Lecture # 17

Derivative Bounds

Following topic -- useful in its own right & important for when we do series -- easy consequence of Cauchy integral formula; in general,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_\Gamma \frac{f(z)dz}{(z-z_0)^{n+1}}.$$

Usually, cannot actually *calculate* anything but can keep making crude estimates till we get something we *can* do:

$$\left|f^{(n)}(z_0)\right| \le \frac{n!}{2\pi}\oint_\Gamma \frac{|f(z)||dz|}{|(z-z_0)|^{n+1}} \le \frac{n!}{2\pi}\mathbb{F}_{\mathrm{Max}}(\Gamma)\oint_\Gamma \frac{|dz|}{|(z-z_0)|^{n+1}}$$

where (we recall) $\mathbb{F}_{\mathrm{Max}}(\Gamma)$ is the maximum value of $|f(z)|$ on the contour Γ and $|dz|$ is our private notation for an integral with respect to arclength -- "ds".

Still, nothing we can readily calculate ... except in the case that the contour is circular.

If Γ is a circle of radius R, centered about z_0, get

$\mathbb{F}_{\text{Max}}(R)$ is notation for the maximum of $f(z)$ on the circle of radius R surrounding z_0.

$$\boxed{\left|f^{(n)}(z_0)\right| \le \frac{n!}{2\pi}\mathbb{F}_{\text{Max}}(R)\frac{1}{R^{n+1}}2\pi R = \frac{n!}{R^n}\mathbb{F}_{\text{Max}}(R).}$$

Key idea (not emphasised in the book) can use any R that you want. Several not particularly interesting things that you can do. Here is one that *is* of intrest.

Q: How big is $n!$

Crude: $n! \le n^n$.

There are n terms, all of them less or equal to n.

Not as bad as you think: $n! \ge \left(\frac{n}{2}\right)^{\frac{n}{2}}$.

There are at least $\frac{\text{n}}{2}$ terms larger or equal to $\frac{\text{n}}{2}$.

So, somehow growth faster than exponential; looks to be more like n^n.

We will show this.

Detailed asymptotics (a little beyond this course). **Stirling's Formula:**

$$n! \sim n^n e^{-n}\sqrt{2\pi n}\,[1 + \tfrac{c_1}{n} + \ldots\,]$$

Accurate in the sense that difference between trunkation of rhs & lhs times appropriate power of n tends to definitive constant.

Above not so easy but claim we can get leading term (as upper bound) using our crude estimates.

Well, e^z nice analytic function. Also has advantage that derivative is itself.

In particular $\left.\dfrac{d^n}{dz^n} e^z\right|_{z=0} = 1.$

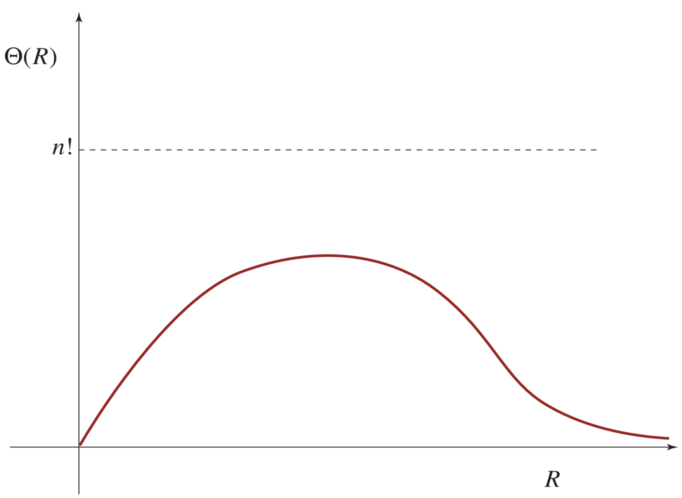
But then letting R denote *any* radius, we learn:

$$\frac{n!}{R^n} \max_{|z| = R} |e^z| \geq 1.$$

Can also easily find this maximum: $|e^z| = |e^{R\cos\theta}||e^{iR\sin\theta}| = |e^{R\cos\theta}| \leq e^R.$

Conclusion: $\boxed{n! \geq R^n e^{-R}}$

¿ What R? If I pick $R << 1$, will be small and same if $R >> 1$. Optimise. $\Theta(R) = R^n e^{-R}$.



Take derivative, set equal to zero:

$$nR^{n-1}e^{-R} - R^n e^{-R} = 0$$

Maximum @ $R = n$.

$$n! \geq n^n e^{-n}.$$

Not a bad bound when you compare with actual asymptotics.