]u_x(x_0,y_0)+[\Delta y]u_y(x_0,y_0)+i[\Delta x]v_x(x_0,y_0)+i[\Delta y]v_y(x_0,y_0)}{\Delta z}$$

$$+$$

$$\lim_{(\Delta x,\, \Delta y)\to 0} \frac{[\Delta x]^2\mathbb{A}+[\Delta y]^2\mathbb{B}+[\Delta x\Delta y]\mathbb{D}}{\Delta z}.$$

should this limit exist.

But second limit certainly does exist; terms of the form that we just showed were zero.

Thus: Existence -- or lack thereof -- of $f'(z)$ boils down to the first order terms.
This, will be settled by satisfaction -- or lack thereof -- of the CR equations.

Final claim: If $u_x = v_y$ and $u_y = -v_x$, then limit *does* exist -- and is equal to, say, $u_x + iv_x$.

Here, to save time, all functions evaluated @ (x_0, y_0).

Eliminate all terms with a y subscript:

$$[\Delta x]u_x + i[\Delta x]v_x + [\Delta y]u_y + i[\Delta y]v_y = [\Delta x]u_x + i[\Delta x]v_x - [\Delta y]v_x + i[\Delta y]u_x$$
$$= [\Delta x + i\Delta y]u_x + i[\Delta x]v_x + i^2[\Delta y]v_x$$
$$= u_x[\Delta x + i\Delta y] + iv_x[\Delta x + i\Delta y]$$

Divide by $\Delta x + i\Delta y$, and take limit; done; $f(z_0)$ exists, equals $u_x(x_0, y_0) + iv_x(x_0, y_0)$
(or $v_y(x_0, y_0) - iu_y(x_0, y_0)$ or, ...).

Reiterate: Upshot very nice (and important): Can just check CR–equations and, if satisfied, function has a complex derivative and if not, doesn't.