Department of Mathematics

Math 132 Section 2 Spring 2021

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Section:

Student ID #

Problem Set # 6

Problem (1) Let $\vec{F} = \langle P, Q \rangle$ denote a conservative vector field, $\vec{F} = \nabla\phi$, which is defined in some region that includes a simple closed contour Γ and its interior. It may be assumed that ϕ is twice differentiable. While it is obvious on the basis of potential theory that

$$0 = \oint_{\Gamma} \vec{F} \cdot d\vec{r}$$

show that this result can also be obtained via Green's theorem.

Problem (2) Let $f(z) = z^\alpha$ where α is real and not equal to -1. Of course when α is non–integer we had better say something about the range of "θ" – to be definitive, $0 \le \theta \le 2\pi$. Let Γ denote the circle of radius R. Derive a formula for

$$\oint_\Gamma z^\alpha dz.$$

Your formula should agree with all known answers including the limit $\alpha \to -1$.

Problem (3) Let $z = z(s)$ denote a path $\mathcal{P}$ in $\mathbb{C}$ that is parameterized by arc–length. Derive a (beautiful) formula for the curvature $\kappa(s)$ along $\mathcal{P}$.

Problem (4) Let $f(z)$ be an analytic function in a circular region Ω wherein lie the points z_1 and z_2. It is known that for all z in Ω, $|f'(z)| < M$. Show that $|f(z_1) - f(z_2)| \leq M|z_1 - z_2|$.

Problem (5) If γ s the straight line segment from $z = R$ to $z = R + 2\pi i$ (with $R > 0$) show that

$$|\int_{\Gamma} \frac{\mathrm{e}^{3z}}{1+\mathrm{e}^{z}} dz| \leq 2\pi \frac{\mathrm{e}^{3R}}{\mathrm{e}^{R}-1}.$$

Problem (6) If Γ is the circle of radius 3 centered at the origin, show that

$$\left|\int_\Gamma \frac{dz}{z^2-1}\right| \leq \frac{3}{4}\pi.$$

Problem (7) Consider the functions $P_n(s)$ defined by the integral formulas

$$P_n(s) = \frac{1}{\pi}\int_0^\pi (s + i\sqrt{1-s^2}\cos\vartheta)^n d\vartheta$$

where $|s| \leq 1$. (These are the famous Legendre polynomials.) Use the integral inequalities which have been discussed to show that for all these s, $|P_n(s)| \leq 1$.

Problem (8) Let $K(z)$ be a *bounded* analytic function of z which is to say that there is a number $B < \infty$ such that for all z, $|K(z)| < B$. Now consider the contour integral

$$I_R := \oint_{C_R} \frac{K(z)dz}{z^2}$$

where C_R is the circle of radius R centered at the origin. Get an estimate on $|I_R|$ and compute $\lim_{R\to\infty} I_R$. Later we will show on the basis of the above and similar argument that such functions are not particularly interesting.

Problem (9) Suppose that in and on the unit circle $f(z)$ is of the form

$$f(z) = \frac{A_k}{z^k} + \frac{A_{k-1}}{z^{k-1}} + \cdots + \frac{A_1}{z} + g(z)$$

where g is analytic (for all z in and on the unit circle). show that

$$\oint_{|z|=1} f(z)dz = 2\pi A_1.$$

Problem (10) Determine the domain of analyticity for the function

$$f(z) = \sec \frac{z}{2}$$

and use this calculate $\oint_{|z|=2} f dz$.

Problem (11) Calculate the following integral:

$$\int_\Gamma \frac{dz}{1+z^2}$$

where Γ is the line segment from $z = 1$ to $z = 1 + i$

Problem (12) Let Γ be the circular contour of radius 2 centered at the origin. Compute

$$\oint_\Gamma \frac{\sin 3z}{z - \frac{\pi}{2}} dz \ \text{ and } \ \oint_\Gamma \frac{z\mathrm{e}^z}{2z-3} dz.$$

Problem (13) Following the procedure implemented in class, compute

$$I_a = \int_0^{2\pi} \frac{\cos\theta d\theta}{1 + a^2 - 2a\cos\theta}$$

for the case $|a| > 1$. Check your answer by deriving the formula for I_a when $|a| > 1$ from the formula for I_a when $|a| < 1$.

Problem (14) Consider again the integral

$$I_a = \int_0^{2\pi} \frac{\cos\theta d\theta}{1 + a^2 - 2a\cos\theta}$$

featured in Problem 13. For $|a| < 1$ this is known to equal $\frac{2\pi a}{1-a^2}$. Regarding a as a *small* parameter, check your answer to order a^3 by direct expansion of the integrand.