UCLA Department of Mathematics

Name: ______ Last ______ First ___ MI

Section:

Student ID # □□□-□□□-□□□

Math 132 Section 2 Spring 2021

Problem Set # 5

Problem (1) It is sometimes thought that failure of differentiability (or definition of the function) is required in order that a function be non–analytic. This is often true but it need not always be the case. Consider the two examples

$$f(z) = z^N \log z \quad \& \quad g(z) = z^{\alpha}.$$

In the first case N is an integer ≥ 2 and in the second case α is a positive real non–integer greater than one. In both cases we will take $-\pi < \theta \leq \pi$ (i.e. branch cut on the negative real axis) and in the first case we must formally define $f(0) = 0$ – its limiting value. Thus the functions are defined everywhere.

Part A. Show that both functions are non–analytic at the origin.

Part B. Show that both functions are complex differentiable at the origin; use an "$f(z + \Delta z) - f(z)$" argument.

Problem (2) Find a parameterization for the line segment joining the points z_0 and z_1 in the complex plane. Express your answer in the favored "$z(t)$" form.

Problem (3) For z_0 a generic point in the complex plane and $a > 0$ a positive real number, find a parameterization $z = z(t)$ for the curve (circle) $|z - z_0| = a$. Express your parameterization using complex exponentials and, similarly, find $\dot{z}(t)$.

Problem (4) Let us find an easy way using complex integration (here, perhaps, integration of a complex valued function of a real variable) to compute the integral

$$\int_a^b \mathrm{e}^{\alpha t} \cos \beta t dt$$

(1) Notice that the integrand is the real part of some fairly simple complex function of t.

(2) Do the indefinite integral of said complex function.

(3) Take the real part of your answer.

Before you commit to your answer you should (i) differentiate your ultimate answer – just to be sure. (ii) Check (or recollect) the standard methods for such computations which involve multiple integrations by parts.

Problem (5) Calculate the following integral:

$$\int_{\Gamma} (3z^2 - 5z - i)dz$$

where Γ is the straight line segment from $z = i$ to $z = 1$

Problem (6) Calculate the following integral:

$$\int_{\Gamma} \sin^2 z \cos z dz$$

where Γ is the strait line from $z = \pi$ to $z = i$

Problem (7) Consider the function $\mathrm{Log} z$ (where we use the principal value meaning that the "argument" runs from $-\pi$ to $+\pi$) and let Γ denote the contour which is the quarter circle of radius R in the positive quadrant connecting the real to the imaginary axis. Compute $\int_\Gamma \mathrm{Log} z dz$.

Problem (8) Calculate the following integral:

$$\int_\Gamma \mathrm{e}^z dz$$

where Γ is the upper half circle of radius one centered at the origin.

Problem (9) Compute the contour integrals

$$\int_{\Gamma_1} z dz \quad \& \quad \int_{\Gamma_2} z dz$$

where Γ_1 is the straight line segment from 0 to $1+i$ and Γ_2 is the parabolic arc $y = x^2$ from 0 to $1+i$

Problem (10) Compute the contour integrals

$$\int_{\Gamma_1} \overline{z} dz \quad \& \quad \int_{\Gamma_2} \overline{z} dz$$

where Γ_1 is the straight line segment from 0 to $1+i$ and Γ_2 is the parabolic arc $y = x^2$ from 0 to $1+i$

Problem (11) Let Γ be the portion of the circle of radius R which lies between the angles α and β. Using ideas of complex integration, derive, for $n > 1$, the formula for the integrals

$$\int_{\Gamma} z^n dz.$$

Problem (12) Let p and q denote positive real numbers with $p > q$. Show that the curve

$$z(t) = p\mathrm{e}^{it} + q\mathrm{e}^{-it}$$

with $0 \leq t \leq 2\pi$ is a parametric description for an ellipse. Find the semi–major and semi–minor axes.