Lecture # 12

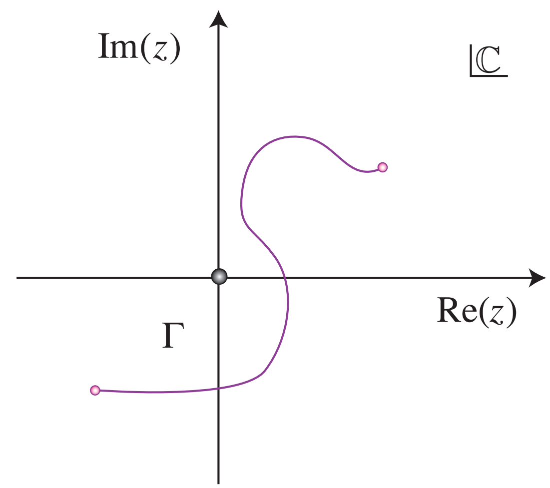
Integral Bounds

Now, low point of exclusive reliance on parametric description of line integration.

-- Want to state (and prove) inequality which is obvious from Riemannian construction.

Here, actually requires slight bit of cleverness.

Recall, from 32B there was *second* type of line integral.



As before, Γ a path and g(x,y) a function (smooth etc.) on path.

Γ parameterized by $x = x(t)$, $y = y(t)$; $t_1 \le t \le t_2$.
These have derivatives $\dot{x}(t)$ and $\dot{y}(t)$ respectively. Then we define

$$\int_\Gamma g\,ds \;=\; \int_{t_1}^{t_2} g(x(t),y(t))\sqrt{\dot{x}^2+\dot{y}^2}\,dt.$$

Reminder: This was the type of path integral which did *not* depend on orientation of path.

Finally, recall that if $g(x,y) \equiv 1$ then $\int_\Gamma ds = |\Gamma|$ is called the *arclength* of the curve Γ.

Now for 132 these second type of line integrals not so important in their own right. But needed to prove other results which *are* important.

Useful claim(s), as far as this course is concerned:

[I] $\left| \int_\Gamma f(z)dz \right| \leq \int_\Gamma |f(z)|\,ds$

[II] $\int_\Gamma |f(z)|\,ds \leq (|\Gamma|)[\mathbb{F}_{max}(\Gamma)]$ where $\mathbb{F}_{max}(\Gamma)$ is the maximum value that $|f(z)|$ takes on along Γ.

First inequality means: (1) do integral on lhs, will get complex number. Take modulus of *that* complex number, will get positive real number. That positive real number is smaller than (or equal to) what you would get if you did "other type" of 32B integral where "$g(x,y)$" is $|f(z)|$.

Most often, we use these in tandem -- eliminating the middle term.

Inequality [II] actually trivial -- should be taught in 32B. Do this for general $g(x,y)$ which happens to be positive. Write down line integral of g according to some parameterization:

$$\int_\Gamma g(x,y)ds = \int_{t_1}^{t_2} g(x(t),y(t))\sqrt{\dot{x}^2+\dot{y}^2}\,dt \leq \mathbb{G}\int_{t_1}^{t_2}\sqrt{\dot{x}^2+\dot{y}^2}\,dt$$

where $\mathbb{G} = \max\limits_{t_1 \leq t \leq t_2} g(x(t), y(t))$ is the maximum value that g achieves on the contour Γ.

But coefficient of this $\mathbb{G}$ is just the length of Γ and so we

$$\int_\Gamma g(x,y)ds \leq \mathbb{G}|\Gamma|$$

just our second inequality for $g = |f(z)|$.

First inequality slightly more serious.

Actually will prove weaker form of [I] with x–tra factor of 2 (or $\sqrt{2}$) out front. More clever derivation gets rid of excess factor. But these numbers not important. will just use these as crude bounds.

Look at real part: $\mathrm{Re}[\int_\Gamma f(z)dz] = \int_{t_1}^{t_2} [u(x(t),y(t))\dot{x} - v(x(t),y(t))\dot{y}]dt.$

Now, in general, $|\int_a^b h(q)dq| \le \int_a^b |h(q)|dq$

so $|\mathrm{Re}[\int_\Gamma f(z)dz]| \le \int_{t_1}^{t_2} |[u(x(t),y(t))\dot{x} - v(x(t),y(t))\dot{y}]|dt.$

So far, everything straight forward. But now, want to borrow from vector theory:

Recall that if $\vec{A}$ and $\vec{B}$ are (2–component) vectors:

$$\vec{A} = (a_1, a_2)\ ;\ \vec{B} = (b_1, b_2)$$

then

$$|\vec{A}\cdot\vec{B}| = |\vec{A}||\vec{B}||\cos\Theta_{AB}| \ \leq\ |\vec{A}||\vec{B}|$$

(where Θ_{AB} is the angle between $\vec{A}$ and $\vec{B}$).

As far as the numbers a_1, a_2, b_1 and b_2 are concerned, all of this says:

$$|a_1b_1 + a_2b_2| \ \leq\ (\sqrt{a_1^2 + a_2^2})(\sqrt{b_1^2 + b_2^2}).$$

So, this is quite general and applies even if the "numbers" a_k and b_k happen to be time dependent, real and imaginary parts of some function, etc.

Thus $|u\dot{x} - v\dot{y}| \ \leq\ [u^2 + v^2]^{½}[\dot{x}^2 + \dot{y}^2]^{½}$ – and this, of course is valid inside the integrand.

So:

$$\int_{t_1}^{t_2} |[u(x(t),y(t))\dot{x} - v(x(t),y(t))\dot{y}]|\,dt \quad\leq\quad \int_{t_1}^{t_2} \underbrace{\sqrt{u^2+v^2}}_{\text{that's } |f|}\ \underbrace{\sqrt{\dot{x}^2+\dot{y}^2}\,dt}_{\text{that's } ds.}$$

That's it for real part. Derivation for imaginary part very similar gets us

$$\left|\int_\Gamma f(z)dz\right| \ \leq\ 2\int_\Gamma |f(z)|\,ds\ .$$