UCLA Department of Mathematics

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Math 132 Section 2 Spring 2021

Problem Set #8

Problem (1) Suppose that $f(z)$ and $g(z)$ are two entire analytic functions and there is a fixed κ such that for all z, $|f(z) - g(z)| < \kappa$. Show that this implies that $f(z) = g(z) + c$ where c is a constant. (And satisfies $|c| \leq \kappa$.)

Problem (2) Suppose that $f(z)$ is an entire function which satisfies $|f(z)| \leq z^2$. Show that this implies that f is a quadratic (albeit possibly trivial) function.

Problem (3) A function $f(z)$ is analytic in the disk $|z| \leq 1$ where the modulus satisfies the bound

$$|f(z)| \leq a + \frac{1}{2} b |z|^3$$

Here $b \geq a > 0$ Find an optimal bound on $|f'(0)|$ in terms of a and b. Complete arguments required. [By optimal it is meant that (1) the bound holds for all functions with the stated property and (2) there actually is a function with the stated property such that the bound holds as an equality. The second part of this problem will be considerably more challenging that the first.]

Problem (4) Determine the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n z^n$ where $c_n = \mathrm{e}^{sn}$.

Problem (5) Determine the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n z^n$ where $c_n = n^4 \mathrm{e}^{\kappa\sqrt{n}}$.

Problem (6) Determine the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n z^n$ where $c_0 = 0$ and, for $n > 0$,

$$c_n = \Delta^n \prod_{k=1}^{n} (1 + \frac{1}{k}).$$

Problem (7) Determine the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n z^n$ where $c_n = \frac{(2n+1)!!}{n!}$ and, we recall, for m odd, $m!! = (1)(3)\ldots(m)$.

Problem (8) The (famous) hypergeometric series is defined by

$$F(a,b,c;z) = 1 + \frac{ab}{1c}z + \frac{[a(a+1)][b(b+1)]}{(1)(2)[c(c+1)]}z^2 + \frac{[a(a+1)(a+2)][b(b+1)(b+2)]}{(1)(2)(3)[c(c+1)(c+3)]}z^3 + \dots$$

where, in general a, b, and c are complex numbers with the exception, of course, that c is cannot be a negative integer. Find the radius of convergence of this series. Assume, here that a and b are also not negative integers. Otherwise, what happens?

Problem (9) Consider the power series $\sum_{n=0}^{\infty} c_n z^n$ where $c_n = 3^n$ if n is even and $c_n = 2^n$ if n is odd. Show that both the ratio test and the root test fail to produce a limit and, nevertheless, find the radius of convergence. Include an explicit demonstration that your answer is right.

Problem (10) Find the radius of convergence of the series

$$\varphi(z) = \sum_{n=0}^{\infty} \frac{n^n}{n!} z^n.$$

Problem (11) Suppose that $\sum_{n=0}^{\infty} c_n z^n$ has a radius of convergence r where, for simplicity, $0 < r < \infty$ and r is "produced" by the ratio test. Show that for any positive number k, $\sum_{n=0}^{\infty} n^k c_n z^n$ has the same radius of convergence.
Remark: This result holds even if r is "existential"; see if you can prove this.

Problem (12) Suppose that $f(z)$ is a *non–constant* function in and on a simple closed contour Γ. Further suppose that for all z on Γ, $|f(z)|$ is identically equal to some constant $b > 0$. Show that this implies that f has at least one zero in $\mathrm{Int}(\Gamma)$. Hint If an analytic function has no zeros in a region, than its log would also be analytic. For the purposes of this exercise, you may consider Γ to be a circular contour but this actually is true in general. I.e., see the next problem

Problem (13) Prove the following *minimum* modulus principle: Suppose $f(z)$ is analytic in and on a simple closed contour Γ and that $f(z)$ has no zeros in $\mathrm{Int}(\Gamma)$. Then $|f(z)|$ achieves its minimum value on Γ. (Note that the condition about no zeros is important. It should enter into your proof and without this condition, it is easy to construct counterexamples like $f(z) = z$.

Problem (14) Let $f(z)$ be an entire analytic function with the property that $\mathrm{Re} f(z)$ is bounded. Show that this necessarily implies that f is constant. Hint: consider the function $\mathrm{e}^{f(z)}$.