UCLA Department of Mathematics

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Math 132 Section 2 Spring 2021

Problem Set # 7

Problem (1) Recall: If Γ is a curve parameterized by arc–length s then the curvature is (to within a sign) defined by

$$\kappa = \frac{d\phi}{ds}$$

where ϕ is the angle of the tangent vector. Using the complex description it was shown that

$$\kappa = -i\frac{z''}{z'} = \mathrm{IM}[\frac{d\log z'}{ds}].$$

Derive the formula for curvature for the general case of general parameterization $z = z(t)$.

Problem (2) Let Γ be the circular contour of radius 2 centered at the origin. Compute

$$\oint_\Gamma \frac{\cos z}{z^3 + 9z} dz \quad \text{and} \quad \oint_\Gamma \frac{\mathrm{e}^{-z}}{(z+1)^2} dz$$

Problem (3) Let Γ be the circular contour of radius 2 centered at the origin. Compute

$$\oint_\Gamma \frac{5z^2 + 2z + 1}{(z-i)^3} dz \quad \text{and} \quad \oint_\Gamma \frac{\sin^2 z}{z^2(z-4)} dz$$

Problem (4) Consider the function

$$f(z) = \frac{1}{1-z}$$

which is certainly analytic in and on any circle of radius $a < 1$ centered at the origin. Using the Cauchy bounds for derivatives with these circular contours obtain “good” upper bounds for $|f^{(n)}(0)|$ by optimizing the radius of the contour.

Problem (5) A function $f(z)$ is analytic in the disk $|z| \leq 1$ where it enjoys the bound

$$|f(z)| \leq a + b|z|^2$$

where $b \geq a \geq 0$. Show that at the origin, the modulus of the derivative, $|f'(0)|$, is bounded by $2\sqrt{ab}$.

Problem (6) Let $p(z) = a_0 + a_1 z + \cdots + a_k z^k$ be a polynomial which, for all z with $|z| = 1$, satisfies $|p(z)| \le M$. Show that this necessarily implies $|a_j| \le M$ for $j = 0, 1, ...k$.

Problem (7) Suppose that Γ is a simple closed contour and that $f(z)$ and $g(z)$ are analytic functions in (and on) Γ which agree at all points z on Γ. Show that, necessarily, $f = g$ at all points *inside* Γ.

Problem (8) Consider the function $f_{[k]}(z)$ given by the formula

$$f_{[k]}(z) = \frac{1}{2-z}\frac{1}{3-z}\cdots\frac{1}{k+1-z}$$

(with $k \geq 2$ an integer) which is analytic except at the points $z = 2, \ldots z = k+1$. (Note: this is a finite product and ends at the integer $k+1$.) Show that, at $z = 0$, the first k derivatives of $f_{[k]}(z)$ all have modulus less than one.

Problem (9) If $g(z)$ is a "nice" (i.e. continuous) function defined on – but not necessarily inside – a simple closed curve Γ, one can always define an analytic function inside Γ via the formula

$$G(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{g(\eta)}{\eta - z} d\eta.$$

Show, however, that it need not be the case that $G(z)$ tends to $g(z)$ as z approaches the contour by considering the case of $g(z) = 1/z$ and Γ a circle of radius a centered at the origin. Hint: use partial fractions to evaluate explicitly $G(z)$.

Problem (10) Let us reconsider a much (over)discussed integral: Following the procedure implemented in class, compute

$$I_a^{[n]} = \int_0^{2\pi} \frac{\cos n\theta d\theta}{1 + a^2 - 2a\cos\theta}$$

where n is an integer. Comment on the large n behavior.

Problem (11) For $0 \leq \theta \leq 2\pi$ there is an unusual looking function $\Phi(\theta)$ which is given by the expression

$$\Phi_a(\theta) = \mathrm{e}^{a\cos\theta}[\cos(a\sin\theta)]$$

where a is a real parameter. considering $z = r\mathrm{e}^{i\theta}$ Focusing attention on the unit circle show that $\Phi(\theta)$ is in fact the real part of a perfectly ordinary analytic function that has been restricted to the unit circle. Then, evaluate the integral

$$\int_0^{2\pi} \Phi_a(\theta) d\theta.$$